Definition

Homeomorphism

Topology

Definition

Continuity via neighbourhoods

Topology

Definition & Proposition

Open, Closed sets and their properties

Topology

Definition, Example

Limit point

Topology

Definition & Theorem w/ proof

Closure

Topology

Definition

Topological space via neighbourhoods

Topology

Definition

Subspace topology

Topology

Definition

Topology via open sets

Topology

Theorem w/ proof

A set is closed if and only if it contains all its limit points.

Topology

Definition

Dense, Interior, Frontier

Topology

A **topological space** is a set X equipped with a nonempty collection $\mathcal{N}_x$ of subsets - a topology - of X (called **neighbourhoods**) for each $x \in X$ such that

1. $x \in N$ for all $N \in \mathcal{N}_x$,
2. $N \cap N' \in \mathcal{N}_x$ for all $N, N' \in \mathcal{N}_x$,
3. for all $N \in \mathcal{N}_x$ and all $U \subset X$, $N \subset U$ implies $U \in \mathcal{N}_x$ ("neighbourhoods are *large enough*"),
4. for all $N \in \mathcal{N}_x$, their **interior** $\mathring{N} := N^\circ := \{z \in N : N \in \mathcal{N}_z\}$ is in $\mathcal{N}_x$.

A function $f\colon X \to Y$ is a **homeomorphism** if f is *bijective* and f as well as f^{-1} are *continuous*. We then write $X \approx Y$ and say that X and Y are **homeomorphic**.

Let X be a topological space and $Y \subset X$ a subset.
Via neighbourhoods: For $y \in Y$ and $N \in \mathcal{N}_y \subset X$ we declare $N \cap Y$ to be a neighbourhood of y with respect to Y.
Via open sets: The open sets of Y with respect to the subspace topology on X are precisely the sets $O \cap Y$, where O is open in X.
This is the **subspace topology** *induced by* Y *on* X.
It is the initial topology with respect to the inclusion $Y \to X$.

A map $f\colon X \to Y$ between topological spaces is **continuous** if for all $x \in X$ and for all neighbourhoods $N \in \mathcal{N}_{f(x)}$, $f^{-1}(N) \in \mathcal{N}_x$.

Let T be a family of subsets of a set X. We call the elements of T **open** and we require that unions, finite intersections and $\varnothing, X$ are open. A set $N \subset X$ is a *neighbourhood* of $x \in X$ if there exists an open set $O \in T$ such that $x \in O \subset N$. Then the collection of neighbourhoods T is a topology on X.
A subset $A \subset X$ is **closed** if $X \backslash A$ is *open*.
Finite unions and arbitrary intersections of closed sets are closed.

A subset $O \subset X$ is **open** if for all $x \in O$, $O \in \mathcal{N}_x$, that is, O is a neighbourhood of x.
Let I be an index set such that $O_i \subset X$ is open for all $i \in I$. Then $\bigcup_{i \in I} O_i$ is open, if I finite, then $\bigcap_{i \in I} O_i$ is open, the sets $\varnothing$ and X are open.

" $\Longrightarrow$ ": Let A be closed. Then $X \backslash A$ is open, so $X \backslash A \in \mathcal{N}_x$ for all $x \in X \backslash A$. Hence no point in $X \backslash A$ can be a limit point, so A contains all its limit points.
" $\Longleftarrow$ ": Suppose A contains all its limit points. Let $x \in X \backslash A$. Then x is not a limit point. Then there exists a neighbourhood $N \in \mathcal{N}_x$ such that $A \cap N = \varnothing$, so $N \subset X \backslash A$. Hence $X \backslash A$ is a neighbourhood of each of its point, so it is open, hence A is closed.

Let $A \subset X$ be a subset. A point $x \in X$ is a **limit point** (or accumulation point) *of* A if $(A \backslash \{x\}) \cap N \neq \varnothing$ for all neighbourhoods $N \in \mathcal{N}_x$ of x.
Every $x \in \mathbb{E}^n$ is a limit point of $\mathbb{Q}^n$. No $x \in \mathbb{E}^n$ is a limit point of $\mathbb{Z}^n$.

Let $A \subset X$.
Then A is **dense** if $\overline{A} = X$.
The **interior** of A, $\mathring{A}$, is the union of all open sets contained in A.
The **frontier** of A is $\overline{A} \cap \overline{X \backslash A}$.

The *closure* of $A \subset X$, $\overline{A}$, is the union of A with all of its limit points.
The set $\overline{A}$ is the *smallest closed set containing* A.
Corollary: A set $A \subset X$ is closed if and only if $A = \overline{A}$.

Definition, Remark

Basis and subbasis

Topology

Definition

Coarser / finer topology

Topology

Definition, 3 Lemmas w/o proof

Disk

Topology

Theorems, Lemma w/o proofs

Compactness

Topology

Definition

One-point compactification

Topology

Definition, Examples, Theorem w/o proof

Continuity (in terms of open sets)

Topology

Definition, Lemma

Hausdorff space

Topology

Definition

Compact

Topology

Theorem w/o proof

Lebesgue lemma

Topology

Definition, Theorem w/o proof

Product topology

Topology

A function $f\colon X \to Y$ is **continuous** (a map) if and only if for all open sets $O \subset Y$, the full preimage $f^{-1}(O) \subset X$ is open.
Composition of maps is a map. Restriction of a map is a map. Identity and inclusion map are maps.
$f\colon X \to Y$ is *map* $\iff$ $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$. $\iff$ $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$. $\iff$ f cts on basis $\iff$ $f^{-1}(B)$ is closed for all closed sets $B \subset Y$.

A topological space X is **Hausdorff** if for any two distinct points $x, y \in X$ there exists *disjoint* open neighbourhoods of x and y, respectively.
A space X is HAUSDORFF if and only if $\{x\} = \bigcap\{\overline{U} : U \in \mathcal{N}_x\}$ holds for all $x \in X$.

A family $\mathcal{F}$ of open subsets of X is an **open cover** of X if $\bigcup_{F\in\mathcal{F}} F = X$. A family $\mathcal{F}'$ is a **subcover** of $\mathcal{F}$ if $\mathcal{F}'$ is a open cover and $\mathcal{F}' \subset \mathcal{F}$.
The space X is **compact** if every open cover of X has a *finite subcover*.

Let (X, d) be a *compact metric space* and $\mathcal{F}$ be an *open cover* of X. Then there exists a $\delta > 0$ (the LEBESGUE number of $\mathcal{F}$) such that any subset of diameter less than δ is contained in some $F \in \mathcal{F}$.

Application: Suppose $U \cup V$ is a open cover of $[0, 1]$. Then there exists a subdivision $0 = t_0 < t_1 < \ldots < t_m = 1$ such that $[t_k, t_{k+1}] \subset U$ or V for all $k \in \{0, \ldots, m-1\}$.

For nonempty topological spaces X and Y, $\mathbb{B} := \{U \times V : U \subset X, V \subset Y \text{ open}\}$ is a basis for the **product topology** on $X \times Y$. The space $X \times Y$ equipped with the product topology is a *product space*.
With respect to the product topology on X and Y, the *projections* $p_X\colon X \times Y \to X$, $(x, y) \mapsto x$ and p_Y (defined analogously) are continuous and *open* (they map open sets to open sets). The product topology is the *coarsest* topology on $X \times Y$ for which p_X and p_Y are maps (so product topology = initial topology with respect to the projections).

Let β be a collection of open subsets of X. If each open set is the union of sets in β, then β is a **basis of the topology** on X.
A function/space is continuous/compact iff it is continuous/compact on the basis.

A **subbasis** $\mathcal{F}$ of a topology on X (in terms of open sets) induced by any family of subsets containing $\varnothing$ and X, is the topology with the basis consisting of all finite intersections of sets in $\mathcal{F}$.

Let X be a set and $T_1, T_2 \subset 2^X$ be topologies on X. If $T_1 \subset T_2$, then T_1 is *coarser* than T_2 and T_2 is *finer* than T_1.
The coarsest topology on any set is $\{\varnothing, X\}$ (called *trivial* topology) and the finest is 2^X (the *discrete* topology).

Coarsest = smallest number of open sets
Finest = largest number of open sets

A **disk** is a topological space homeomorphic to $\mathbb{B}^2$.

If A is a disk and $h\colon A \to \mathbb{B}^2$ is a homeomorphism, then $h^{-1}(\mathbb{S}^1) = \partial A$. In particular $h^{-1}(\mathbb{S}^1)$ does not depend on h.

Any homeomorphism from the frontier of a disk to itself can be extended to a homeomorphism of the entire disk.

Let $A, B \subset X$ with $A, B \approx \mathbb{B}^d$ and *intersect only along their frontiers* ($\approx \mathbb{S}^{d-1}$) in a homeomorphic copy of $\mathbb{B}^{d-1}$. Then $A \cup B \approx \mathbb{B}^d$.

The **continuous image** of a compact space is compact.

A closed subset of a compact space is compact.

Let X be HAUSDORFF space and $A \subset X$ be a **compact** subset. Then for each point $x \in X \backslash A$, there exist **disjoint neighbourhoods of** x **and** A. In particular A is closed.

Let X be compact and Y a **Hausdorff** space. If $f\colon X \to Y$ is a bijective map, then f is a homeomorphism.

(BOLZANO-WEIERSTRASS) An **infinite** subset of a **compact** space has a **limit point**.

Let (X, τ_X) be a *non-compact* topological space. The **one-point compactification** of X the set $X^+ := X \cup \{\infty\}$ with the topology

$$\tau^+ := \tau_X \cup \{(X \backslash C) \cup \{\infty\} : C \subset X \text{ closed and compact}\}.$$

The one-point compactification (X^+, τ^+) is compact and $X \subset X^+$ is dense.
The one-point compactification X^+ is HAUSDORFF if and only if X is a locally compact HAUSDORFF space.

Theorems w/o proofs

Product of Hausdorff/compact spaces

Topology

Definition, 2 Theorems w/o proofs

Path-connected

Topology

Definition, Lemma w/ proof

Loop

Topology

Definition, Theorem w/o proof

Fundamental group

Topology

Theorem w/o proof

$_*$ is covariant functor

Topology

Definition

Connected, connected component

Topology

Definition

Real projective space

Topology

Definition, Lemma w/o proof, Remark

Homotopy (relative to a set)

Topology

Theorem w/ proof

Fundamental group of path-connected space

Topology

Fundamental group of $\mathbb{S}^1$

Topology

A space X is **connected** if for all nonempty subsets $A, B \subsetneq X$ with $X = A \cup B$ we have $\overline{A} \cap B \neq \varnothing$ *or* $A \cap \overline{B} \neq \varnothing$ *or:* if it is *not the union of two nonempty disjoint proper open subsets.*
The **connected components** of a topological space X are its *maximally connected subsets.*
Connected components are closed.

The product of two HAUSDORFF/compact spaces is HAUSDORFF/compact.

$$\mathbb{R}\,\mathrm{P}^n \approx (\mathbb{E}^{n+1} \setminus \{0\})/_{x \sim \lambda x} \approx \mathbb{S}^n/_{p \sim -p} \approx \mathbb{B}^n/_{x \sim -x \in \mathbb{S}^{n-1}}$$

We have $\mathbb{R}\,\mathrm{P}^1 \approx \mathbb{S}^1$. As the quotient map is open, $\mathbb{R}\,\mathrm{P}^n$ is HAUSDORFF, compact and path-connected.

A space X is **path-connected** if any two points $x, y \in X$ can be connected by a path, that is, there exists a *map* $\gamma\colon [0,1] \to X$ *with* $\gamma(0) = x$ *and* $\gamma(1) = y$.

Path-connected spaces are connected.

Let $X \subset \mathbb{E}^n$ be *open* and *connected.* Then X is *path-connected.*

Let $f, g\colon X \to Y$ be maps. Then f is *homotopic* to g if there exists a map $F\colon X \times I \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. Then F is a **homotopy** and we write $f \simeq_F g$. If additionally $A \subset X$ and $F(a,t) = f(a)$ for all $a \in A$ and $t \in I$, then F is a *homotopy relative to* A and we write $f \simeq_F g$ rel A (then $f|_A \equiv g|_A$).
The relation $\simeq$ rel A is an *equivalence relation.*
If $f, g\colon X \to Y$ are homotopic maps and $u\colon Y \to Z$ is a map, then $u \circ f \simeq u \circ g$. If $v\colon Y \to Z$ is homotopic to u, then $u \circ f \simeq v \circ g$. This also holds for relative homotopy.

Let $\alpha\colon I \to X$ be a *path* in X. Then α is a **loop** *based at* $p \in X$ if $\alpha(0) = p = \alpha(1)$.

A loop in X is the same as a map $\mathbb{S}^1 \to X$.
We have quotient map $q\colon [0,1] \to [0,1]/\{0,1\} \approx \mathbb{S}^1$ and a path $\alpha\colon [0,1] \to X$. Now α respects the equivalence relation $0 \sim 1$ as $\alpha(0) = \alpha(1)$. By the mapping property of the final topology, there must exist a map $\tilde{\alpha}\colon \mathbb{S}^1 \to X$ such that the diagram commutes.
Given a map $f\colon \mathbb{S}^1 \to X$, let $\alpha := f \circ q$.

If X is *path-connected* and $p, q \in X$, then $\pi_1(X,p) \cong \pi_1(X,q)$.
Let γ be a path from p and q. Then γ^{-1} is a path from q to p. For any loop $\alpha\colon I \to X$ based at p, the path $\gamma_*(\alpha) := \gamma^{-1} \bullet \alpha \bullet \gamma$ is a loop based at q.
If $\alpha \simeq_F \alpha'$, then $\tilde{\gamma}_*(\alpha) \simeq_G \tilde{\gamma}_*(\alpha')$ via $G(\cdot,t) := \gamma \bullet F(\cdot,t) \bullet \gamma^{-1}$
Hence γ_* is constant on equivalence classes $\langle \alpha \rangle$. We can hence view γ_* as

$$\gamma_*\colon \pi_1(X,p) \to \pi_1(X,q), \qquad \langle \alpha \rangle \mapsto \langle \gamma_*(\alpha) \rangle,$$

which is a well-defined *group homomorphism.* It is bijective with inverse $(\gamma_*)^{-1} = (\gamma^{-1})_*$.

For a loop $\alpha\colon I \to X$ based at p, the *homotopy class* of α,

$$\langle \alpha \rangle := \{\alpha' : \alpha' \text{ is a loop based at } p \text{ with } \alpha \simeq \alpha' \text{ rel } \{0,1\}\},$$

is the equivalence class of α with respect to the equivalence relation $\simeq$ rel $\{0,1\}$. For $p \in X$ the set

$$\pi_1(X,p) := \{\langle \alpha \rangle : \alpha \text{ is a loop in } X \text{ based at } p\}$$

equipped with the above multiplication $\langle \alpha \rangle \cdot \langle \beta \rangle := \langle \alpha \bullet \beta \rangle$, where $\bullet$ denotes concatenation, is a *group*, the **fundamental group** *of* X *with base point* p.

Let $\pi(t) := e^{2\pi i t}$, $\gamma_n\colon [0,1] \to [0,n]$, $t \mapsto nt$ and $\pi_n := \pi \circ \gamma_n$.
Thm. $\Phi\colon (\mathbb{Z}, +) \to (\pi_1(\mathbb{S}^1, 1), \cdot)$, $n \mapsto \langle \pi_n \rangle$ is **isomorphism**.
Φ is **homomorphism**: $\sigma := \gamma_n + m$. Then $\pi \circ \sigma = \pi \circ \gamma_n$, so $\gamma_m \cdot \sigma \simeq \gamma_{m+n}$ rel $\{0,1\}$ and so $\Phi(m+n) = \Phi(m) \cdot \Phi(n)$.
Φ is **onto**: α loop in $\mathbb{S}^1$ at 1. Then $\exists$ lift $\tilde{\alpha}$, $\Phi(\tilde{\alpha}(1)) = \langle \alpha \rangle$.
Φ is **injective**: $\exists n \in \mathbb{Z}$ s.t. $\Phi(n) = \langle e \rangle$. Let γ path in $\mathbb{R}$ from 0 to n s.t. $\pi \circ \gamma \simeq_F e$ rel $\{0,1\}$. Lift F to homotopy $\tilde{F}\colon I^2 \to \mathbb{R}$, $\pi \circ \tilde{F} = F$, $\tilde{F}(0,\cdot) = 0$. We have $\tilde{F}(1,1) = 0$. Then $\tilde{F}(\cdot,1)$ is path in $\mathbb{R}$, lift of $\pi \circ \gamma$. Uniqueness of lifts: $\tilde{F}(\cdot,1) = \gamma$ So $n = \gamma(1) = 0$.

Let $f\colon X \to Y$ be a map. For any $\alpha \in \langle \alpha \rangle \in \pi_1(X,p)$ we have $f \circ \alpha \in \pi_1(Y, f(p))$. For $\alpha' \in \langle \alpha \rangle$ we have $f \circ \alpha' \in \langle f \circ \alpha \rangle$. Hence we obtain a function $f_*\colon \pi_1(X,p) \to \pi_1(Y, f(p))$, $\langle \alpha \rangle \mapsto \langle f \circ \alpha \rangle$. Then $f \circ (\alpha \bullet \beta) = (f \circ \alpha) \bullet (f \circ \beta)$, so f_* is a *group homomorphism.* Further, $(g \circ f)_* = g_* \circ f_*\colon \pi_1(X,p) \to \pi_1(Z, g(f(p)))$.
So π_1 can be applied to commutative diagrams.
If h is a homeomorphism, then h_* and h_*^{-1} are group isomorphisms.
Hence π_1 is a topological invariant.

Corollary w/ proof

$\pi_1(\mathbb{S}^n)$, $n \geqslant 2$

Topology

Theorem w/ proof idea

Fundamental group of a product

Topology

Definition, 2 Remarks, 2 Examples

Homotopy equivalence

Topology

Definition, Example

Deformation retract(ion)

Topology

Theorem w/o proof

Functorial properties of π_1

Topology

Theorem w/ proof

Let X and Y be path connected. $X \simeq Y$
$\implies \pi_1(X) \cong \pi_1(Y)$

Topology

Definition, Lemma

Contractible

Topology

Definitions, Thm w/ proof

(Universal) covering map

Topology

Theorem w/ proof

Brouwer's fixed point theorem

Topology

2 Theorems w/o proofs

Jordan's curve theorem

Topology

Let X and Y be topological spaces, $p \in X$ and $q \in Y$. Then $\pi_1(X \times Y, (p,q)) \cong \pi_1(X,p) \times \pi_1(Y,q)$.

Projection maps induce homomorphisms $(p_1)_*$, $(p_2)_*$. Then $(p_1)_* \times (p_2)_*$ is isomorphism.

A homotopy $G\colon X \times I \to X$ *relative to* $A \subset X$ is a **deformation retraction** if $G(a,\cdot) = a\ \forall a \in A$, $G(\cdot,0) = \mathrm{id}_X$ and $G(\cdot,1) \subset A$.
For $f\colon A \hookrightarrow X$, $a \mapsto a$ and $g\colon X \to A$, $x \mapsto G(x,1)$ (the *retract*) we have $f \circ g \simeq \mathrm{id}_X$ and $g \circ f = \mathrm{id}_A$. Hence $X \simeq A$, so A is a *deformation retract* of X and we say that X *retracts to* A. Hence in order to compute the fundamental group of X it suffices to find the fundamental group of any deformation retract of X.
$\mathbb{S}^1 \times I$ retracts to the circle $\mathbb{S}^1$, $\mathbb{E}^n \setminus \{0\}$ retracts to $\mathbb{S}^{n-1}$.

π_1 even is a functor on the category of top. spaces modulo homotopy equivalence.
Let $f\colon X \to Y$ and $g\colon Y \to X$ be maps with $\mathrm{id}_X \simeq_F g \circ f$ and $\mathrm{id}_Y \simeq_G f \circ g$. Let qY and $p := g(q)$ be base points. Define $\gamma\colon I \to X$, $s \mapsto F(p,s)$. Then $\gamma(0) = F(p,0) = \mathrm{id}_X(p) = p$ and $\gamma(1) = F(p,1) = g(f(p))$, so γ is a path in X joining p with $g(f(p))$. By functorial properties of π_1, $(g \circ f)_* = (g_* \circ f_*) = \gamma_*\colon \pi_1(X,p) \to \pi_1(X, g(f(p)))$ is an isomorphism. Hence f_* is injective (its left inverse up to conjugation is g_*). A similar argument using G instead of F shows that f_* is surjective, so f_* is bijective.

Let X and Y be topological spaces. A map $p\colon Y \to X$ is a **covering map** or *cover* (of X) if every point $x \in X$ has an open neighbourhood V for which $p^{-1}(V)$ decomposes into a disjoint union of open sets $U_i \subset Y$ such that $p|_{U_i}\colon U_i \to V$ is a *homeomorphism*.
The morphisms are the homeomorphisms $f\colon Y \to Z$, where $p_1\colon Y \to X$ and $p_2\colon Z \to X$ are covers of X, with $p_2 \circ f = p_1$.
In particular, a covering map is always surjective.
A cover $p\colon \tilde{X} \to X$ is *universal* if $\tilde{X}$ is simply connected, e.g. $\mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$.
We have $\mathrm{Deck}(\tilde{X} \to X) := \{f\colon \tilde{X} \to \tilde{X} : f \text{ homeo},\ p \circ f = p\} \cong \pi_1(X)$
Let $J \subset \mathbb{E}^2$ be a JORDAN *curve*, that is $J \approx \mathbb{S}^1$. Then $\mathbb{E}^2 \setminus J$ has *exactly two (path)components.*

Let $A \subset \mathbb{E}^2$ be a curve which doesn't closed up ("arc"), that is, $A \approx [0,1]$. Then $\mathbb{E}^2 \setminus A$ is path-connected.

Let $q \in \mathbb{S}^n \setminus \{p\}$. As $\mathbb{S}^n \setminus \{p\} \approx \mathbb{E}^n$, $\mathbb{S}^n \setminus \{p\}$ is simply connected. We can decompose $\mathbb{S}^n$ as a union of open simply connected subsets:

$$\mathbb{S}^n = (\mathbb{S}^n \setminus \{p\}) \cup (\mathbb{S}^n \setminus \{q\}).$$

As $n \geqslant 2$, their intersection

$$(\mathbb{S}^n \setminus \{p\}) \cap (\mathbb{S}^n \setminus \{q\}) = \mathbb{S}^n \setminus \{p,q\}$$

is path-connected. By VAN-KAMPEN, $\pi_1(\mathbb{S}^n, p)$ is trivial.

X and Y are **homotopy equivalent** ($X \simeq Y$) if $\exists$ maps $f\colon X \to Y$ and $g\colon Y \to X$ s.t. $g \circ f \simeq \mathrm{id}_X$, $f \circ g \simeq \mathrm{id}_Y$.
Homeomorphic spaces are homotopy equivalent.
Homotopy equivalence is an **equivalence relation**.
Convex subset of $\mathbb{E}^n$ are homotopy equivalent to $\{\bullet\}$.
$\mathbb{E}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$.

Let $f, g\colon X \to Y$ be maps with $f \simeq_F g$ and $p \in X$. Then $g_*\colon \pi_1(X,p) \to \pi_1(Y, g(p))$ equals the composition $\gamma_* \circ f_*$, where $f_*\colon \pi_1(X,p) \to \pi_1(Y, f(p))$ and the path joining $f(p)$ and $g(p)$,

$$\gamma\colon I \to Y, \qquad s \mapsto F(p,s),$$

induces a map

$$\gamma_*\colon \pi_1(Y, f(p)) \to \pi_1(Y, g(p)), \qquad \langle \alpha \rangle \mapsto \langle \gamma^{-1} \bullet \alpha \bullet \gamma \rangle.$$

A space X is **contractible** if $\mathrm{id}_X \simeq e_p$ for some $p \in X$, where $e_p\colon X \to X$, $x \mapsto p$ denotes the constant map at $p \in X$ on X.
Let X be contractible. Then X is simply connected and id_X is homotopic to e_x for all $x \in X$.
Let Y be a space with maps $f, g\colon Y \to X$. Then $f \simeq g$.

Every map $f\colon \mathbb{B}^n \to \mathbb{B}^n$ has a fixed point for $n \geqslant 1$.
$n = 2$: Suppose $f(x) \neq x$ for all $x \in \mathbb{B}^2$. For $x \in \mathbb{B}^2$ let $g(x)$ be the *unique* point of intersection of $\mathbb{S}^1$ and the ray from $f(x)$ to x in the direction of x. Then $g\colon \mathbb{B}^2 \to \mathbb{S}^1$ is a map as a composition of maps. For $x \in \mathbb{S}^1$ we have $g(x) \in \mathbb{S}^1$. Hence g is a *retraction* from $\mathbb{B}^2$ to $\mathbb{S}^1$. Then $\mathbb{B}^2 \simeq g(\mathbb{B}^2) = \mathbb{S}^1$, which is a contradiction as $\pi_1(\mathbb{B}^2) \neq \pi_1(\mathbb{S}^1)$.
For $n = 1$ replace π_1 by π_0 in the above proof, where π_0 is the set of path-connected components (which is not a group).
For $n > 2$ replace π_1 by H_{n-1}.

Definition

Surface, topological k-manifold

Topology

Definition

Affine hull, general position

Topology

Definition, Lemma w/o proof

Realisation of a geometric simplicial complex

Topology

Definition, Lemma w/o proof

Cone of a simplicial complex

Topology

Definition

Abstract simplicial complex

Topology

Definitions, 2 Theorems

Interior, boundary, closed

Topology

Definitions

Simplex, (proper) face, geometric simplicial complex + dim

Topology

Definitions, Theorem w/o proof

Triangulation / triangulable space

Topology

Definition

Simplicial map, simplicial isomorphism

Topology

Definitions

Barycentric coordinates, barycentre, first barycentric subdivision

Topology

Let S be a surface. The **interior** of S is the set of all $x \in \mathbb{S}$ such that there exists a neighbourhood $N \in \mathcal{N}$ with $N_x \approx \mathbb{E}^2$.
The **boundary** of S is the set of all $x \in \mathbb{S}$ such that there exists a homeomorphism $f\colon \mathbb{E}^2_+ \to N$, where $N \in \mathcal{N}_x$ and $f(0) = x$.
Thm. The interior and boundary of a surface are disjoint.
Thm. Let S_1 and S_2 be surfaces which are homeomorphic via h. Then h maps the interior of S_1 homeomorphically onto the interior of S_2 (and likewise for the boundaries).
A surface is **closed** if it is *compact* and has empty boundary.

A **(topological)** k**-manifold** is a HAUSDORFF space such that *each point has a neighbourhood* either homeomorphic to $\mathbb{E}^k$ or $\mathbb{E}^k_+ := \{(x_1, \ldots, x_k) \in \mathbb{E}^k : x_k \geqslant 0\}$.
A **surface** is a 2-manifold.

The convex hull $\operatorname{conv}(\{v_0, \ldots, v_k\})$ is a k**-simplex** if $\{v_0, \ldots, v_k\}$ is in general position.
Let $\sigma = \operatorname{conv}(V)$ be a k-simplex. If $W \subset V$, then $\tau := \operatorname{conv}(W)$ is a simplex, too, called a **face of** σ. We write $\tau \leqslant \sigma$. Further, τ *proper face* of σ if $W \notin \{\varnothing, V\}$.
A *finite* collection of simplices in $\mathbb{E}^n$ is a *geometric simplicial complex* if any two simplices from the collection meet in a common face (which may be empty). Its dimension is the maximal dimension of a simplex.

The *affine hull* of $V := \{v_0, \ldots, v_k\} \subset \mathbb{E}^n$ is $\operatorname{aff}(V) := \left\{\sum_{j=0}^k \lambda_j v_j : \sum_{j=0}^k \lambda_j = 1\right\}$.
The set V is in *general position* if $\dim(V) = k$, that is, $(v_j - v_0)_{j=1}^k$ are *linearly independent*. We say that the point in V are *affinely independent*.
The set $\{x_0, \ldots, x_m\} \subset \mathbb{R}^n$ is affinely dependent if and only if the set $\{(1, x_0), \ldots, (1, x_m)\} \subset \mathbb{R}^{n+1}$ is linearly dependent.

A space is triangulable if it is homeomorphic to the realisation of a geometric simplicial complex.
Let X be a topological space. A pair (K, X) is a **triangulation** *of* X if K is a simplicial complex and $h\colon |K| \to X$ is a homeomorphism.
The space X is *triangulable* if it has a triangulation.

Every closed surface is triangulable.

The realisation of a simplicial complex $K := (\sigma_j)_{j=1}^k$ in $\mathbb{E}^n$ is

$$|K| := \bigcup_{j=1}^k \sigma_j \subset \mathbb{E}^n .$$

$|K| \subset \mathbb{E}^n$ is *compact*.
$x \in |K|$ is contained in the relative interior of a unique simplex, called the **carrier** of x.
K is connected $\iff$ K is path-connected $\iff$ $K^{\leqslant 1}$ is a connected graph.

Let K and L be geometric simplicial complexes. A function $\varphi\colon \operatorname{Vert}(K) \to \operatorname{Vert}(L)$ is a **simplicial map** if for all $\sigma \in K$ we have $\varphi(\sigma) \in L$ (extend φ via barycentric coordinates).
A *bijective* simplicial map whose inverse is a simplicial map is a **simplicial isomorphism**.

Let K be a geometric simplicial complex in $\mathbb{E}^n \cong \mathbb{E}^n \times \{0\} \subset \mathbb{E}^{n+1}$. The *cone of* K *with apex* $v \in \mathbb{E}^{n+1} \setminus (\mathbb{E}^n \times \{0\})$ is

$$CK := \left\{\sigma * v : \sigma \in K \cup \{\varnothing\}\right\} \cup K,$$

where

$$\sigma * v := \operatorname{conv}(\sigma \cup \{v\})$$

is a $(n+1)$-simplex if $\sigma \subset \mathbb{E}^n$ is a n-simplex.

We have $|CK| \approx C|K|$.

Let $\tau := \operatorname{conv}(v_0, \ldots, v_k) \subset \mathbb{R}^n$ be a k-simplex. Then each point $x \in \tau$ can be *uniquely* written as $x = \sum_{j=0}^k \lambda_j v_j$, where $\lambda_j \geqslant 0$ for all $j \in \{0, \ldots, k\}$ and $\sum_{j=0}^k \lambda_j = 1$. The coefficients $(\lambda_j)_{j=0}^k$ are the **barycentric coordinates** of x. The **barycentre** of τ is the point $\beta_\tau := \frac{1}{k+1} \sum_{j=0}^k v_j \in \tau$.
Let K be a GSC. The **barycentric subdivision** K^1 of K is the GSC with $\operatorname{Vert}(K^1) = \{\beta_\sigma : \sigma \in K \setminus \{\varnothing\}\}$ such that $\operatorname{conv}(\beta_{\sigma_0}, \ldots, \beta_{\sigma_k})$ is a face if and only if there exists a $\varphi \in \operatorname{Sym}(\{0, 1, \ldots, k\})$ such that $\sigma_{\varphi(0)} < \sigma_{\varphi(1)} < \ldots < \sigma_{\varphi(k)}$.
The m-th **barycentric subdivision** is $K^m := (K^{m-1})^1$.
We have $\operatorname{Vert}(K) \subset \operatorname{Vert}(K^1)$. k faces gets divided in $(k+1)!$ k-faces.

Let V be a *finite* set. A subset $K \subset 2^V$ is an **abstract simplicial complex** *with vertex set* V if for all $\sigma \in K$ and all $\tau \subset \sigma$ we have $\tau \in K$.

Definition

Barycentric subdivision of abstract SC

Topology

Definition, Lemma w/ proof

Simplicial approximation

Topology

Definition

Equivalent Edge paths / loops

Topology

Theorems w/o proofs

Mixing cross-caps and handles, Poincaré conjecture

Topology

Definitions

Orientation of a simplex, induced orientation, orientable triangulation

Topology

Definition, Lemma w/o proof

Mesh

Topology

Theorem w/o proof, remark

Simplicial approximation

Topology

Definition, Theorem w/o proof

Edge group

Topology

Theorem w/o proof, Lemma

Properties of triangulations of closed surfaces

Topology

Lemma w/ proof

Orientation of a triangulation and the corresponding surface

Topology

The **mesh** (size) of a complex K is $\mu(K) := \max(\{\operatorname{diam}(\sigma) : \sigma \in K\})$, where $\operatorname{diam}(\sigma) := \max(\{\|x-y\| : x, y \in \sigma\})$.
Each simplex of K^1 is contained in a simplex of K.
We have $|K^1| = |K|$.
If $\dim(K) = n$, then $\mu(K^1) \leqslant \frac{n}{n+1}\mu(K)$.

The (first) barycentric subdivision of an abstract simplicial complex K has as vertices the faces of K, that is, $K \backslash \{\varnothing\}$, and as faces the flags of K.

The following theorem states that we can replace continuous maps between realisations of GSCs by simplicial maps.

Let $f\colon |K| \to |L|$ be a map. Then there exists a $m \in \mathbb{N}$ such that there exists a simplicial approximation $s\colon K^m \to L$ to f.

A simplicial map $s\colon K \to L$ is a **simplicial approximation** of the continuous function f if $s(x)$ lies in the *carrier* of $f(x)$ for each $x \in |K|$.

If s is a simplicial approximation of f, then $|s| \simeq f$.

Let $L \subset \mathbb{E}^n$. Define $F\colon |K| \times I \to \mathbb{E}^n$, $(x,t) \mapsto (1-t)|s(x)| + tf(x)$. For $x \in |K|$, there exists a face $\sigma \in L$ such that $|s|(x), f(x) \in \sigma$. Since σ is convex, the straight line homotopy F stays inside σ. Hence the image of the F is contained in $|L|$ and $|s| \simeq_F f$.

The **edge group** $E(K,v)$ of K based at $v \in V$ consists of equivalence classes of edge loops at v with respect to the multiplication

$$[v_0, \dots, v_k][v_k, v_{k+1}, \dots, v_m] := [v_0, \dots, v_m],$$

where $v_0 = v_k = v_m = v$.

Thm. $E(K,v) \cong \pi_1(|K|, v) \cong G(K,L)$, where L is a simply connected subcomplex of K, e.g. a spanning tree of $K^{\leqslant 1}$.

An **edge path** in a GSC K is a sequence $(v_0, \dots, v_k)$ in V such that the edge (or point: we allow $v_{i-1} = v_i$) $v_{i-1}v_i := \operatorname{conv}(\{v_{i-1}, v_i\})$ lies in K for all $i \in [k]$.
If $v_0 = v_k$, then this sequence is an **edge loop** based at v_0.
Two edge paths are **equivalent** if they can be transformed into another by finitely many operations of the following kind:

- $(u,v,w) \leftrightarrow (u,w)$ if $\operatorname{conv}(u,v,w) \in K$ ("shortcut").
- $(u,u) \leftrightarrow u$.

Rado (1923): Every closed surface is triangulable: $S \approx |K|$.
$\dim(K) = 2$.
K is pure, that is, each facet (maximal face with respect to inclusion) is 2-dimensional.
Each edge of K is contained in exactly two triangles.
Any two vertices of K can be joined by an edge-path.
Each vertex v is contained in at least three triangles, which together form a cone with apex v.

Modifying the sphere $\mathbb{S}^2$ by adding m handles and $n > 0$ disjoint cross-caps is homeomorphic to $\mathbb{S}^2$ with $2m+n$ disjoint cross-caps.

Every *simply connected, closed 3*-manifold is homeomorphic to $\mathbb{S}^3$.

Let $S := |K|$. If S is orientable, then K is orientable.

Assume S is orientable. Pick any triangle and orient it. In dual graph, take a spanning tree rooted at the triangle chosen, along which we can uniquely spread the orientation. The only thing that can go wrong is that two nodes in this tree have incompatible orientations. If this were the case, then there is a sequence of triangles $\sigma_1, \dots, \sigma_k$ which are adjacent such that σ_i and σ_{i+1} are adjacent and have compatible orientations for $i \in \{1, \dots, k-1\}$, but σ_1 and σ_k are adjacent but have no compatible orientations. We join the barycentres β_{σ_i} to the barycentres of the edges $\beta_{\sigma_i \cap \sigma_{(i+1) \mod k}}$ and $\beta_{\sigma_{i-1} \cap \sigma_i}$, obtaining a simple closed polygonal path in the polyhedron $|K^1|$. "Thickening" that path yields a Möbius strip, contradicting that S is orientable.

An **orientation** of a simplex is an *ordering* of its vertices up to an *even permutation*.
The orientation of a $\operatorname{conv}(v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$ induced by an orientation $v_0, \dots, v_k$ is ($-$ denotes to opposite orientation)

$$\begin{cases} v_0 \dots v_{i-1} v_{i+1} \dots v_k, & \text{if } i \text{ is even,} \\ -v_0 \dots v_{i-1} v_{i+1} \dots v_k, & \text{if } i \text{ is odd..} \end{cases}$$

A triangulation K is **orientable** if there exists orientations of all triangles such that each edge receives opposite orientations from its two triangles.

Definition, Remark, Lemma w/o proof

Thickening

Topology

Lemma w/ proof

Euler characteristic of a simplicial complex

Topology

Explanation

Surgery

Topology

Theorem w/o proof

Fundamental group and Euler characteristic of $H(p)$ and $M(q)$

Topology

Definition, Remark

Chain module

Topology

Definition, Lemma w/o proof

f-vector, Euler characteristic

Topology

Theorem w/ proof

First part of the Classification Theorem

Topology

Theorem w/ proof

Surgery increases Euler characteristic

Topology

Definitions

Abelianisaton, (free) R-module

Topology

Definitions

Boundary operator

Topology

Let L be a d-dimensional simplicial complex. Then $f_k(L)$ is the number of k-dimensional simplices of L, $(f_j)_{j=0}^d$ is the **f-vector** (or face vector) of L and the EULER *characteristic* of L is $\chi(L) := \sum_{k=0}^d (-1)^k f_k(L)$.

Let L and M be cell complexes which intersect in a *common subcomplex* $L \cap M$. Then $\chi(L \cup M) = \chi(L) + \chi(M) - \chi(L \cap M)$ by the *inclusion-exclusion principle*.

$\chi(L^1) = \chi(L)$.

Every polygonal curve in $|K|$ made up of edges in K^1 separates $|K| \iff \chi(K) = 2 \iff |K| \approx \mathbb{S}^2$.

"① $\implies$ ②". As in the proof of EULER's theorem, it follows that Γ is a tree. Hence $\chi(K) = \chi(T) + \chi(\Gamma) = 1 + 1 = 2$.
"② $\implies$ ③". If $\chi(T) = 2$, then $\chi(\Gamma) = 1$ by the above formula and hence Γ is a tree. Thus $|K| = |N(T)| \cup |N(\Gamma)|$ is a union of two disks glued at their boundaries, so $|K| \approx \mathbb{S}^2$.
"③ $\implies$ ①". This comes from the proof of the JORDAN theorem.

We have $\chi(K_*) > \chi(K)$.
Case 1: $|N|$ is a cylinder. Then, as L_1 and L_2 are disjoint,

$$\begin{aligned}\chi(K_*) &= \chi(M) + \chi(CL_1) + \chi(CL_2) - \chi(L_1) - \chi(L_2)\\ &= \chi(M) + 1 + 1 - 0 - 0 = \chi(M) + 2.\end{aligned}$$

Case 2: $|N|$ is a Möbius strip. Then

$$\chi(K_*) = \chi(M) + \chi(CL) - \chi(L) = \chi(M) + 1.$$

Lastly, in both cases we have

$$\chi(K) = \chi(K^2) = \chi(M) + \chi(N) - \chi(M \cap N) = \chi(M) + 0 - 0 = \chi(M),$$

as $M \cap N$ is a circle.

The **abelisation** of G is $G^{\mathrm{Ab}} := G/G'$, the largest ABELIAN factor of G, where $G' := \langle \{[a,b] : a, b \in G\} \rangle$, where $[a,b] := aba^{-1}b^{-1}$.
An **R-module** is an ABELIAN *group* $(G, *)$ together with a *ring* R and a *ring homomorphism* $\varphi\colon R \to \mathrm{End}(G)$, which is the (not unique) *R-module structure* on G.
Let X be a set. Then the **free $R-$module** *with basis* X is

$$\bigoplus_{x \in X} R := \left\{ \sum_{i=1}^n r_i x_i : r_i \in R, x_i \in X, n \in \mathbb{N} \right\},$$

where $\sum_{i=1}^n r_i x_i + \sum_{i=1}^n s_i x_i := \sum_{i=1}^n (r_i + s_i) x_i$, is the set of unique formal linear combinations.

The q-faces form (by construction) an *R-basis* of $C_q(K; R)$. Hence

$$\partial_q \underbrace{(w_0 w_1 \dots w_q)}_{\text{oriented } q\text{-face}} := \sum_{j=0}^q (-1)^j \underbrace{w_0 w_1 \dots \widehat{w_j} \dots w_q}_{\text{oriented } (q-1)\text{-face}} \in C_{q-1}(K; R)$$

defines an R-linear map by extension.
For $q \geqslant 1$, $\partial_q\colon C_q(K; R) \to C_{q-1}(K; R)$ is the q-th simplicial **boundary operator** of K.
Let $\partial_0 := 0$ (the boundary of a point is zero) *or* define $C_{-1}(K; R) = R \cdot \varnothing$ (all R-multiples of the empty set) and $\partial_0(v) = \varnothing$ for $v \in V$ (latter one = reduced).

Let L be a *one-dimensional* subcomplex in K^1. The **thickening** of L is the subcomplex of K^2 of the triangles (and their faces) which *meet* L.

The thickening of L is a closed neighbourhood of $|L|$ in $|K|$ whose polyhedron is homotopy equivalent to $|L|$.

The thickening of a tree is homeomorphic to a disk. The thickening of a simple closed polygonal curve is either a cylinder or a MÖBIUS strip.

We have $\chi(K) \leqslant 2$ for a simplicial complex K.

Choose a spanning tree T in K and construct the complementary graph Γ, whose vertices are the (barycentres of) triangles and whose edges correspond to edges in K which are not edges in T. Referring to the barycentres means that we can realise this a geometric simplicial complex $\Gamma^1 \leqslant K^1$. Then $\chi(K) = \chi(T) + \chi(\Gamma)$, because each face of the triangulation either contributes to T (if it is a vertex or an edge of T) or to Γ (if it is a triangle or if it bijectively corresponds an edge which is not in T) in such a way that the signs match. As T is a tree, $\chi(T) = 1$ and as Γ is a connected simple graph, $\chi(\Gamma) \leqslant 1$ by CoMa.

K combinatorial surface. Assume that L is a simple closed polygonal curve which does not separate $|K|$. Then $|K| \not\approx \mathbb{S}^2$.
Let N be the thickening of L in K^2, which is either a cylinder or a MÖBIUS strip. Let M be the subcomplex complementary to N in K^2 (cf. thickening of dual graph Γ).
If $|N| \approx \mathbb{S}^1 \times I$, then $\partial|N| = \partial|M| \approx \mathbb{S}^1 \sqcup \mathbb{S}^1$. Let $L_1, L_2 \leqslant K^2$ support those circles. Let $K_* := M \cup CL_1 \cup CL_2$.
If $|N| \approx$ Möbius, then $\partial|N| = \partial|M| \approx \mathbb{S}^1$. Let $L \leqslant K^2$ be that circle. Let $K_* := M \cup CL$.
Then K_* is obtained from K by doing surgery along L resp. $L_1 \cup L_2$.

For $p \geqslant 0$ we have

$$\pi_1(H(p)) \cong \left\langle a_1, b_1, \dots, a_p, b_p \;\middle|\; \prod_{k=1}^p a_k b_k a_k^{-1} b_k^{-1} \right\rangle$$

and for $q \geqslant 1$

$$\pi_1(M(q)) \cong \left\langle a_1, \dots, a_q \;\middle|\; \prod_{k=1}^p a_k^2 \right\rangle.$$

as well as $\chi(H(p)) = 2 - 2p$ and $\chi(M(q)) = 2 - q$.

Let K be a simplicial complex, whose vertices $V = \mathrm{Vert}(K) = \{v_1, \dots, v_n\}$ are *totally ordered*. Fix a *commutative ring* R with *multiplicative unit* 1.
The q-th simplicial **chain module** of K with coefficients in R is $C_q(K; R)$, the set of all *formal linear combinations* of *q-dimensional faces* of K with *coefficients in R*.

The chain module of K is a *free R-module*, where the addition and multiplication are inherited coefficient-wise from the addition and multiplication in R, respectively.

Lemma w/ proof

$$\partial^2 = 0$$

Topology

Definition

Homology module, class, Betty number

Topology

Theorem

Euler characteristic and Betty numbers

Topology

Theorem w/ proof

Characterising H_0

Topology

Theorem w/o proof, Remark

Hurewicz: $H_1(K;\mathbb{Z}) \cong \pi_1(|K|, v)^{\mathrm{Ab}}$

Topology

Remark

Integral homology of closed surfaces

Topology

Lemma, Theorems w/ and w/o proofs, Corollary

Topological invariance of homology

Topology

Theorems w/ proof

Homotopy invariance of homology

Topology

Definition, Remark, Theorem w/o proof

Nerve complex

Topology

Definition, Remark

Čech complex

Topology

Since ∂_{q+1} and ∂_q are R-linear, $B_q(K;R)$ and $Z_q(K;R)$ are *free* R-submodules of the R-module $C_q(K;R)$. As $\partial^2 = 0$, $B_q(K;R) \leqslant Z_q(K;R)$, so we can take the quotient.
The *q-th* **simplicial homology module** is

$$H_q(K;R) := Z_q(K;R)/B_q(K;R).$$

The *q-th* **Betty number** β_q *of* K is the free rank of $H_q(K;\mathbb{Z})$.
For $c \in Z_q(K;R)$ the **homology class** *of* c is

$$[c] := c + B_q(K;R).$$

We have $\partial_q \partial_{q+1} = 0$ for all $q \geqslant 0$.

Consider $q+2$ vertices $w_0, \dots, w_{q+1}$ of a $(q+1)$-dimensional simplex. Then

$$\begin{aligned}\partial^2(w_0 \dots, w_{q+1}) &= \partial\left(\sum_{k=0}^{q+1}(-1)^k w_0 \dots \widehat{w_k} \dots w_{q+1}\right) \overset{(\mathrm{L})}{=} \sum_{k=0}^{q+1}(-1)^k \partial\left(w_0 \dots \widehat{w_k} \dots w_{q+1}\right) \\ &= \sum_{k=0}^{q+1}(-1)^k \Bigg(\sum_{j=k+1}^{q+1}(-1)^{j-1} w_0 \dots \widehat{w_k} \dots \widehat{w_j} \dots\dots w_{q+1} \\ &\quad + \sum_{k=0}^{q+1}(-1)^k \sum_{j=0}^{k-1}(-1)^j w_0 \dots \widehat{w_j} \dots \widehat{w_k} \dots\dots w_{q+1}\Bigg)\end{aligned}$$

where we use that ∂ is linear in (L) and the hat indicates that this vertex is omitted. Each ordered q-simplex occurs twice, but with opposite sign, hence the term is zero.

We have $H_0(K;R) \cong R^C$, where C is the number of connected components of K.

If v and w are vertices of K in the same connected component, then the equivalence classes in the 0-th homology agree ("they are 0-homologous": $[v] = [w]$. Indeed, we can joint v and w by an edge path $vv_1 \dots v_k w$ in which no consecutive vertices are equal. Then $\partial((vv_1) + (v_1v_2) + \dots + (v_kw)) = w - v$. Furthermore, vertices which lie in different components of $|K|$ are not homologous and R-multiples of a single vertex can never be a boundary.

$$\chi(K) = \sum_{k=0}^{n}(-1)^k f_i(K) = \sum_{k=0}^{n}(-1)^k \dim(H_k(K;\mathbb{F})),$$

where $\mathbb{F}$ is any field of characteristic zero, such as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Let K be a combinatorial surface of genus g. Then K is connected, so $H_0(K;\mathbb{Z}) = \mathbb{Z}$ and we have the following table:

	K is orientable	K is non-orientable
$H_0(K;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$
$H_1(K;\mathbb{Z})$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2$
$H_2(K;\mathbb{Z})$	$\mathbb{Z}$	$\{0\}$

Table 1: The first three $\mathbb{Z}$-homology groups of surfaces with genus g. Note that a non-orientable surface of genus 0 does not exist, so everything is well-defined. ($H_k = \{0\}$, $k > 2$).

Let K be a connected ASC with totally ordered vertex set and with a vertex v. Each edge loop $\alpha = vv_1v_2 \dots v_kv$ based at v gives rise to a simplicial 1-chain with integer coefficients $z(\alpha) := (vv_1) + (v_1v_2) + \dots + (v_kv) \in C_1(K;\mathbb{Z})$ provided that subsequent vertices are distinct, i.e. $v_i \neq v_{i+1}$. The order matters: $(v_i, v_{i+1}) = -(v_{i+1}v_i)$. We have $\partial(z(\alpha)) = 0$ because α is closed and thus each vertex appears exactly twice in the linear combination. Hence $z(\alpha) \in Z_1(K;\mathbb{Z})$.
For another chain β, which is equivalent to α in the edge path group of K, we get $z(\beta) - z(\alpha) \in B_1(K;\mathbb{Z})$, i.e. $z(\beta) - z(\alpha)$ is a 1-boundary of K.
This yields a homomorphism of groups

$$\varphi\colon \pi_1(|K|, v) \to H_1(K;\mathbb{Z}), \qquad [\alpha] \mapsto [z(\alpha)],$$

where $[\alpha]$ is the homotopy class in the edge group and $[z(\alpha)]$ is the homology class. The homomorphism is onto and its kernel is the *commutator subgroup* of $\pi_1(|K|, v)$.

If $f, g\colon |K| \to |L|$ are *homotopic*, then the induced maps f_* and g_* in homology are equal. Hence if $|K| \simeq |L|$, then $H_q(K;R) \cong H_q(L;R)$ for all $q \geqslant 0$.

Simplicial maps induce homomorphisms in homology.
Subdivision preserves homology.
Let K and L be SCs. The map $f\colon |K| \to |L|$ induces a *homomorphism of R-modules* $f_*\colon H_q(K;R) \to H_q(L;R)$ for each $q \in \mathcal{N}_0$.
If $f\colon |K| \to |K|$ is the identity map, then for all $q \geqslant 0$, $f_*\colon H_q(K;R) \to H_q(K;R)$ is the identity, too.
If M is another SC and $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$ are maps, then $(g \circ f)_* = g_* \circ f_*\colon H_q(K;R) \to H_q(M;R)$.
Hence $H_q(\cdot;R)$ is a **covariant functor**.

If $f, g\colon |K| \to |L|$ are *homotopic*, then the induced maps f_* and g_* in homology are equal. Hence if $|K| \simeq |L|$, then $H_q(K;R) \cong H_q(L;R)$ for all $q \geqslant 0$.

If $m \neq n$, then $\mathbb{S}^m \not\simeq \mathbb{S}^n$.

We have $H_m(\mathbb{S}^m) = \mathbb{Z} \neq \{e\} = H_m(\mathbb{S}^n)$.

We have $\mathbb{E}^m \approx \mathbb{E}^n$ if and only if $m = n$.

Let $h\colon \mathbb{E}^m \to \mathbb{E}^n$ be a homeomorphism preserving the origin. Then $\mathbb{S}^{m-1} \simeq \mathbb{E}^m \setminus \{0\} \overset{h}{\approx} \mathbb{E}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ can only be true if $n = m$.

Let $S \subset \mathbb{R}^d$ be a *finite* set of points and $r \geqslant 0$. Then

$$\check{\mathrm{C}}\mathrm{ech}(r) := \left\{\sigma \subset S : \bigcap_{x \in \sigma} B_x(r) \neq \varnothing\right\}$$

is the **Čech complex** of S with respect to the radius r. Here, $B_x(r)$ is the ball of all points with distance at most r from x. The Čech complex is an abstract simplicial complex (as the nerve complex is) on the vertex set S. Then nerve theorem implies that $\check{\mathrm{C}}\mathrm{ech}(r) \simeq \bigcup_{x \in S} B_x(r)$.

Let F be a *finite* collection of *closed convex* sets in $\mathbb{E}^d$. Then the **nerve complex** is an abstract simplicial complex on F as vertex set:

$$\mathrm{Nrv}(F) := \left\{X \subset F : \bigcap X \neq \varnothing\right\}.$$

If $Y \subset X$ and $\bigcap X \neq \varnothing$, then $\bigcap Y \neq \varnothing$, so the nerve complex is an abstract simplicial complex.
Thm. We have

$$\mathrm{Nrv}(F) \simeq \bigcup \{x : x \in F\} \subset \mathbb{E}^d.$$

Lemma w/o proof

Properties of Čech complexes

Topology

Definitions

Homology class is born / dies

Topology

Definition

Persistence diagram

Topology

Definition, Theorem w/o proof

Vietoris-Rips complex

Topology

Definition, Lemma

Elementary collapse

Topology

Definition

Persistent homology module

Topology

Definition, Lemma w/o proof

Persistent Betti number, $\mu_p^{i,j}$

Topology

Definition

Clique complex

Topology

Definition

Free face, regular pair

Topology

Definition, Examples, Counterexamples

Collapsible

Topology

Let $K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n = K$ be some *filtration* of a complex K. Then (by definition) for $0 \leqslant i \leqslant j \leqslant n$, we have $K_i \leqslant K_j$ and the induced homomorphisms $f_p^{i,j} \colon H_p(K_i; R) \to H_p(K_j; R)$, where R is a commutative ring with 1. The p-th **persistent homology module** of the filtration $K_\bullet$ with respect to R is $H_p^{i,j} := H_p^{i,j}(K_\bullet; R) := \mathrm{im}(f_p^{i,j})$.
We have $H_p^{i,j} = Z_p(K_i)/[B_p(K_j) \cap Z_p(K_i)]$ and thus in particular$H_p^{i,i} = H_p(K_i)$.

We have $\check{\mathrm{C}}\mathrm{ech}(0) = S$ as a 0-dimensional complex.
$\check{\mathrm{C}}\mathrm{ech}(\infty)$ is a $(|S|-1)$-dimensional simplex on the vertex set S, where ∞ denotes a sufficiently large radius $r > \mathrm{diam}(S)$.
If $r \leqslant r'$, then $\check{\mathrm{C}}\mathrm{ech}(r) \leqslant \check{\mathrm{C}}\mathrm{ech}(r')$, where $\leqslant$ denotes "is a subcomplex of".
Hence we get a *filtration* of the final complex $\check{\mathrm{C}}\mathrm{ech}(\infty)$, a sequence of subcomplexes that is contained in each other. In between $r = 0$ and "$r = \infty$" we get some things which depend on the geometry of S, while $\check{\mathrm{C}}\mathrm{ech}(0)$ and $\check{\mathrm{C}}\mathrm{ech}(\infty)$ only depend on $|S|$.

Let $\beta_p^{i,j} := \dim_R(H_p^{i,j})$ be the *p-th persistent* Betti *number* of $K_\bullet$ with respect to R. Further, if $i < j$, let

$$\mu_p^{i,j} := \left(\beta_p^{i,j-1} - \beta_p^{i,j}\right) - \left(\beta_p^{i-1,j-1} - \beta_p^{i-1,j}\right),$$

which is the number of independent p-dimensional homology classes born in K_i which die entering K_j. For all $0 \leqslant k \leqslant \ell \leqslant n$ we have

$$\beta_p^{k,\ell} = \sum_{i \leqslant k} \left(\sum_{j=\ell+1}^{n} \mu_p^{i,j} + \mu_p^{i,\infty} \right),$$

where $\mu_p^{i,\infty}$ is the number of homology classes still alive in $K = K_n$.

A homology class $\gamma \in H_p(K_i)$ is *born* at K_i if $\gamma \notin H_p^{i-1,i}$.
If γ is born at K_i, then it *dies entering* K_j if $f_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $f_p^{i,j-1}(\gamma) \in H_p^{i-1,j}$.

Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E. The **clique complex** is the simplicial complex

$$C(G) := \left\{\sigma \subset V : \forall u, v \in \sigma \text{ with } u \neq v : \{u, v\} \in E\right\}$$

on V.

The clique complex of (V, E) contains both V and E as 0 and 1-dimensional faces. Its two-dimensional faces are the triangles as in a triangle each vertex is connected to each other vertex, whereas for a quadrangle this is not the case.

The p-th **persistence diagram** of the filtration with respect to R is the point configuration

$$\{(i, j) : \mu_p^{i,j} \geqslant 1\} \in \overline{\mathbb{R}}^2 := \mathbb{R} \times (\mathbb{R} \cup \{\infty\})$$

with multiplicities.

Let K be a finite ASC. A face $\sigma \in K$ is **free** if there is a *unique* $\tau \in K$ such that $\sigma \lessdot \tau$, that is σ *is a facet of* τ, it has exactly one dimension less: $\dim(\sigma) = \dim(\tau) - 1$. In that case, (σ, τ) is a **regular pair** of K.

Let $S \subset \mathbb{R}^d$ be a finite point set and fix a radius $r \geqslant 0$. Consider the 1-dimensional simplicial complex $G(r) := \check{\mathrm{C}}\mathrm{ech}(r)^{\leqslant 1}$ as a graph.
The **Vietoris-Rips** complex is $\mathrm{VR}(r) := C(G(r))$
This yields a filtration by choosing radii r.

If (τ, σ) is a *regular pair*, then $K \backslash \{\tau, \sigma\}$ is again a simplicial complex due to the uniqueness of the larger face τ ("nobody else is missing σ other than τ").

We have $\check{\mathrm{C}}\mathrm{ech}(r) \leqslant \mathrm{VR}(r) \leqslant \check{\mathrm{C}}\mathrm{ech}(\sqrt{2}r)$.

Contractible spaces which have a sequence of regular pairs such that removing them from the space yields a point are **collapsible**. (Hence collapsible $\implies$ contractible.)
The dunce hat is contractible, but not collapsible. The Möbius strip, $\mathbb{S}^n$, $n \geqslant 1$ and $M(p)$ are not contractible and thus not collapsible.
Any tree and any convex set is collapsible.

The complex $K \backslash \{\tau, \sigma\}$ is the **complex obtained from K by an elementary collapse**.
If (σ, τ) is a regular pair of K, then complex obtained from K by an elementary collapse $K \backslash \{\sigma, \tau\}$ is homotopy equivalent to K.

This combinatorial operation, if it is possible, *simplifies the complex* because it reduces the number of faces but *retains the topological information.*