Lemma (Wintner)

$$\sigma(T) \subset \overline{W(T)} \text{ for } T \in L(H)$$

FA II

Lemma

Numerical Range of Self-adjoint / Positive Operators

FA II

Continuous functional calculus

For every $T = T^* \in L(\mathcal{H})$ $\exists!$ cts., lin., multiplicative, involutive homomorphism of algebras $\Phi_T : \mathcal{C}(\sigma(T)) \to L(\mathcal{H})$ with $\Phi_T(\mathrm{id}) = T$, $\Phi_T(\mathbb{1}) = \mathrm{id}$.

FA II

Properties of the Continuous Functional Calculus

Let $T = T^* \in L(\mathcal{H})$ and $f \mapsto f(T)$ be the CFC for $f \in \mathcal{C}(\sigma(T))$. 1. Then $\|f(T)\| = \|f\|_\infty := \sup_{\lambda \in \sigma(T)} |f(\lambda)|$ holds. 2. If $f|_{\sigma(T)} \geqslant 0$, then $f(T)$ is positive. 3. If $Tx = \lambda$ for some $x \in \mathcal{H}$, then $f(T)x = f(\lambda)x$. 4. The spectral mapping theorem holds for all $f \in \mathcal{C}(\sigma(T))$. 5. $\{f(T)\}_{f \in \mathcal{C}(\sigma(T))}$ is a commutative Banach operator-algebra. 6. All $f(T)$ are normal; if f is real, then $f(T)$ is self-adjoint.

FA II

Theorem

Riesz-Markov-Kakutani

FA II

All you need to know about...

Orthogonal projections

FA II

Definition

(complex) signed Radon measure

FA II

Idea of Borel measurable Calculus

Let $T = T^* \in L(\mathcal{H})$. 1. For every $x \in \mathcal{H}$ there exists a non-negative Radon-measure E^x such that $\langle f(T)x, x \rangle = \int_{\sigma(T)} f \, \mathrm{d}E^x$ holds for all $f \in \mathcal{C}(\sigma(T))$. 2. For every on $\sigma(T)$ bounded Borel-measurable function g there exists a unique $G \in L(\mathcal{H})$ such that $\langle Gx, x \rangle = \int_{\sigma(T)} g \, \mathrm{d}E^x$ holds for all $x \in \mathcal{H}$. If g is real (non-negative), G is self-adjoint (positive).

FA II

Definitions

Unbounded Operator

FA II

Theorem

Hellinger-Toeplitz

FA II

For $T = T^* \in L(\mathcal{H})$ we have $\sigma(T) \subset [\inf(W(T)), \sup(W(T))] = \big[\min(\sigma(T)), \max(\sigma(T))\big]$ and for $T \geqslant 0$: $\sigma(T) \subset [0, \infty)$.
$\|T\| = \sup_{\|x\|=1} \langle Tx, x \rangle$ for self-adjoint operators.
Self-adjoint operators with $\sigma(T) \subset [0, \infty)$ are positive. Positive operators are self-adjoint.
$T \in L(\mathcal{H})$ is self-adjoint if and only if $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.

Let $\lambda \notin \overline{W(T)}$ and $d := \operatorname{dist}(\lambda, W(T)) > 0$. Then $d \leqslant |\lambda - \langle Tx, x \rangle \leqslant \|(\lambda \operatorname{id} - T)x\| \cdot \|x\|$ for $\|x\| = 1$. Thus T injective, $(\lambda \operatorname{id} - T)^{-1} : \mathcal{R}(\lambda \operatorname{id} - T) \to \mathcal{H}$ bounded below. Hence $\mathcal{R}(\lambda \operatorname{id} - T)$ closed. Assume $\exists x_0 \in \mathcal{R}(\lambda \operatorname{id} - T)^{\perp}$, $\|x_0\| = 1$. Then $0 = \langle (\lambda \operatorname{id} - T)x_0, x_0 \rangle = \lambda - \langle Tx_0, x_0 \rangle$. Thus $\mathcal{R}(\lambda \operatorname{id} - T) = \mathcal{H}$ and $\lambda \in \rho(T)$.
Case 1: $\lambda \in \sigma_p(T)$. Then there exists a $v \in H$ such that $Tv = \lambda v$. Thus $\langle v, Tv \rangle = \lambda \|v\|^2$ holds, implying $\langle \frac{v}{\|v\|}, T \frac{v}{\|v\|} \rangle = \lambda$, which means $\lambda \in W(T)$.
Case 2: $\lambda \in \sigma_r(T)$. Since the range is not dense, we have $(\overline{\mathcal{R}(T - \lambda)})^{\perp} \neq \{0\}$. For $v \in (\overline{\mathcal{R}(T - \lambda)})^{\perp}$ we have $\langle v, (T - \lambda)v \rangle = 0$. Thus $0 = \frac{\langle v, (T-\lambda)v \rangle}{\|v\|^2} = \frac{\langle v, Tv \rangle}{\|v\|^2} - \lambda$ holds, implying $\lambda \in W(T)$. Case 3: $\lambda \in \sigma_c(T)$. There exist a sequence of unit vectors z_n with $(T - \lambda)z_n \to 0$, otherwise $T - \lambda$ would be *bounded from below* and would necessarily have a closed range. Thus $\frac{\langle z_n, Tz_n \rangle}{\|z_n\|^2} - \lambda = \frac{\langle z_n, (T-\lambda)z_n \rangle}{\|z_n\|^2} \to 0$, holds, implying $\lambda \in \overline{W(T)}$.

---

2. Let $f \geqslant 0$ and $g \in \mathcal{C}(\sigma(T))$ with $g^2 = f$ and $g \geqslant 0$. Then

$$\langle f(T)x, x \rangle = \langle g(T)x, \overline{g}(T)x \rangle = \langle g(T)x, g(T)x \rangle = \|g(T)x\|^2 \geqslant 0.$$

4. "$\subset$": Let $\mu \notin f(\sigma(T))$. Then $g(x) := \frac{1}{f(x)-\mu} \in \mathcal{C}(\sigma(T))$ and $g(f - \mu) = (f - \mu)g = 1$. Hence we get $g(T)(f(T) - \mu \operatorname{id}) = (f(T) - \mu \operatorname{id})g(T) = \operatorname{id}$, hence $\mu \in \rho(f(T))$.
"$\supset$": Let $\mu = f(\lambda)$, $\lambda \in \sigma(T)$ and choose polynomials f_n with $\|f_n - f\|_\infty \leqslant \frac{1}{n}$. Then $|f(\lambda) - f_n(\lambda)| \leqslant \frac{1}{n}$ and $\|f(T) - f_n(T)\| \leqslant \frac{1}{n}$. We know that $f_n(\lambda) \in \sigma(f_n)T))$, i.e. $\exists x_n \in \mathcal{H}$: $\|x_n\| = 1$, $\|(f_n(T) - f_n(\lambda) \operatorname{id})x_n\| \leqslant \frac{1}{n}$. Thus, $\|(f(T) - \mu \operatorname{id})x_n\| \leqslant \|(f(T) - f_n(T))x_n\| + \|(f_n(T) - f_n(\lambda) \operatorname{id})x_n\| + \|(f_n(\lambda) - \mu \operatorname{id})x_n\| \leqslant \frac{1}{n} + \frac{1}{n} + \frac{1}{n}$ holds, implying that $(f(T) - \mu \operatorname{id})$ is not boundedly invertible, i.e. $\mu \in \sigma(f(T))$. 6.: $f(T)^* = \overline{f}(T)$ and $f(T)^* f(T) = \overline{f} f(T) = f \overline{f}(T) = f(T) f(T)^*$. 1 and 3: approximation.

Uniqueness: $\Phi_T(z \mapsto z^n) = T^n$, hence Φ_T implies uniqueness on polynomials. $\sigma(T) \subset \big[m(T), M(T)\big]$ is compact and the polynomials dense in $\mathcal{C}(\sigma(T))$ by Stone-Weierstrass. Due to continuity, Φ_T is unique on $\mathcal{C}(\sigma(T))$.
Existence: We set $\Phi_T(f) = \sum_{k=0}^n a_k T^k$ for a polynomial $f(z) = \sum_{k=0}^n a_k z^k$. If we show continuity of Φ_T on polynomials, there would be an unique extension (cf. above) to $\mathcal{C}(\sigma(T))$, which we denote by Φ_T again. By the SMT for polynomials f we obtain

$$\begin{aligned} \|\Phi_T(f)\|^2 &= \|\Phi_T(f)^* \Phi(f)\| = \|\Phi_T(\overline{f} f)\| \\ &= \sup_{\lambda \in \sigma(\Phi_T(\overline{f} f))} |\lambda| = \sup_{\lambda \in \sigma(T)} |(\overline{f} f)(\lambda)| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|^2 = \|f\|_\infty^2. \end{aligned}$$

$E = \mathcal{N}(P) \oplus \mathcal{R}(P)$, $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are closed and $\operatorname{id} - P$ is a projection, too, with $\mathcal{N}(\operatorname{id} - P) = \mathcal{R}(P)$, $\mathcal{R}(\operatorname{id} - P) = \mathcal{N}(P)$.
$P \in L(\mathcal{H}) \neq \{0\}$ projection is orthogonal $\iff \|P\| = 1 \iff P$ self-adjoint $\iff P$ normal $\iff P$ positive.
P_1, P_2 orthogonal projections onto U_1, U_2. TFAE: $U_1 \subset U_2$, $\mathcal{R}(P_1) \subset \mathcal{R}(P_2)$, $\mathcal{N}(P_2) \subset \mathcal{N}(P_1)$, $P_1 P_2 = P_2 P_1 = P_1$, $P_2 - P_1$ is positive.
If $T = T^* \in L(\mathcal{H})$ and $\sigma(T) = \{0, 1\}$, T is an ortho projection.

For every functional $\varphi : \mathcal{C}(K) \to \mathbb{C}$ there exists a Radon measure $\mu \in \mathcal{M}(K)$ such that

$$\varphi(f) = \int_K f(x) \, \mathrm{d}\mu(x) \qquad \forall f \in \mathcal{C}(K).$$

The isometric isomorphism $\Phi : \mathcal{C}(K)^* \to \mathcal{M}(K)$ maps positive functionals to non-negative measures.

---

1. is Riesz
For 2.: Uniqueness from Exercise as G self-adjoint.
Existence: $Q(x) := \int_{\sigma(T)} g \, \mathrm{d}E^x$. Show Q fulfills lemma requirements. Use Riesz uniqueness for parallelogram-like equality and $Q(\lambda x) = |\lambda|^2 Q(x)$ to show $E^{x+y} + E^{x-y} = 2E^x + 2E^y$ etc.

A *finite* measure μ on $(K, \mathcal{B})$ is called Radon measure if

$$\mu(B) = \inf_{\substack{B \subset G \\ G \text{ open}}} \mu(G) \quad \forall B \in \mathcal{B} \qquad \text{(outer regularity)}$$

$$\mu(G) = \inf_{\substack{T \subset G \\ G \text{ comp.}}} \mu(T) \quad \forall G \text{ open} \qquad \text{(inner regularity)}$$

hold.
A σ-additive mapping $\mu = \mu^+ - \mu^-$ on $\mathcal{B}$ with $\mu^{\pm}$ Radon is called signed Radon measure.

Let T be a symmetric operator with $\operatorname{dom}(T) = \mathcal{H}$. Then $T \in L(\mathcal{H})$.
Proof. Let $x_n \to 0$ and $Tx_n \to y$. By the CGT it suffices to show $y = 0$.

$$\langle y, y \rangle = \langle \lim_{n \to \infty} Tx_n, y \rangle = \lim_{n \to \infty} \langle Tx_n, y \rangle = \lim_{n \to \infty} \langle x_n, Ty \rangle = 0.$$

A linear operator $T : \mathcal{H} \supset \operatorname{dom}(T) \to \mathcal{H}$, where $\operatorname{dom}(T)$ is a linear subspace is called densely defined if $\operatorname{dom}(T)$ is dense. The scalar product on its graph $G(T) := \{(Tx, x) : x \in \operatorname{dom}(T)\}$ is $\langle (u, v), (x, y) \rangle_{\mathcal{H} \times \mathcal{H}} := \langle u, x \rangle + \langle v, y \rangle$.
T is closable if $\overline{G(T)}$ is the graph of some linear operator $T_0 =: \overline{T}$. S is an extension of T (denote $T \subset S$) if $G(T) \subset G(S) \iff \operatorname{dom}(T) \subset \operatorname{dom}(S)$ and $S \equiv T$ on $\operatorname{dom}(T)$.
$\operatorname{dom}(S + T) := \operatorname{dom}(S) \cap \operatorname{dom}(T)$ and $\operatorname{dom}(ST) := \{x \in \operatorname{dom}(T) : Tx \in \operatorname{dom}(S)\}$.
T symmetric $\iff \langle Tx, y \rangle = \langle x, Ty \rangle \; \forall x, y \in \operatorname{dom}(T)$.

Definition

Adjoint of densely defined operator

FA II

Borel-measurable Calculus

As CFC but also $f_n \in \mathcal{B}(\sigma(T))$ with $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f_n(t) \to f(t)$ on $\sigma(T)$ implies $\langle \Phi_T(f_n)x, y \rangle \to \langle \Phi_T(f)x, y \rangle$.

FA II

Lemma

Let $Q : \mathcal{H} \to \mathbb{C}$. Then $\exists! A \in L(\mathcal{H})$: $Q(x) = \langle Ax, x \rangle$ iff $\exists C > 0$: $|Q(x)| \leqslant C\|x\|$, $Q(x+y) + Q(x-y) = 2(Q(x) + Q(y))$ and $Q(\lambda x) = |\lambda|^2 Q(x)$.

FA II

Definition

Spectral measure

FA II

Lemma

Properties of spectral measures

FA II

Lemma

Properties of spectral integrals

FA II

Applications

Spectral theorem (bounded, self-adjoint)

FA II

Theorem

Spectral theory for unbounded operators

FA II

Theorem

Spectral theorem for unbounded operators

FA II

Basic properties

Fourier transform

FA II

Uniqueness: as above + Werner lemma.
For bounded measurable g on $\sigma(T)$ define $\Phi_T(g) = G$. CFC guarantees $\Phi_T(\mathrm{id}) = T$, $\Phi(\mathbb{1}) = \mathrm{id}$.
For real-valued g we have $\|\Phi_T(g)\| = \|G\| = \sup_{\|x\|=1} |\langle Gx, x \rangle| \leqslant \sup_{\|x\|=1} \|g\|_\infty \|x\|^2 = \|g\|_\infty$. The last property follows from $\langle \Phi_T(f_n)x, x \rangle = \int_{\sigma(T)} f_n \,\mathrm{d}E^x \to \int_{\sigma(T)} f \,\mathrm{d}E^x = \langle \Phi(f)x, x \rangle$, Lebesgue, polarisation.
(ii) Prinzip der guten Menge: $g \in \mathcal{C}(\sigma(T))$, $U := \{f \in \mathcal{B}(\sigma(T)) :$ property holds $\}$. U closed under pointwise limits of uniformly bounded sequences. Thus $U = \mathcal{B}(\sigma(T))$.

$\mathrm{dom}(T^*) := \{y \in \mathcal{H} : x \mapsto \langle Tx, y \rangle$ cts on $\mathrm{dom}(T)\}$. By RIESZ we can extend (uniquely due denseness) the above mapping for $y \in \mathrm{dom}(T^*)$ and can be represented as $x \mapsto \langle x, z \rangle$, $T^*y := z$.
We have $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in \mathrm{dom}(T)$, $y \in \mathrm{dom}(T^*)$.

Let Σ be the σ-Algebra of BOREL sets on $\mathbb{R}$. $E : \Sigma \to L(\mathcal{H})$, $A \mapsto E_A$ is called SM if E_A is a orthogonal projection for all $A \in \Sigma$, $E_\varnothing = 0$, $E_\mathbb{R} = \mathrm{id}$ and $\sum_{k\geqslant 1} E_{A_k}(x) = E_{\bigcup_{k\geqslant 1} A_k}(x)$ for pw. disjoint $(A_k)_k \subset \Sigma$.
E has compact support if $\exists K \subset \mathbb{R}$ compact with $E_K = \mathrm{id}$.

$2 \implies 1$:

$$\Phi(x, y) := \frac{Q(x + y) - Q(x - y) + iQ(x + iy) - iQ(x - iy)}{4}$$
$$Ax := \sum_{k\in O} \Phi(x, e_k)e_k$$

for an ONB $(e_k)_{k\in O}$ of $\mathcal{H}$. We have $\Phi(x, y) = \langle Ax, y \rangle$ (proven as in FA I (parallelogram identity)).

f simple $\implies \|\int f \,\mathrm{d}E\| \leqslant \|f\|_\infty \implies$ well-definedness.
$f \mapsto \int f \,\mathrm{d}E$ linear, continuous, $\|\int f \,\mathrm{d}E\| \leqslant \|f\|_\infty$. f real $\implies \int f \,\mathrm{d}E$ self-adjoint

- $E_A + E_B = E_{A\cap B} + E_{A\setminus B} = E_{A\cap B} + E_{A\cup B}$.
- $A \subset B$. Then $E_{B\setminus A} = E_B - E_A$ projection. $E_BE_A = E_AE_B = E_A$.
- $A \cap B = \varnothing$. Then $E_A + E_B = E_{A\cup B}$, $E_A^2 + E_AE_B = E_AE_{A\cup B} = E_A$, hence $E_AE_B = 0$.
- Thus $E_AE_B = E_AE_{A\cap B} + E_AE_{B\setminus A} = E_{A\cap B}$.
- $A \cap B = \varnothing$. Then $\langle E_Ax, E_By \rangle = \langle E_B^*E_Ax, y \rangle = \langle E_BE_Ax, y \rangle = \langle 0, y \rangle = 0$.

$\lambda \in \rho(T) \iff \lambda - T : \mathrm{dom}(T) \to \mathcal{H}$ boundedly invertible. Resolvent mapping: $R(\lambda) := (\lambda - T)^{-1}$ is analytic and $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$. $\sigma(T) := \rho(T)^\complement$ closed. $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$.
Let $z \in \mathbb{C}\setminus\mathcal{R}$, T dense, symmetric. $T = T^* \iff T$ closed and $\mathcal{N}(T^*-z) = \mathcal{N}(T^*-\overline{z}) = \{0\} \iff \mathcal{R}(T-z) = \mathcal{R}(T-\overline{z}) = \mathcal{H}$.
$T = T^*$ dense $\implies \sigma(T) \subset \mathbb{R}$. (By HELLINGER-TOEPLITZ it suffices to show: everywhere defined.)

$T = T^* \in L(\mathcal{H})$, $g : \sigma(T) \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ B-meas., bd. $(f \circ g)(S) = f(g(S))$.
$T \in L(\mathcal{H})$ positive. $\exists! S \in L(\mathcal{H})$ positive, $S^n = T$.
Proof. $f_n(t) := t^{\frac{1}{n}}$ cts. bd. non-negative on $\sigma(T) \subset [0, \infty)$. $g_n(t) := t^n$. Then $(f_n \circ g_n)(t) = t$, $t \in \sigma(S) \subset [0, \infty)$. Uniqueness: $S = (f_n \circ g_n)(S) = f_n(g_n(S)) = f_n(S^n) = f_n(T)$.
$T \in L(\mathcal{H})$, $|T| := \sqrt{T^*T}$. $\exists$ partial isometry $U : T = U|T|$.
$\||T|x\|^2 = \|Tx\|^2$. $U(|T|x) := Tx : \mathcal{R}(|T|) \to \mathcal{R}(T)$ isometry, extended to $\overline{\mathcal{R}(|T|)}$, $U \equiv 0$ on $\mathcal{R}(|T|)^\perp = \mathcal{N}(|T|) = \mathcal{N}(T)$.

- $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ linear, bd.
- $\mathcal{F}(f(\cdot - y)) = e^{-i\langle y,\cdot\rangle}\hat{f}(\cdot)$, $\mathcal{F}(e^{i\langle y,\cdot\rangle}f(\cdot)) = \hat{f}(\cdot - y)$
- $\widehat{f(\lambda^{-1}\cdot)} = \lambda^n\widehat{f(\lambda\cdot)}$, $g(x) = \overline{f(-x)} \implies \hat{g} = \overline{\hat{f}}$
- $f, g \in L^1(\mathbb{R}^n) \implies f * g \in L^1(\mathbb{R}^n)$, $\widehat{f * g} = (2\pi)^{\frac{n}{2}}\hat{f}\hat{g}$
- $g(x) = -ix_kf(x)$, $g \in L^1(\mathbb{R}^n) \implies \partial_k\hat{f} = \hat{g}$.

Let $T = T^* : \mathrm{dom}(T) \to \mathcal{H}$. $\exists!$ SM E on BOREL sets of $\sigma(T) : T = \int_{\sigma(T)} \lambda \,\mathrm{d}E_\lambda$, i.e. $\langle Tx, y \rangle = \int_{\sigma(T)} \lambda \,\mathrm{d}\langle E_\lambda x, y \rangle$ for all $x \in \mathrm{dom}(T)$, $y \in \mathcal{H}$.
$\sigma(T) \subset \mathbb{R}$ might be unbounded!

THEOREM

RIEMANN-LEBESGUE Lemma

FA II

THEOREM

FOURIER inversion formula

FA II

THEOREM

Plancherel

FA II

THEOREM

First Uncertainty principle

FA II

THEOREM

Second Uncertainty Principle

FA II

DEFINITION

SCHWARTZ space

FA II

THEOREM

Properties of SCHWARTZ convergence

FA II

THEOREM

FOURIER transform on $\mathcal{S}(\mathbb{R}^n)$

FA II

THEOREM

SCHWARTZ FOURIER inversion formula

FA II

DEFINITION & PROPERTIES

Tempered distributions

FA II

$f, \hat{f} \in L^1(\mathbb{R}^n) \implies f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\langle x, \xi \rangle} \,\mathrm{d}\xi = \mathcal{F}^{-1}(\mathcal{F} f)(x)$.

FEJÉR-kernels $F_\lambda := \lambda D_{\lambda^{-1}} F$, where $\lambda > 0$, $F(x) := \frac{1}{2\pi}\int_{-1}^{1}(1-|t|)e^{ixt}\,\mathrm{d}t$ satisfy $\|f - f * F_\lambda\|_1 \to 0$ and $f * F_\lambda(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)\left[\lambda \int_{-1}^{1}(1-|\theta|)e^{i(x-t)\theta\lambda}\,\mathrm{d}\theta\right]\mathrm{d}\lambda \to f(x)$ pointwise a.e. We show $f * F_\lambda(x) \to \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\theta)e^{ix\theta}\,\mathrm{d}\theta$ for $\lambda \to \infty$.

$f \in L^1(\mathbb{R}^n) \implies \hat{f} \in \mathcal{C}_0(\mathbb{R}^n, \mathbb{R}^n)$.

Continuity of $\hat{f}$: LEBESGUE Let $f := \mathbb{1}_{\prod_{k=1}^n [a_k, b_k]}$. Then $|\hat{f}(\xi)| = \frac{1}{(2\pi)^{\frac{n}{2}}}\prod_{k=1}^n \frac{|e^{-ib_k\xi_k} - e^{-ia_k\xi_k}|}{|\xi_k|} \xrightarrow{\|\xi\|\to\infty} 0$. holds.

By denseness $\exists$ step function h arbitrarily close to f:

$$|\mathcal{F}(f(\xi))| \overset{\triangle\neq}{\leqslant} |\mathcal{F}(f-h)(\xi)| + |\mathcal{F}(h(\xi)| \leqslant \frac{\|f-h\|_1}{(2\pi)^{\frac{n}{2}}} + |\mathcal{F}(h(\xi)|$$
$$\leqslant \varepsilon(2\pi)^{-\frac{n}{2}} + |\mathcal{F}(h(\xi)| \xrightarrow{|\xi|\to\infty} \varepsilon(2\pi)^{-\frac{n}{2}}.$$

$f \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$. $g := (\cdot - a)f$, $h := (\cdot - b)\hat{f}$. $\|g\|_2\|h\|_2 \geqslant \frac{\|f\|_2^2}{2}$.

1. Density $(\mathcal{S})$. W.l.o.g $\|f\|_2 = 1$, $a = b = 0$ ($\tilde{f} := f(\cdot + a)e^{-ib\cdot}$). PI: $1 = -\int_{\mathbb{R}} x \frac{\mathrm{d}}{\mathrm{d}x}|f(x)|^2\,\mathrm{d}x = -\int_{\mathbb{R}} x f'(x)\overline{f(x)} + x\overline{f'(x)}f(x)\,\mathrm{d}x$. Hence $1 \leqslant 2\int_{\mathbb{R}} |x|f'(x)||f(x)|\,\mathrm{d}x \leqslant 2\|g\|_2\|f'\|_2$. Plancherel: $\|f'\|_2 = \|\xi\hat{f}\|_2$.
2. $(Sf)(x) := xf(x)$, $f \in L^2(\mathbb{R}^n)$. $(Tf)(x) := if'(x)$ for diff'ble $f \in L^2(\mathbb{R}^n)$. $([S,T]f)(x) = -if(x)$. By UP II $\frac{1}{2}\|f\|_2^2 \leqslant \|(S-aI)f\|_2\|(T-bI)f\|_2$. Plancherel analogous to the above.

For $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$.

$h(x) := (f * (g - \cdot))(x)$. $\hat{h}(\xi) = (2\pi)^{\frac{n}{2}}\hat{f}(\xi)\overline{\hat{g}(\xi)}$. $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$ (dense!) $\implies \hat{h} \in L^1(\mathbb{R}^n)$. FOURIER-inversion: $\langle f, g \rangle = h(0) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\mathrm{d}\xi = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}(2\pi)^{\frac{n}{2}}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi = \langle \hat{f}, \hat{g} \rangle$.

Extension to L^2: $f_k := f\mathbb{1}_{[-k,k]^n} \in L^1 \cap L^2$. $\|f - f_k\| \to 0$. PLANCHEREL: $\|\hat{f}_k - \hat{f}_\ell\|_2 = \|f_k - f_\ell\|_2$. Thus CAUCHY in L^2. Completeness $\implies f_k \to g =: \hat{f}$.

$$\mathcal{S}(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|_{(k,\ell)} < \infty \ \forall k, l \in \mathbb{N}_0\}$$
$$= \{f : \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \ \forall \alpha, \beta \in \mathbb{N}_0^n\}$$
$$\|f\|_{(k,\ell)} := \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{\frac{k}{2}}\sum_{|\alpha|\leqslant\ell}|D^\alpha f(x)|,$$
$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n}|x^\alpha D^\beta f(x)|,$$
$$\|f\|_{(N)} := \max_{|\alpha|\leqslant N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{\frac{N}{2}}|D^\alpha f(x)|$$
$$f_j \xrightarrow{\mathcal{S}} f \iff \|f_j - f\|_{(k,\ell)} \to 0 \ \forall k, \ell \in \mathbb{N}_0.$$

$S : \mathrm{dom}(S) \to \mathcal{H}$, $T : \mathrm{dom}(T) \to \mathcal{H}$ self-adjoint, $a, b \in \mathbb{R}$. $\|(S-aI)f\|\|(T-bI)f\| \geqslant \frac{1}{2}|\langle [S,T] \rangle| \forall f \in \mathrm{dom}(ST) \cap \mathrm{dom}(TS)$.

Easy: $[S-aI, T-bI] = [S,T]$. $S - aI$, $T - bI$ self-adjoint.

$$\begin{aligned}\langle [S,T]f, f \rangle &= \langle (S-aI)(T-bI) - (T-bI)(S-aI)f, f \rangle \\ &= \langle (T-bI)f, (S-aI)f \rangle - \langle (S-aI)f, (T-bI)f \rangle \\ &= 2i \cdot \Im(\langle (T-bI)f, (S-aI)f \rangle)\end{aligned}$$

holds. Now Cauchy-Schwarz-inequality.

Let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ SCHWARTZ-converge to $f \in \mathcal{S}(\mathbb{R}^n)$.

1. It holds that $\hat{f}, \check{f} \in \mathcal{S}(\mathbb{R}^n)$.
2. For all $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ we have
$$D^\alpha \hat{f}(\xi) = (-i)^{|\alpha|}\widehat{x^\alpha f}(\xi) \text{ and } \xi^\alpha \hat{f}(\xi) = (-i)^{|\alpha|}\widehat{D^\alpha f}(\xi)$$
3. It holds that $\hat{f}_j \xrightarrow{\mathcal{S}} \hat{f}$ and $\check{f}_j \xrightarrow{\mathcal{S}} \check{f}$.
4. $\widehat{\varphi \otimes \psi}(\xi, \eta) = \hat{\varphi}(\xi)\hat{\psi}(\eta)$, $\widehat{\varphi\psi} = (2\pi)^{-\frac{n}{2}}\hat{\varphi} * \hat{\psi}$ and $\hat{f} = f$ for $f(x) = e^{-\frac{|x|^2}{2}}$.

Let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ with $f_j \xrightarrow{\mathcal{S}} f \in \mathcal{S}(\mathbb{R}^n)$.

1. $f_j \to f$ in $L^p(\mathbb{R}^n)$ holds for all $p \in (0, \infty)$.
2. $D^\alpha f_j \xrightarrow{\mathcal{S}} D^\alpha f$ holds for all $\alpha \in \mathbb{N}_0^n$.
3. $(x \mapsto x^\alpha f_j(x)) \xrightarrow{\mathcal{S}} (x \mapsto x^\alpha f(x))$ holds for all $\alpha \in \mathbb{N}_0^n$.
4. Then $T_h f \xrightarrow[h\to 0]{\mathcal{S}} f$ holds.
5. For $g \in \mathcal{S}(\mathbb{R}^n)$ also $fg, f * g \in \mathcal{S}(\mathbb{R}^n)$.
6. $D^\alpha(f * g) = (D^\alpha f) * g = f * (D^\alpha g)$ holds for all $\alpha \in \mathbb{N}_0^n$.

$\mathcal{S}'(\mathbb{R}^n)$ = cts dual space of $\mathcal{S}(\mathbb{R}^n)$, i.e. $Tf_k \to Tf \ \forall f_k \xrightarrow{\mathcal{S}} f$.

$T_k \xrightarrow{\mathcal{S}} T \iff T_k f \to Tf \ \forall f \in \mathcal{S}(\mathbb{R}^n)$.

$T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ linear. $T \in \mathcal{S}'(\mathbb{R}^n) \iff \exists c > 0, k, \ell \in \mathbb{N}_0 : |Tf| \leqslant c\|f\|_{(k,\ell)} \ \forall f \in \mathcal{S}(\mathbb{R}^n)$.

"1. $\implies$ 2.": Assume $\forall c > 0$, $k, \ell \in \mathbb{N}_0$ $\exists f_{c,k,\ell}$ s.t. $1 = |Tf_{c,k,\ell}| > c\|f_{c,k,\ell}\|_{(k,\ell)}$. $f_k := f_{k,k,k} \xrightarrow{\mathcal{S}} 0$, by cts: $Tf_k \to 0$. A contradiction.

$T_f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, $g \mapsto \int_{\mathbb{R}^n} fg\,\mathrm{d}x$. $|T_f g| \leqslant \|g\|_{(0,0)}\|f\|_1$. Also $T_f \in \mathcal{S}'(\mathbb{R}^n)$. Extendable to $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$.

Finite BOREL measure μ via $\mu(g) = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}\mu(x)$.

$\check{\hat{f}} = \hat{\check{f}} = f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are bijective on $\mathcal{S}(\mathbb{R}^n)$.

$I(\varepsilon) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\hat{f}(\xi)e^{i\langle x,\xi \rangle}e^{-\varepsilon^2\frac{|\xi|^2}{2}}\,\mathrm{d}\xi \xrightarrow{\varepsilon\to 0} \check{\hat{f}}$ by (L).

$g(x) := \exp\left(-\varepsilon^2\frac{|x|^2}{2}\right)$. $\hat{g}(\xi) = \varepsilon^{-n}\exp\left(-\frac{|\xi|^2}{2\varepsilon^2}\right)$ by FP, $D_{\varepsilon^{-1}}$.

$$(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\hat{f}(\xi)e^{i\langle x,\xi \rangle}h(\xi)\,\mathrm{d}\xi = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}T_{-x}f(\xi)\hat{h}(\xi)\,\mathrm{d}\xi$$
$$= (2\pi\varepsilon^2)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(x+y)\exp\left(-\frac{1}{2}\left|\frac{y}{\varepsilon}\right|^2\right)\mathrm{d}y$$
$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(x+\varepsilon z)\exp\left(-\frac{|z|^2}{2}\right)\mathrm{d}z \xrightarrow{\varepsilon\to 0} f(x).$$

Definition & Lemma

Regular distribution

FA II

Context

Convolution and Fourier transform of measures

FA II

Definition, Corollary

Tempered distributions and derivative / Fourier transforms.

FA II

Definition

The spaces $D(\Omega)$ and $D'(\Omega)$

FA II

Definition

Decomposition of the spectrum

FA II

Properties

Normal operator

FA II

Definition & more

Weighted L^2 space

FA II

Definition & more

Bessel potential spaces H^s

FA II

Definition

Locally compact abelian topological group

FA II

Definition & theorem

Haar measure

FA II

Measure μ on $\mathbb{R}^n$ with density $\varphi \in \mathcal{S}(\mathbb{R}^n)$, λ, ψ analog. $(\varphi * \psi)(f) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(z+y) \,\mathrm{d}\mu(z) \,\mathrm{d}\lambda(y)$ (uniqueness by RIESZ), $f \in \mathcal{C}_0(\mathbb{R}^n)$. Extendable to B-meas. functions, i.e. $\widehat{\mu * \lambda}(\xi) = (2\pi)^{\frac{n}{2}} \hat{\mu}(\xi)\hat{\lambda}(\xi)$ $\|\mu * \lambda\| \leqslant \|\mu\| \cdot \|\lambda\|$ (total variation), $\hat{\mu}(\xi) := \int e^{-i\langle x, \xi \rangle} \,\mathrm{d}\mu(x)$.
$(\mu * \lambda)(E) = (\mu \otimes \lambda)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x + y \in E\})$ for all B-meas. $E \subset \mathbb{R}^n$.

$T \in \mathcal{S}'(\mathbb{R}^n)$ regular if $\exists f \in L^1(\mathbb{R}^n)$ s.t. $T = T_f$.
$f, g \in L^1(\mathbb{R}^n)$, $T_f \equiv T_g$. Then $f = g$ a.e.
Suffices: $T_h \equiv 0 \implies h = 0$ a.e. Mollifier ω_ε (symmetric, unit int., unit supp.) $h * \omega_\varepsilon \in \mathcal{C}(\mathbb{R}^n)$. $0 = \int_{\mathbb{R}^n} h\omega_\varepsilon * f \,\mathrm{d}x = \int_{\mathbb{R}^n} fh * \omega_\varepsilon \,\mathrm{d}x$. $x \in U_x$ s.t. $(\omega_\varepsilon * h)(x) > 0$. $\varphi_x \in \mathcal{S}(\mathbb{R}^n)$ s.t. $\operatorname{supp}(\varphi_x) \subset U_x$. Denseness of cts compactly supported fct in L^1: $h = h_1 + h_2$, h_1 cts. compact supp, $\|h_2\|_1 \leqslant t$. $\|h\|_1 \leqslant \|h_1 - \omega_\varepsilon * h_1\|_1 + \|h_2\|_1 + \|\omega_\varepsilon * h_2\|_1$.
$\|h_1 - \omega_\varepsilon h_1\|_1 \leqslant \sup_{|y| \leqslant \varepsilon} \|h_1(\cdot) - h_1(\cdot - y)\|_1 \to 0$ using FUBINI, $\varepsilon, t > 0$ small enough.

$\mathcal{D}(\Omega) \cong \mathcal{C}_c^\infty(\Omega)$. $f_j \xrightarrow{\mathcal{D}} f \in \mathcal{D}(\Omega) \iff \exists K \subset \Omega$ compact, $\operatorname{supp}(f_j) \subset K$, $D^\alpha f_j \to D^\alpha f$ $\forall \alpha \in \mathbb{N}_0^n$.
Distributions $D'(\Omega)$ like $\mathcal{S}'$.
$T_j \to T$ in $\mathcal{D}(\Omega)$ if $T_j(f) \to T(f)$ $\forall f \in \mathcal{D}(\Omega)$.
FOURIER transforms or convolutions do not have an easy counterpart on $\mathcal{D}(\Omega)$.

$T \in \mathcal{S}'(\mathbb{R}^n)$, $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$.

- $(D^\alpha T)(f) = (-1)^{|\alpha|} T(D^\alpha f)$
- $\widehat{T}(f) = T(\hat{f})$, $\check{T}(f) = T(\check{f})$
- $(fT)(g) = T(fg)$.

$D^\alpha T$, fT, $\widehat{T}$, $\check{T} \in \mathcal{S}'(\mathbb{R}^n)$.
$\mathcal{F}$, $\mathcal{F}^{-1}$ bijective, cts., $\widehat{D^\alpha T} = i^{|\alpha|} x^\alpha \widehat{T}$, $\widehat{x^\alpha T} = |i|^\alpha D^\alpha \widehat{T}$, $\widehat{D_\lambda T} = \lambda^{-n} \widehat{T(\lambda^{-1} \cdot)}$, $\widehat{\tau_h T} = e^{-i\langle h, \xi \rangle} \widehat{T}$, $\widehat{M_h T} = \tau_h \widehat{T}$.

T normal $\iff$ T^* normal $\iff$ $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.
For normal T the following statements hold:

1. $\mathcal{N}(T) = \mathcal{N}(T^*)$, $\overline{\mathcal{R}}(T) = \overline{\mathcal{R}}(T^*)$ and $\mathcal{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}}(T)$.
2. If $\alpha \neq \beta$ are eigenvalues, $\mathcal{N}(T - \alpha) \perp \mathcal{N}(T - \beta)$.
3. $\sigma_r(T) = \varnothing$
4. $r(T) = \|T\|$, which follows from $\|T^*T\| = \|T\|^2 = \|T^2\|$.
5. $\sigma_{\text{app}}(T) = \sigma(T)$

Let $R := \mathcal{R}(T_\lambda := T - \lambda)$. $\sigma(T)$ is the *disjoint* union of the point spectrum $\sigma_p(T) := \{\lambda \in \sigma(T) : T_\lambda \text{ not injective}\}$, continuous spectrum $\sigma_c(T) := \{\lambda \in \sigma(T) \backslash \sigma_p(T) : R \subsetneq \mathcal{H} \text{ dense}\}$, residual spectrum $\sigma_r(T) := \sigma(T) \backslash (\sigma_p(t) \cup \sigma_c(T))$ or

$$\sigma_r(T) = \{\lambda \in \sigma(T) : T_\lambda \text{ injective},\ R \subset \mathcal{H} \text{ not dense}\},$$

Furthermore the approximate point spectrum is

$$\sigma_{\text{app}}(T) := \{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|T_\lambda x\| = 0\} \supset \sigma_p(T).$$

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : w_s \hat{f} \in L^2(\mathbb{R}^n)\}, \quad s \in \mathbb{R}.$$

$$\langle f, g \rangle_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} w_s(x) \hat{f}(x) \overline{w_s(x) \hat{g}(x)} \,\mathrm{d}x.$$

$$\mathcal{S}(\mathbb{R}^n) \overset{\mathrm{d}}{\subset} H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

w cts positive fct $L^2(\mathbb{R}^n, w) := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : wf \in L^2(\mathbb{R}^n)\}$ $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n, w)} := \langle w \cdot, w \cdot \rangle_{L^2(\mathbb{R}^n)}$.
$L^2(\mathbb{R}^n, w) \to L^2(\mathbb{R}^n)$, $f \mapsto wf$ unitary.
$w_s : \mathbb{R}^n \to [0, \infty)$, $x \mapsto (1 + |x|^2)^{\frac{s}{2}}$, $s \in \mathbb{R}$.

$$\mathcal{D}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n) \overset{\mathrm{d}}{\subset} L^2(\mathbb{R}^n, w_s) \subset \mathcal{S}'(\mathbb{R}^n).$$

$\mathcal{F}(W_2^k(\mathbb{R}^n)) = \mathcal{F}^{-1}(W_2^k(\mathbb{R}^n)) = L^2(\mathbb{R}^n, w_k)$ unitary.

A left (right) HAAR *measure* on a LCG G is a nonzero RADON measure μ satisfying $\mu(xE) = \mu(E)$ ($\mu(Ex) = \mu(E)$) for all BOREL sets $E \subset G$ and all $x \in G$. (e.g. integral on $\mathcal{C}_c^\infty$)
(HAAR, 1933) Every locally compact group possesses a left (right) HAAR measure uniquely determined up to rescaling by a positive number.

A *topological group* is a group G equipped with a topology such that the group operations $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.
A topology is *locally compact* if every point has a compact neighbourhood. In a HAUSDORFF space points can be separated by open sets.
If the topology of G is locally compact and HAUSDORFF, G is a locally compact group.
$\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$ and $\mathbb{Z}_k$ are LCAGs.

Definition & theorem

PONTRYAGIN duality

FA II

Definition & Properties

FOURIER transform on $L^1(G)$

FA II

Theorem

FOURIER inversion formula on $L^1(G)$

FA II

Definition & properties

Convolution on $L^1(G)$

FA II

Lemma

Werner

FA II

Definition

Derivative, FOURIER-transform in $\mathcal{S}'(\mathbb{R}^n)$

FA II

Theorem

FOURIER transform of SCHWARTZ functions under dilation etc.

FA II

Definition

Least-squares and minimal norm solution

FA II

Definition & theorem

MOORE-PENROSE inverse

FA II

Theorem

Normal equation

FA II

Let $f \in L^1(G)$ and μ be the left-invariant HAAR measure on G. For $\gamma \in \widehat{G}$ the FOURIER transform of f is

$$\mathcal{F} f(\gamma) := \hat{f}(\gamma) := \int_G f(x)\overline{\gamma(x)}\,\mathrm{d}\mu(x).$$

Denote by $\mathcal{C}_0(\widehat{G})$ the set of continuous and bounded functions on $\widehat{G}$. We have $\mathcal{F} : L^1(G) \to \mathcal{C}_0(\widehat{G})$.
PLANCHEREL: The FOURIER transform on $L^1(G) \cap L^2(G)$ uniquely extends to a unitary isomorphism from $L^2(G)$ to $L^2(\widehat{G})$.

A continuous homomorphism $\gamma : G \to \mathbb{T}$ is a *character* of G. The *dual group* of G, $\widehat{G}$, is the set of all its characters.
$\widehat{\mathbb{R}} \cong \mathbb{R}$ via $x \mapsto \gamma_x$, where $\gamma_x(y) := e^{ixy}$.
$\widehat{\mathbb{T}} \cong \mathbb{Z}$ via $m \mapsto \gamma_m$, where $\gamma_m(\theta) := \theta^m$.
$\widehat{\mathbb{Z}} \cong \mathbb{T}$ via $\theta \mapsto \gamma_\theta$, where $\gamma_\theta(m) := \theta^m$.
$\widehat{\mathbb{Z}}_k \cong \mathbb{Z}_k$ via $m \mapsto \gamma_m$, where $\gamma_m(n) := \exp\left(2\pi i \frac{mn}{k}\right)$.
For LCAGs $(G_k)_{k=1}^n$, $\widehat{\bigotimes_{k=1}^n G_k} \cong \bigotimes_{k=1}^n \widehat{G}_k$ holds.
The map $\Phi : G \to \widehat{\widehat{G}}$, $(\Phi(x))(\gamma) := \gamma(x)$ is an isomorphism of topological groups.

For $f, g \in L^1(G)$

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)\,\mathrm{d}\mu(x) \quad \text{a.e. in } G.$$

For $f, g \in L^1(G))$ we have $\|f * g\|_1 \leqslant \|f\|_1\|g\|_1$. For $f, g \in L^2(G)$ we have $\widehat{fg} = \hat{f} * \hat{g}$.

Let $f \in L^1(G)$ such that $L^1(\widehat{G})$. Then $f(x) = \hat{\hat{f}}(x^{-1})$ holds for almost all $x \in G$, i.e.

$$f(x) = \int_{\widehat{G}} \hat{f}(\gamma(\gamma(x)\,\mathrm{d}\mu(\gamma) \quad \text{a.e. in } G,$$

where μ is the appropriately normalised left-invariant HAAR invariant HAAR measure on $\widehat{G}$.

For $T \in \mathcal{S}'(\mathbb{R}^n)$, $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ we define

$$(D^\alpha T)(f) := (-1)^{|\alpha|} T(D^\alpha f), \qquad (\mathcal{F} T)(f) := T(\mathcal{F} f),$$
$$(\mathcal{F}^{-1} T)(f) := T(\mathcal{F}^{-1} f), \qquad (fT)(g) := T(fg).$$

Let $K \subset \mathbb{C}$ be compact and $(\mathcal{B}(K), \|\cdot\|_\infty)$ the BANACH space of bounded BOREL-measurable functions on K and $\mathcal{C}(K) \subset U \subset \mathcal{B}(K)$ a set of functions with the following property: for all $(f_n)_{n\in\mathbb{N}} \subset$ with $f(t) := \lim_{n\to\infty} f_n(t)$ existing everywhere and $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ implies that $f \in U$. Then $U = \mathcal{B}(K)$.

Let $y \in Y$ and (P) be given as $Tx = y$. Then $x \in X$ is a

- *least-square-solution* of (P) if $x = \arg\min_{z\in X} \|Tz - y\|$.
- *minimal norm solution* of (P) if x is a least-square solution $x = \arg\min_{z\in X} \|z\|$.

1. Let $T \in \mathcal{S}'(\mathbb{R}^n)$. $\mathcal{F}^{-1} \mathcal{F} T = \mathcal{F} \mathcal{F}^{-1} T = T$ holds. $\mathcal{F}^{(-1)}$ map $\mathcal{S}'(\mathbb{R}^n)$ bijectively and continuously onto itself.
2. $\mathcal{F}(D^\alpha T) = i^{|\alpha|} x^\alpha \mathcal{F} T$ and $\mathcal{F}(x^\alpha T) = i^{|\alpha|} D^\alpha \mathcal{F} T$ holds.
3. For $\varepsilon > 0$ let $T_\varepsilon(f) := T(\varepsilon^{-n} f(\varepsilon^{-1}\cdot))$ for $f \in \mathcal{S}(\mathbb{R}^n)$ be the *dilation of* T. Then $\mathcal{F} T_\varepsilon = \varepsilon^{-n} \mathcal{F}(T)(\varepsilon^{-1}\cdot)$ holds.
4. For $h \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$, the *translation of* T is $(\tau_h T)(f) := T(f(\cdot + h))$ and $\mathcal{F}(\tau_h T) = e^{-i\langle h,\cdot\rangle} \mathcal{F} T$ holds.
5. The *modulation of* T is $(M_h T)(f) := T(e^{i\langle h,\cdot\rangle} f)$ and $\mathcal{F}(M_h T) = \tau_h(\mathcal{F} T)$ holds.

For $y \in \mathcal{D}(T^+)$, $x \in X$ is a least-square solution of $Tx = y$ if and only if $x \in X$ satisfies the *normal equation* $T^*Tx = T^*y$.
If in addition $x \in \mathcal{N}(T)^\perp$, we have $x = x^+$.
By the proof of the previous theorem, (1) is equivalent to $Tx = P_{\overline{\mathcal{R}(T)}} y$. By the properties of orthogonal projections this is equivalent to $Tx \in \overline{\mathcal{R}(T)}$ and $Tx - y \in \overline{\mathcal{R}(T)}^\perp = \mathcal{N}(T^*)$, i.e. $T^*(Tx - y) = 0$.

Set $\tilde{T} := T|_{\mathcal{N}(T)^\perp} : \mathcal{N}(T)^\perp \to \mathcal{R}(T)$. The MOORE-PENROSE Pseudoinverse T^+ is the unique linear extension of $\tilde{T}^{-1}$ with $\mathcal{D}(T^+) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ and $\mathcal{N}(T^+) = \mathcal{R}(T)^\perp$.
T^+ satisfies $\mathcal{R}(T^+) = \mathcal{N}(T)^\perp$ and we have (these four equations characterise T^+ uniquely.)

1. $TT^+T = T$
2. $T^+TT^+ = T^+$
3. $T^+T = \mathrm{id} - P_{\mathcal{N}(T)} = P_{\mathcal{N}(T)^\perp}$
4. $TT^+ = (P_{\overline{\mathcal{R}(T)}})|_{\mathcal{D}(T^+)}$

hold.

THEOREM

$T^+ \in L(\mathcal{D}(T^+), X)$ implies that $\mathcal{R}(T)$ is closed.

FA II

THEOREM

If $\mathcal{R}(T)$ is closed, $T^+ \in L(\mathcal{D}(T^+), X)$

FA II

KOROLLAR

Let $K \in \mathcal{K}(X, Y)$ with $\dim(\mathcal{R}(K)) = \infty$. Then K^+ is not continuous.

FA II

DEFINITION

SVD of a compact operator

FA II

THEOREM

For $K \in \mathcal{K}(X, Y)$ there exists a singular value decomposition.

FA II

THEOREM

PICARD condition

FA II

DEFINITION

Types of ill-conditionedness

FA II

CONTEXT

Functional calculus with SVD

FA II

DEFINITION

Regularisation of T^+

FA II

THEOREM

If $y \in \mathcal{D}(T^+)$, then $Tx = y$ has a unique minimal norm solution ...

FA II

First we show that T^+ is closed. Let $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T^+)$ converge to $y \in Y$ with $T^+ y_n \to x \in X$. By the fourth MOORE-PENROSE formula $TT^+ y_n = P_{\overline{\mathcal{R}(T)}} y_n \to P_{\overline{\mathcal{R}(T)}} y$ holds by the continuity of orthogonal projections. Since T is continuous, $P_{\overline{\mathcal{R}(T)}} y = \lim_{n\to\infty} P_{\overline{\mathcal{R}(T)}} y_n = \lim_{n\to\infty} TT^+ y_n = Tx$, implying that x is a least-square solution to $Tx = y$. As $T^+ y_n \in \mathcal{R}(T^+) = \mathcal{N}(T)^\perp$, which is closed, holds for all $n \in \mathbb{N}$ we have that $T^+ y_n \to x \in \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}$. thus x is a minimal norm solution to $Tx = y$, so T^+ is closed. The closed graph theorem finishes the proof.

As $\mathcal{D}(T^+) \subset Y$ is dense, T^+ can be uniquely and continuously extended to Y by $\overline{T^+} \in L(Y, X)$ defined by $\overline{T^+ y} := \lim_{n\to\infty} T^+ y_n$ for some sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T^+)$ converging to $y \in Y$. Let $(y_n)_{n\in\mathbb{N}} \subset \mathcal{R}(T)$ be a sequence converging to $y \in \overline{\mathcal{R}(T)}$. By the fourth MOORE-PENROSE equation and the continuity of T, $y = P_{\overline{\mathcal{R}(T)}} y = \lim_{n\to\infty} P_{\overline{\mathcal{R}(T)}} y_n = \lim_{n\to\infty} TT^+ y_n = TT^+ y \in \mathcal{R}(T)$, hence $\mathcal{R}(T) = \overline{\mathcal{R}(T)}$.

A sequence $((\sigma_n, u_n, v_n))_{n\in\mathbb{N}}$ is the singular value decomposition of K if $(\sigma_n)_n \subset \mathbb{R}^+$ is a decreasing sequence converging to 0, $(u_n)_{n\in\mathbb{N}} \subset Y$ an ONB of $\overline{\mathcal{R}(K)}$ and $(v_n)_{n\in\mathbb{N}} \subset X$ an ONB of $\overline{\mathcal{R}(K^*)}$ such that

1. $Kv_n = \sigma_n v_n$ and $K^* u_n = \sigma_n v_n$ holds for all $n \in \mathbb{N}$
2. $Kx = \sum_{n\in\mathbb{N}} \sigma_n \langle x, v_n \rangle u_n$ holds for all $x \in X$.

Towards contradiction assume that K^+ is continuous. Now $\mathcal{R}(K)$ is closed. Let $\tilde{K} := K|_{\mathcal{N}(K)^\perp} : \mathcal{N}(K)^\perp \to \mathcal{R}(K)$, which is bijective. Then $\tilde{K}^{-1} \in L(\mathcal{R}(K), \mathcal{N}(K)^\perp)$ holds by the inverse mapping theorem.
As K is compact, so is $K \circ \tilde{K}^{-1}$, which is the identity on $\mathcal{R}(K)$. In FA I this way shown to imply $\dim(\mathcal{R}(K)) < \infty$, a contradiction.

Let $((\sigma_n, u_n, v_n))_{n\in\mathbb{N}}$ be a singular system for K and $y \in \overline{\mathcal{R}(K)}$. Then $y \in \mathcal{R}(K)$ holds if and only if the PICARD condition

$$\sum_{n\in\mathbb{N}} \sigma_n^{-2} |\langle y, u_n \rangle|^2 < \infty$$

is satisfied. In this case we have

$$K^+ y = \sum_{n\in\mathbb{N}} \sigma_n^{-1} \langle y, u_n \rangle v_n.$$

As $K^*K \in \mathcal{K}(X, X)$ is SA, $\exists (\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}\backslash\{0\} \to 0$, decreasing in $|\cdot|$, ONS $(v_n)_{n\in\mathbb{N}} \subset X$: $K^*Kx = \sum_{n\in\mathbb{N}} \lambda_n \langle x, v_n \rangle v_n$. Thus $\lambda_n = \lambda \|v_n\|^2 = \langle \lambda_n v_n, v_n \rangle = \langle K^*Kv_n, v_n \rangle = \langle Kv_n, Kv_n \rangle = \|Kv_n\|^2 > 0$. Set $\sigma_n := \sqrt{\lambda_n} > 0$, $u_n := \sigma_n^{-1} K v_n \in Y$. Then $(u_n)_{n\in\mathbb{N}} \subset Y$ is ONS: $\langle u_i, u_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle Kv_i, Kv_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle K^*Kv_i, v_j \rangle = \frac{\lambda_i}{\sigma_i \sigma_j} \langle v_i, v_j \rangle = \delta_{i,j}$. Thus$K^* u_n = \sigma_n^{-1} K^*K v_n = \sigma_n^{-1} \lambda_n v_n = \sigma_n v_n$ By spectral theorem, $(v_n)_{n\in\mathbb{N}}$ is ONB for $\overline{\mathcal{R}(K^*K)} = \overline{\mathcal{R}(K)}$ Hence $(v_n)_{n\in\mathbb{N}}$ extendable to ONB V for X, as the rest must be in $\mathcal{N}(K) = \overline{\mathcal{R}(K^*)}^\perp$. Thus $Kx = \sum_{v\in V} \langle x, v \rangle Kv = \sum_{n\in\mathbb{N}} \langle x, v_n \rangle Kv_n = \sum_{n\in\mathbb{N}} \langle x, v_n \rangle \sigma_n u_n = \sum_{n\in\mathbb{N}} \langle x, K^* u_n \rangle u_n = \sum_{n\in\mathbb{N}} \langle Kx, u_n \rangle u_n$ thus $(u_n)_{n\in\mathbb{N}}$ is ONB for $\overline{\mathcal{R}(K)}$.

SVD allows us to define functions of compact operators: Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous (locally bounded?) function. For $K \in \mathcal{K}(X, Y)$ with singular system $((\sigma_n, u_n, v_n))_{n\in\mathbb{N}}$ and $x \in X$ define $f(K^*K) : X \to X$, $x \mapsto \sum_{n\in\mathbb{N}} f(\sigma_n^2) \langle x, v_n \rangle v_n + f(0) P_{\mathcal{N}(K)} x$ This series converges in X, as f is evaluated on the compact interval $[0, \sigma_1^2] = [0, \|K\|^2]$. We have $f(K^*K) \in L(X)$: $\|f(K^*K)\| = \sup_{n\in\mathbb{N}} |f(\sigma_n^2)| \leqslant \sup_{\lambda\in[0,\sigma_1^2]} |f(\lambda)| < \infty$. Let $f = \sqrt{\cdot}$. The *absolute value of* K is $|K| := f(K^*K) = \sum_{n\in\mathbb{N}} \sigma_n \langle \cdot, v_n \rangle v_n$.

1. $Tx = y$ is *moderately ill-conditioned* if the decay of the singular values is at most polynomial, i.e there exist $c, r > 0$ such that $\sigma_n \geqslant cn^{-r}$ for all $n \in \mathbb{N}$.
2. If 1. is not the case, $Tx = y$ is *strongly ill-conditioned.*
3. $Tx = y$ is called *exponentially ill-conditioned* if there exists $c, r > 0$ such that $\sigma_n \leqslant ce^{-nr}$ $\forall n \in \mathbb{N}$.

$\ldots x^+ = T^+ y$. The set of all least squares solutions is given by $x^+ + \mathcal{N}(T)$.

A family $(R_a)_{a>0} \subset L(Y, X)$ is called *regularisation of* T^+ if $R_a y \xrightarrow{a\to 0} T^+ y$ holds for all $y \in \mathcal{D}(T^+)$.

Theorem

E compactly supported spectral measure, $T = \int \lambda \, \mathrm{d}E_\lambda \in L(\mathcal{H})$ is self-adjoint. Then $\Psi : \mathcal{B}(\sigma(T)) \to L(\mathcal{H})$, $f \mapsto \int_{\sigma(T)} f \, \mathrm{d}E$ is the BM FC, especially $E_{\sigma(T)} = \mathrm{id}$

spectral Theorem for bounded SA operators

$T = T^* \in L(\mathcal{H})$. $\exists! E$ compactly supported SM: $T = \int_{\sigma(T)} \lambda \, \mathrm{d}E_\lambda . \Psi : \mathcal{B}(\sigma(T)) \to L(\mathcal{H})$, $f \mapsto f(T) = \int f(\lambda) \, \mathrm{d}E_\lambda$ coincide with BM FC, $\langle f(T)x, y \rangle = \int_{\sigma(T)} f(\lambda) \, \mathrm{d}\langle E_\lambda x, y \rangle$, where $\langle E_\lambda x, y \rangle$ is the complex-valued measure $A \mapsto \langle E_A x, y \rangle$.

The double adjoint

Let T dense.

- T^* closed. T^* dense $\implies T \subset T^{**}$ and $\overline{T} = T^{**}$.
- T symmetric $\iff T \subset T^*$. Then $T \subset T^{**} \subset T^* = T^{***}$, T^{**} symmetric.
- T closed, symmetric $\iff T = T^{**} \subset T^*$.
- T self-adjoint $\iff T = T^* = T^{**}$.