Definition & Examples

Weak n-th Derivative (1D)

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Theorem + Proof Idea

Fundamental Theorem of the Calculus of Variations (1D)

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Theorem statement

Properties of the mollifier

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Proof

Fundamental Theorem of the Calculus of Variations (1D)

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Statement of the Theorem

$W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$

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Definition

Embeddings

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Definition

$W_0^{1,p}$

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Statement of the Theorem

Poincaré-Friedrichs-Inequality (onedimensional)

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Proof of the Theorem

Poincaré-Friedrichs-Inequality (onedimensional)

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Statement of the Theorem

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Let $u \in L^1_{\text{loc}}(I)$ be a function such that $\int_a^b u(x)\varphi(x) = 0$ for all $\varphi \in \mathcal{C}_0^\infty(I)$. Then, $u|_I \equiv 0$ almost everywhere.

If $\varphi(x) := \operatorname{sign}(u(x))$ were in $\mathcal{C}_0^\infty(I)$, we could test with it:

$$0 = \int_a^b u(x)\varphi(x)\,\mathrm{d}x = \int_a^b |u(x)|\,\mathrm{d}x = \|u\|_1 \implies u \equiv 0 \text{ a.e.}$$

We modify can φ so that it is compactly supported: consider $\psi := \varphi \cdot \mathbb{1}_{[c,d]}$ for $a < c < d < b$.

Let $u, v \in L^1_{\text{loc}}(I)$ and $n \in \mathbb{N}$. If the equation

$$\int_a^b u(x)\varphi^{(n)}\,\mathrm{d}x = (-1)^n \int_a^b v(x)\varphi(x)\,\mathrm{d}x$$

holds for all $\varphi \in \mathcal{C}_0^\infty(I)$, we call u n times *weakly differentiable* with the *weak n-th derivative v*.
The weak derivative of the absolute value is the sign function.
The HEAVISIDE function is not weakly differentiable.
$x \mapsto x^2 \sin\left(\frac{1}{x}\right)$ is continuous but not weakly differentiable.

Let $[c,d] \subset (a,b)$, $w := \operatorname{sign}(u)\,\mathbb{1}_{[c,d]} \in L^1_{\text{loc}}(a,b)$, $\operatorname{supp}(w) \subset [c,d]$. Let $w_\varepsilon := \mathcal{J}_\varepsilon * w$. Then $w_\varepsilon(x) \to w(x)$ and $\operatorname{supp}(w_\varepsilon) \subset [c-\varepsilon, d+\varepsilon]$, hence $w_\varepsilon \in \mathcal{C}_0^\infty(a,b)$ for small enough ε. With $\varphi = w_\varepsilon \in \mathcal{C}_0^\infty(a,b)$, obtaining

$$0 = \int_a^b \underbrace{u(x)w_\varepsilon(x)}_{\xrightarrow{\text{a.e.}} u(x)w(x)}\,\mathrm{d}x = \int_{c-\varepsilon}^{d+\varepsilon} u(x)w_\varepsilon(x)\,\mathrm{d}x.$$

For $\varepsilon_0 < \min(c-a, b-d)$ and all $\varepsilon < \varepsilon_0$ we get

$$|u(x)w_\varepsilon(x)| \leqslant |u(x)|\,\mathbb{1}_{[c-\varepsilon_0, d+\varepsilon_0]}(x),$$

which is an integrable majorant. LEBESGUE's Theorem shows

$$0 = \int_a^b u(x)w(x)\,\mathrm{d}x = \int_c^d |u(x)|\,\mathrm{d}x.$$

Let $u \in L^p(I;\mathbb{R})$ and $p \in [1,\infty)$. Then $u_\varepsilon := u * \mathcal{J}_\varepsilon$ is well defined and

1. $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ and $u_\varepsilon^{(k)}(x) = \int_{\mathbb{R}} \mathcal{J}_\varepsilon^{(k)}(x-y)u(y)\,\mathrm{d}y\ \forall k \in \mathbb{N}$.
2. If $\operatorname{supp}(u) \subset I$ and $\varepsilon < \operatorname{dist}(\operatorname{supp}(u), \delta I)$, then $\operatorname{supp}(u_\varepsilon) \subset I$ and therefore, $u_\varepsilon \in \mathcal{C}_0^\infty(I)$.
3. $\|u - u_\varepsilon\|_p \xrightarrow{\varepsilon \searrow 0} 0$.
4. $\|u_\varepsilon\|_{L^p(\mathbb{R})} \leqslant \|u\|_p$ (also holds for $p = \infty$).
5. $u_\varepsilon \xrightarrow{\varepsilon \searrow 0} u$ almost everywhere on I.
6. $\|u_\varepsilon - u\|_{\mathcal{C}(K)} \xrightarrow{\varepsilon \searrow 0} 0$ for compact subsets $K \subset I$ if $u \in \mathcal{C}(I)$.

1. X is *embedded* into Y if and only if there exists an a *injective linear* function $\iota\colon X \to Y$ and X can identified with a subspace of Y.
2. X is *continuously / compactly embedded* into Y and we write $X \hookrightarrow Y$ / $X \overset{c}{\hookrightarrow} Y$ if ι is continuous / compact. Then $\exists c > 0$ such that $\|\iota(x)\|_Y \leqslant c\|x\|_X$ for all $x \in X$ / then a bounded sequence in X has a convergent subsequence with respect to $\|\cdot\|_Y$.
3. X is *densely embedded* into Y and we write $X \overset{d}{\hookrightarrow} Y$ if $\iota(X)$ is dense in Y with respect to $\|\cdot\|_Y$.

Let $u \in W^{1,1}((a,b);\mathbb{R})$. Then u coincides almost everywhere with a function, which is *absolutely continuous* on (a,b) and which can then be extended (as absolutely continuous functions are LIPSCHITZ continuous) to an absolutely continuous function on $[a,b]$ ("u is absolutely continuous"). (Alternatively: there exists an absolutely continuous function on $[a,b]$ whose restriction to (a,b) is in the equivalence class of u.) We have

$$\|u\|_{\mathcal{C}([a,b];\mathbb{R})} \leqslant \frac{\max(1, b-a)}{b-a}\|u\|_{1,1}$$

For $u \in W_0^{1,p}((a,b);\mathbb{R})$ we have

$$\|u\|_{0,p} \leqslant (b-a)|u|_{1,p}$$

This is not true for $W^{1,p}(a,b)$, but on $\left\{u \in H^1((a,b);\mathbb{R}) : \int_\Omega u(x)\,\mathrm{d}x = 0\right\}$ Thus on $W_0^{1,p}$, the norms $\|\cdot\|_{1,p}$ and $|\cdot|_{1,p}$ are *equivalent* and $(W_0^{1,p}((a,b);\mathbb{R}), |\cdot|_{1,p})$ is a closed and therefore *complete* subspace of $W^{1,p}((a,b);\mathbb{R})$. For $p = 2$ we even have $\|u\|_{0,2} \leqslant \frac{b-a}{\sqrt{2}}|u|_{1,2}$, as $\int_a^b |x-a|^{\frac{2}{2}}\,\mathrm{d}x = \frac{1}{2}(b-a)^2$ and we can even instead have $\frac{b-a}{\pi}$.

$$W_0^{1,p}((a,b);\mathbb{R}) := \overline{\mathcal{C}_0^\infty(a,b)}^{\|\cdot\|_{1,p}} \subset W^{1,p}((a,b);\mathbb{R})$$

is a closed subspace. We have

$$W_0^{1,p}((a,b);\mathbb{R}) = \left\{u \in W^{1,p}((a,b);\mathbb{R}) : u(a) = u(b) = 0\right\}.$$

As $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$, this makes sense. This not true in $\mathbb{R}^d$ for $d > 1$.

Let $(V, (\cdot,\cdot), \|\cdot\|)$ be a (real) HILBERT space and $A\colon V \to V^*$ LIPSCHITZ continuous and strongly monotone. Then A is bijective.

We have $u(a) = 0$ and by the Integral Mean Value Theorem

$$\begin{aligned}
\|u\|_p^p &= \int_a^b |u(x)|^p\,\mathrm{d}x \leqslant \int_a^b \left(\int_a^x 1 \cdot |u'(y)|\,\mathrm{d}y\right)^p \mathrm{d}x \\
&\overset{(\mathrm{H})}{\leqslant} \int_a^b \left(\left(\int_a^x 1^q\,\mathrm{d}y\right)^{\frac{1}{q}} \left(\int_a^x |u'(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\right)^p \mathrm{d}x \\
&= \int_a^b \underbrace{\int_a^x |u'(y)|^p\,\mathrm{d}y}_{\leqslant \|u'\|_p^p} |x-a|^{\frac{p}{q}}\,\mathrm{d}x \leqslant |b-a|^{1+\frac{p}{q}}|u|_{1,p}^p.
\end{aligned}$$

and $\left(|b-a|^{1+\frac{p}{q}}\right)^{\frac{1}{p}} = |b-a|^{\frac{1}{p}+\frac{1}{q}} = b-a$.

Proof of the Theorem

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Statement of the Theorem

Lax-Milgram

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Proof of the Theorem

Lax-Milgram

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Definition

Properties of a bilinear form

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Definition

Properties of (non)linear operator

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Theorem & Proof

Corollary of the Theorem of Lax-Milgram

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Statement of the Theorem

Lemma of Céa

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Proof of the Theorem

Lemma of Céa

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Definition

Galerkin scheme and Galerkin basis

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Statement of Theorem

Convergence of linear FEM: Interpolation error

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Let $(V, (\cdot,\cdot), \|\cdot\|)$ be a (real) HILBERT space and $A\colon V \to V^*$ a *linear, strongly positive, bounded* operator. Then A is bijective.

Let $\iota\colon V^* \to V$ be the RIESZ *isomorphism* and define $\Phi\colon V \to V,\ v \mapsto v + \tau\iota(f - Av)$. where $\tau > 0$ is chosen such that $1 - 2\tau\mu + \tau^2\beta^2 < 1$. For $u, v \in V$ we have

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|^2 &= \|u - v + \tau\iota(Av - Au)\|^2\\ &= \|u - v\|^2 + 2\tau\,(\iota(Av - Au), u - v)\\ &\quad + \tau^2\|\iota(Au - Av)\|^2\\ &= \|u - v\|^2 - 2\tau\langle Au - Av, u - v\rangle + \tau^2\|Au - Av\|_*^2\\ &\leqslant \underbrace{(1 - 2\tau\mu + \tau^2\beta^2)}_{<1}\|u - v\|^2.\end{aligned}$$

Let $a\colon V \times V \to \mathbb{R}$ be a *bilinear* form. We call a

- *symmetric* if $a(u,v) = a(v,u)$ holds for all $u, v \in V$.
- *strongly positive* $\exists\mu > 0$ such that $a(u,u) \geqslant \mu\|u\|^2$ for all $u \in V$.
- *positive* if $a(u,u) \geqslant 0$ for all $u \in V$.
- *bounded* if $\exists\beta > 0$ such that $a(u,v) \leqslant \beta\|u\|\|v\|$ holds for all $u, v \in V$.

Let $\iota\colon V^* \to V$ be the isometric isomorphism (RIESZ *map*), such that $\langle f, v\rangle = (\iota(f), v)$ and $\|f\|_* = \|\iota(f)\|$ for all $f \in V^*$ and all $v \in V$. Fix $f \in V^*$, choose $\tau > 0$ such that $1 - 2\mu\tau + \tau^2\beta^2 < 1$ and define $\Phi\colon V \to V$, $v \mapsto v + \tau\iota(f - Av)$. Then $f = Au$ if and only if $\Phi(u) = u$. To use the BANACH fixed point theorem it remains to show that Φ is a contraction: for $u, v \in V$ we have

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|^2 &= \|u - v + \tau\iota(f - Au - f + Av)\|^2\\ &= \|u - v\|^2 + 2\tau(u - v, \iota(A(v - u))) + \tau^2\|\iota(A(u - v))\|^2\\ &= \|u - v\|^2 - 2\tau(\iota(A(u - v)), u - v) + \tau^2\|\iota(A(u - v))\|^2\\ &= \|u - v\|^2 - 2\tau\langle A(u - v), u - v\rangle + \tau^2\|A(u - v)\|_*^2\\ &\leqslant (1 - 2\mu\tau + \tau^2\beta^2)\|u - v\|^2.\end{aligned}$$

Under the above conditions the bijectivity of A implies the existence of a unique solution $u \in V$ to the problem $Au = f$ for all $f \in V^*$ as well as the existence of the *solution operator* $A^{-1}\colon V^* \to V$, which is *linear*, *bounded* and *strongly positive* (hence we have continuous dependence on the right side f).
By the Theorem of LAX-MILGRAM A is bijective, implying the existence of the linear A^{-1}. Its boundedness i.e follows from the inverse mapping theorem but can be show with much more elementary means: For all $f \in V^*$ we have

$$\mu\|A^{-1}(f)\|_V^2 \leqslant \langle AA^{-1}f, A^{-1}f\rangle = \langle f, A^{-1}f\rangle \leqslant \|f\|_{V^*}\|A^{-1}f\|_V.$$

Finally, the strict positivity follows from

$$\|f\|_{V^*}^2 = \|AA^{-1}f\|_{V^*}^2 \leqslant \beta^2\|A^{-1}f\|_V^2 \leqslant \frac{\beta^2}{\mu}\langle AA^{-1}f, A^{-1}f\rangle = \frac{\beta^2}{\mu}\langle f, A^{-1}f\rangle.$$

Let $(V, \|\cdot\|)$ be a real BANACH space. We call a *linear* operator $A\colon V \to V^*$

- *symmetric* if $\langle Au, v\rangle = \langle Av, u\rangle$ holds for all $u, v \in V$.
- *strongly positive* if $\exists\mu > 0$ such that $\langle Au, u\rangle \geqslant \mu\|u\|^2$ for all $u \in V$.
- *positive* if $\langle Au, u\rangle \geqslant 0$ for all $u \in V$.
- *bounded* if it maps bounded sets to bounded sets. Since A is linear, this is equivalent to requiring that $\exists\beta > 0$ such that $\|Au\|_* \leqslant \beta\|u\|$ holds for all $u \in V$.
- LIPSCHITZ *continuous* if $\exists\beta > 0$ such that $\|Au - Av\|_* \leqslant \beta\|u - v\|\ \forall u, v \in V$.
- *monotone* if $\langle Au - Av, u - v\rangle \geqslant 0$ for all $u, v \in V$.
- *strongly monotone* if $\exists\mu > 0$ if $\langle Au - Av, u - v\rangle \geqslant \mu\|u - v\|^2\ \forall u, v \in V$.

By the Theorem of LAX-MILGRAM both problems have unique solutions $u \in V$ and $u_h \in V_h$, respectively. For any $v_h \in V_h$ we have

$$\alpha(u_h, v_h) = \langle f, v_h\rangle = \alpha(u, v_h).$$

Hence $\alpha(u - u_h, v_h) = 0$ for all $v_h \in V_h$, i.e $u - u_h \perp V_h$ with respect to the inner product $\alpha(\cdot,\cdot)$ (s. diagram). Hence for all $v_h \in V_h$ we have

$$\begin{aligned}\mu\|u - u_h\|^2 \leqslant \alpha(u - u_h, u - u_h) &= \alpha(u - u_h, u) - \alpha(u - u_h, u_h)\\ &= \alpha(u - u_h, u) - 0 = \alpha(u - u_h, u) - \alpha(u - u_h, v_h)\\ &= \alpha(u - u_h, u - v_h) \leqslant \beta\|u - u_h\|\|u - v_h\|.\end{aligned}$$

Let V be a real HILBERT space and V_h a *closed* subspace (e.g. a finite dimensional subspace). Let $\alpha\colon V \times V \to \mathbb{R}$ be *bilinear*, *strongly positive* and *bounded*. Then, $\alpha\colon V_h \times V_h \to \mathbb{R}$ is, too. Let $f \in V^*$ and $u \in V$ be the solution of $\alpha(u,v) = \langle f, v\rangle\ \forall v \in V$. Then there exists a solution $u_h \in V_h$ of $\alpha(u_h, v_h) = \langle f, v_h\rangle\ \forall v_h \in V_h$. Then we have

$$\|u - u_h\| \leqslant \frac{\beta}{\mu}\inf_{u_h \in V_h}\|u - u_h\| = \frac{\beta}{\mu}\operatorname{dist}(u, V_h),$$

The sequence of *(linear) FEM spaces* $(V_h)_{h\in(0,1)}$ with an *equidistant grid* is a GALERKIN scheme in V. For each $m \in \mathbb{N}$, $h = \frac{b-a}{m+1}$ and $v \in H_0^1 \cap H^2(a,b)$ we have

$$\|v - I_h v\|_{1,2} \leqslant ch\|v\|_{2,2} \qquad \text{(linear convergence rate)}$$
$$\|v - I_h v\|_{0,2} \leqslant ch^2\|v\|_{2,2} \qquad \text{(quadratic convergence rate)}$$

The family $(V_n \subset V)_{n\in\mathbb{N}}$ of finite-dimensional subspaces is a GALERKIN *scheme* if it is complete in the limit, that is, the approximation error vanishes:

$$\lim_{n\to\infty}\operatorname{dist}(V_n, v) = 0 \quad \forall v \in V.$$

A pairwisely linearly independent sequence $(\Phi_k)_{k\in\mathbb{N}} \subset V$ is called GALERKIN *basis* if $\overline{\bigcup_{n\in\mathbb{N}} V_n} = V$, where $V_n := \operatorname{span}((\Phi_k)_{k=1}^n)$.

Statement of the Theorem

Sobolev Embedding Theorem

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Definition

Sobolev space

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Statement of the Theorem

Meyers-Serrin (”H = W”)

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Definition & Remarks

Dual space of $W_0^{1,p}(a,b)$

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Statement of the Theorem

Rellich

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Definition and Lemma

Hölder-Continuity and Embeddings

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Proof of the Theorem

$W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$

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Proof of the Theorem

$W^{1,p}((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}([a,b])$

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Proof

$W^{k,p}((a,b);\mathbb{R})$ is separable for $p \in [1,\infty)$

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Definition & Remark

Lipschitz domain

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$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall |\alpha| \leqslant k\}$$
$$\|u\|_{k,p}^p := \sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{0,p}^p.$$

The space $W^{k,p}(\Omega)$ is a Banach space, which is separable for $p \in [1,\infty)$ and reflexive for $p \in (1,\infty)$. We define $H^k := W^{k,2}$, which is a Hilbert space with the *inner product*

$$(u,v)_{k,2} := \sum_{|\alpha|\leqslant k} (D^\alpha u, D^\alpha v)_{0,2}.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If kp

1. $< d$ and $\ell \leqslant k$ we have $W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega)$ if $\frac{1}{q} \geqslant \frac{1}{p} - \frac{k-\ell}{d}$.
2. $> d$, then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^{\beta,\alpha}(\overline{\Omega})$, where $\beta := k - \left\lfloor \frac{d}{p} \right\rfloor - 1$ and $\alpha \in (0,1)$ if $\frac{d}{p} \in \mathbb{N}$ and $\alpha \in \left(0, \left\lfloor \frac{d}{p} \right\rfloor + 1 - \frac{d}{p}\right]$ else.
3. $= d$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1,\infty)$.

Also holds for *fractional* Sobolev *spaces*. Rellich: Those embeddings are *compact* if we are not in the boundary case.

We set $W^{-1,q}(a,b) := \left(W_0^{1,p}(a,b)\right)^*$, where p and q are Hölder conjugates. It is equipped with the norm

$$\|f\|_{-1,q} := \sup_{\substack{u\in W_0^{1,p}\\ u\neq 0}} \frac{\langle f, u\rangle}{|u|_{1,p}}$$

$W^{-1,q}(a,b) \supsetneq \left(W^{1,p}(a,b)\right)^*$. We have $L^q \hookrightarrow W^{-1,q}$.
For all $f \in W^{-1,q}$ there exists a not necessarily unique $u_f \in L^q(a,b)$ so that $\langle f, v\rangle_{W^{-1,q}\times W_0^{1,p}} = \int u_f v' \,\mathrm{d}x$, where $v \in W_0^{1,p}(a,b)$.

For any $\Omega \subset \mathbb{R}^d$ and $1 \leqslant p < \infty$ we have

$$W^{k,p}(\Omega) = \overline{\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)}^{\|\cdot\|_{k,p}}$$

For $\alpha \in (0,1)$, $u\colon [a,b] \to \mathbb{R}$ is α-Hölder continuous if

$$\exists c \geqslant 0 : |u(x) - u(y)| \leqslant c|x-y|^\alpha\ \forall x,y \in [a,b].$$

$$\mathcal{C}^{0,\alpha}([a,b]) := \left\{ v \in \mathcal{C}([a,b]) : |u|_\alpha := \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}$$

equipped with the norm $\|u\|_{\mathcal{C}^{0,\alpha}} := \|u\|_\infty + |u|_\alpha$ is complete.

- For $0 < \alpha < \beta < 1$ we have $\mathcal{C}^{0,\beta}([a,b]) \overset{c}{\hookrightarrow} \mathcal{C}^{0,\alpha}([a,b])$.
- We have $H^1((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}^{0,\alpha}([a,b])$ for $\alpha \in \left(0, \frac{1}{2}\right)$

$H^1((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} L^2([a,b];\mathbb{R})$.

$$\|u\|_{0,2}^2 \leqslant \|u\|_{0,2}^2 + \|u'\|_{0,2}^2 = \|u\|_{1,2}^2.$$

For compactness show prove requirements of Fréchet-Kolmogorov-Riesz Theorem.

For $p > 1$ we have $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$. Let $A \subset W^{1,p}((a,b);\mathbb{R})$ be bounded by $M > 0$. As $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b])$, there exists a $c > 0$ such that $\|u\|_\infty \leqslant c\|u\|_{1,p} \leqslant cM$ for all $u \in A$. We now show that A is equicontinuous. For $u \in A$ and $x_1, x_2 \in [a,b]$

$$\begin{aligned}
|u(x_1) - u(x_2)| &= \left| \int_{x_1}^{x_2} u'(t)\,\mathrm{d}t \right| \\
&\overset{\mathrm{H}}{\leqslant} \left(\int_{x_1\wedge x_2}^{x_1\vee x_2} |u'(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}} \left(\int_{x_1\wedge x_2}^{x_1\vee x_2} 1^q \,\mathrm{d}t \right)^{\frac{1}{q}} \\
&\leqslant \|u\|_{1,p} |x_1 - x_2|^{\frac{1}{q}} \leqslant M|x_1 - x_2|^{\frac{1}{q}}.
\end{aligned}$$

The *Theorem of* Arzelá-Ascoli yields the claim since the identity maps bounded set to relatively compact sets and therefore is compact.

Set $v(x) := \int_a^x u'(y)\,\mathrm{d}y$. As $u' \in L^1((a,b);\mathbb{R})$, v is absolutely continuous and $v' = u'$ almost everywhere on (a,b). Therefore, we obtain

$$\int_a^b u\varphi' \,\mathrm{d}x = -\int_a^b u'\varphi \,\mathrm{d}x = -\int_a^b v'\varphi \,\mathrm{d}x = \int_a^b v\varphi' \,\mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$ and hence $u \equiv v + c$ for some $c \in \mathbb{R}$ almost everywhere on (a,b), so u is almost everywhere equal to an absolutely continuous function, which we will call u, too.
By the *Integral Mean Value Theorem* there exists a $x_0 \in [a,b]$ so that $\int_a^b u(x)\,\mathrm{d}x = u(x_0)(b-a)$. This implies

$$|u(x)| \leqslant |u(x_0)| + \left| \int_{x_0}^x u'(x)\,\mathrm{d}x \right| \leqslant \frac{1}{b-a} \int_a^b |u(x)|\,\mathrm{d}x + \int_a^b |u'(x)|\,\mathrm{d}x.$$

This doesn't hold in higher dimensions, u must not even by continuous.

A domain $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and we write $\partial\Omega \in C^{0,1}$, if for every $x_0 \in \partial\Omega$ there exists a $r > 0$ and a Lipschitz continuous function $g\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that (up to a rotation of the coordinate system)

$$B(x_0,r) \cap \Omega = \{(x_1,\dots,x_d) \in B(x_0,r) : x_d > g(x_1,\dots,x_{d-1})\}$$

Then we also have $B(x_0,r) \cap \partial\Omega = \{x \in B(x_0,r) : x_d = g(x_1,\dots,x_{d-1})\}$. As Ω is bounded, $\partial\Omega$ is compact, and thus we only need finitely many g to "describe" the boundary.

Define $T\colon W^{1,p}(a,b) \to L^p(a,b)^2,\ u \mapsto (u,u')^\mathsf{T}$. Then, T is well defined. Further, we have

$$\|Tu\|_{L^p(a,b)^2} = \left(\|u\|_{L^p((a,b);\mathbb{R})}^p + \|u'\|_{L^p((a,b);\mathbb{R})}^p \right)^{\frac{1}{p}} = \|u\|_{1,p}.$$

Hence $W^{1,p}(a,b)$ *isometrically* coincides with a subspace $(L^p(a,b))^2$. This subspace is *closed* as $W^{1,p}(a,b)$ is complete. As $L^p(a,b)$ *is separable*, so is $(L^p(a,b))^2$ and hence the closed subspace, and hence $W^{1,p}(a,b)$.

Definition & Properties

Trace operator

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Statement of the Theorem

+2 Regularity on any bounded domain

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Statement of the Theorem

Existence for a nonlinear problem

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Explanation

FEM

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Proof

Let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain. Then we have

$$W^{1,p}(\Omega) = \{u \in L^1(\Omega) : \partial_j u \in L^p(\Omega)\ \forall j \in \{1, \dots, j\}\}.$$

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Lemma

Classical and weak derivatives

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Inhomgeneous Poisson equation with homogeneous boundary conditions

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Statement of the Theorem

Poincaré-Wirtingner inequality

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Corollary

To the Theorem of Gauss, "Partial Integration"

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Let $\Omega \subset \mathbb{R}^d$ be a *bounded* domain. We consider

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

Then if $f \in H^k(\Omega)$ for $k \in \mathbb{N}$, then the unique solution $u \in H_0^1(\Omega)$ satisfies $u \in H^{k+2}(\Omega')$ for any $\Omega' \subset\subset \Omega$ and there exists an $c_{\Omega'} > 0$ such that we have

$$\|u\|_{H^{k+2}(\Omega')} \leqslant c_{\Omega'} \left(\|u\|_{H_0^1(\Omega)} + \|f\|_{H^k(\Omega)} \right).$$

Let $\Omega \subset \mathbb{R}^d$ be a LIPSCHITZ-domain. Then

$$\operatorname{tr}\colon \mathcal{C}^\infty(\overline{\Omega}) \to L^p(\partial\Omega), \; u \mapsto u|_{\partial\Omega}$$

is the *trace operator* of u. (Makes sense because for LIPSCHITZ domains $\mathcal{C}^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega)$ is dense for $1 \leqslant p < \infty$.) The trace operator is *linear*, *bounded* and hence *uniquely extendable* to an operator $\operatorname{tr}\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$. It is neither injective nor surjective.

$$W_0^{1,p} := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{k,p}} = \{u \in W^{1,0}(\Omega) : \operatorname{tr}(u) = 0\} \subsetneq W^{k,p}(\Omega).$$

Consider the nonlinear problem

$$\begin{cases} -\nabla \cdot (\alpha(x, u(x)) \nabla u(x)) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain and $\alpha\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a CARATHÉODORY function. Furthermore there exists constants $m, M > 0$ such that $m \leqslant \alpha(x,y) \leqslant M$ holds for almost all $x \in \Omega$ and all $y \in \mathbb{R}$. The problem has at least one solution.

Let $u \in \mathcal{C}^1([a,b];\mathbb{R})$. Then the *weak derivative* of u *coincides* with its *classic derivative*.
Let u' be the weak derivative of u on (a,b). Then for all intervals $(\alpha,\beta) \subset (a,b)$ it holds that $u'|_{(\alpha,\beta)}$ is also the weak derivative of $u|_{(\alpha,\beta)}$ on (α,β).

"$\supset$": Let $u \in L^1(a,b)$ with $\partial_j u \in L^p(a,b) \hookrightarrow L^1(a,b)$ for all $j \in \{1,\dots,d\}$. Then $u \in W^{1,1}(\Omega)$.

- If $d = 1$, we have $kp = 1 = d$ and hence $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$..
- If $d > 1$, we have $kp = 1 < d$ and hence $W^{1,1}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, q_0]$, where $q_0 := \frac{d}{d-1} \in (1,2]$.
  - If $q_0 \geqslant p$, we are finished.
  - If $q_0 < p$, we have $u' \in L^p(\Omega) \hookrightarrow L^{q_0}(\Omega)$ and hence $u \in W^{1,q_0}(\Omega)$.
    - If $d = 2$, then $kp = q_0 = 2 = d$ and thus $u \in W^{1,q_0}(\Omega) \hookrightarrow L^p(\Omega)$
    - If $d \geqslant 3$, we have $q_0 < d$ and thus $W^{1,q_0}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \leqslant \frac{d}{d-2} =: q_1 \in (1,3]$.

  This can be inductively continued until $q_k \geqslant p$.

For $p \in [1,\infty]$ and a LIPSCHITZ domain $\Omega \subset \mathbb{R}^d$, there exists a $C = C(\Omega, p) \geqslant 0$ such that

$$\|u - u_\Omega\|_{0,p} \leqslant C \|\nabla u\|_{0,p},$$

for all $u \in W^{1,p}(\Omega)$, where $u_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x) \,\mathrm{d}x$.

For homogeneous DIRICHLET boundary conditions the problem is uniquely solvable by LAX-MILGRAM.
For the variational formulation set $V := H_0^1(\Omega)$.
By SOBOLEV we have $H_0^1(\Omega) \subset H^1(\Omega) \hookrightarrow L^q(\Omega)$ with $q \leqslant \frac{2d}{d-2}$ for $d \geqslant 3$ (for $d = 2$ we have $H^1 \hookrightarrow L^q$ for all $q \in [1,\infty)$). With HÖLDER we have $|\langle f, v \rangle| \leqslant \|f\|_p \|v\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, that is, $p = \frac{2d}{d+2} \in [1,2)$, so f induces $\tilde{f} \in H^{-1}(\Omega)$.

Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be a vector field (i.e. the gradient of $u\colon \Omega \to \mathbb{R}$) and $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ a scalar valued function and $\Omega \subset \mathbb{R}^d$ a LIPSCHITZ domain. Then

$$\int_\Omega \underbrace{(\nabla \cdot F)(x)}_{\operatorname{div}(F)} \varphi(x) \,\mathrm{d}x = -\int_\Omega F(x) \cdot \nabla\varphi(x) \,\mathrm{d}x + \underbrace{\int_{\partial\Omega} \varphi F \cdot \nu \,\mathrm{d}\sigma}_{\text{boundary term}},$$

where ν is the outer normal and $\cdot$ is the scalar product on $\mathbb{R}^d$.