Technische Universität Berlin

Lecture Notes

Differentialgleichungen III

read by Dr. Robert Lasarzik in the summer semester 2021

Contents

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Introduction

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This course differs from the previous courses as we consider differential equations which also incorporate time derivatives. In the first chapter we introduce the BOCHNER integral for functions from a time interval into a BANACH space. We can then, in the second chapter, explore time derivatives and the space $W(0,T)$, the correct space for considering linear first order evolution equations where we prove existence and uniqueness results. We show existence results for nonlinear first order evolution equations with numerical schemes. While we focused on the stationary NAVIER-STOKES equation in the course Differential Equations II B, we will now consider the instationary case including the famous Millennium problem. Lastly, we consider systems of nonlinear evolution equations. Depending on the different couplings of different PDEs this can mean a whole different world for the solvability of these systems.

There are different approaches to evolution equations. Insights into the linear semigroup approach, which we will not study here, can be found in [Paz12] or [Lun12], the idea being the generalisation of e^{-tA} from matrices A to general linear operators A. This works well for linear, but not so well for nonlinear PDEs. For the nonlinear case, one can consult [Bar76] or [Bré73], which only covers HILBERT spaces.

We will take the variational approach shown in [Emm13] and [RR06]. A great further resources is [Rou13]. The variational approach follows the modelling of PDEs and is not restricted to local existence results in the nonlinear case by using generalised solvability notions.

0 Introductory Examples and Preliminaries

0.1 Introductory examples

We first give an overview of the kinds of problems we will tackle in this course. Let $T > 0$.
Example. (Nonlinear heat equation) Consider

$$\begin{cases} C(\theta)\partial_t\theta - \nabla\cdot(\kappa(\theta)\nabla\theta) + g(\theta) = f, & \text{on } \Omega\times(0,T),\\ n\cdot\kappa(\theta)\nabla\theta + b_1\theta + b_2|\theta|^3\theta = 0, & \text{on } \partial\Omega\times(0,T),\\ \theta(0) = \theta_0 & \text{on } \Omega \end{cases}$$

where θ is the temperature, C is the heat capacity, κ is the heat conductivity, g represents heat sources or sinks (e.g. from a chemical reaction) and is a nonlinear function, f is an external heat source and b_1 and b_2 are constants of the convective and radiative heat conduction. The first line is the differential equation, the second is the boundary condition and the third line is the initial value condition. ◇

The previous problem is difficult to solve, but the following one is even harder.
Example. (Navier-Stokes equation) Consider

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p - \nu\Delta v = f, & \text{on } \Omega\times(0,T),\\ \nabla v = 0, & \text{on } \Omega\times(0,T),\\ v(0) = v_0, & \text{on } \Omega,\\ v = 0, & \text{on } \partial\Omega\times(0,T). \end{cases}$$

where v is the velocity field, p is the pressure and f is an external force, e.g gravity, and ν is the viscosity. This models flow in incompressible fluids. ◇

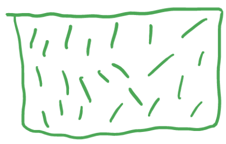

Fig. 1: Molecules in a container aligning along a magnetic field.

Example. (Complex fluids - Liquid Crystals) Liquid crystals can be found in displays of phones. They can be modelled by the Ericksen-Leslie equation

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p - \nu\Delta v + \nabla\cdot(\nabla d^{\mathsf{T}}\nabla d) = f & \text{on } \Omega\times(0,T),\\ \partial_t d + (v\cdot\nabla)d + (I - d\otimes d)(-\Delta d) = 0, & \text{on } \Omega\times(0,T),\\ |d| = 1,\ \nabla v = 0, & \text{on } \Omega\times(0,T),\\ d = d_1,\ v = 0, & \text{on } \partial\Omega\times(0,T),\\ d(0) = d_0,\ v(0) = v_0 & \text{on } \Omega, \end{cases}$$

where the director d (a vector) models the averaged direction of the molecules.

We will see that we will have to follow the modelling and the physics (energetic principles of the system) to show some kind of general solvability of this system. ◇

0.2 Notation and functional analytic preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space, and $\langle\cdot,\cdot\rangle$ denote the dual pairing of X^* and X. Weak convergence of $(x_n)_{n\in\mathbb{N}} \subset X$ to $x \in X$ is denoted by $x_n \rightharpoonup x$ and weak* convergence of $(f_n)_{n\in\mathbb{N}} \subset X^*$ to $f \in X^*$ is denoted by $f_n \overset{*}{\rightharpoonup} f$.

We equip

$$\mathcal{C}([0,T];X) := \{f\colon [0,T] \to X : f \text{ continuous}\}$$

with the norm $\|u\|_{\mathcal{C}([0,T];X)} := \sup_{t\in[0,T]} \|u(t)\|_X$, which makes it a Banach space.

Theorem: Weierstrass Density Theorem

The space spanned by all polynomials

$$[0,T] \ni t \mapsto \sum_{k=0}^{N} c_k t^k, \qquad c_k \in X, \; N \in \mathbb{N}$$

is dense in $\mathcal{C}([0,T];X)$.

Corollary. If X is separable, then so is $\mathcal{C}([0,T];X)$. ◇

For X we can choose ℓ^p, $L^p(\Omega)$ for $p \in [1,\infty)$ or $\mathcal{C}([a,b];\mathbb{R})$ for $a < b \in \mathbb{R}$.

Lemma 0.2.2 (Exercise 2.3)
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $(u_n)_{n\in\mathbb{N}} \subset L^r(\Omega)$ with $1 \leqslant p < r \leqslant \infty$. If $(u_n)_{n\in\mathbb{N}} \subset L^r(\Omega)$ is bounded and convergent in $L^p(\Omega)$, then it is also convergent in $L^q(\Omega)$ with $p \leqslant q < r$.

Lemma 0.2.3 ((Exercise 4.2))
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, which are subspaces of a vector space V. Prove that the space

$$X + Y := \{x + y : x \in X, y \in Y\} \subset V$$

equipped with the norm

$$\|z\|_{X+Y} := \inf_{\substack{x\in X, y\in Y \\ z=x+y}} \max(\|x\|_X, \|y\|_Y)$$

is a Banach space.

Lemma 0.2.4 (Differenzielles Lemma von Gronwall)
Seien $a\colon [t_0,t] \to \mathbb{R}$ eine absolut stetige Funktionen und $\lambda, g \in L^1(t_0,T)$ integrierbar. Gilt für fast alle $t \in (t_0,T)$

$$a'(t) \leqslant g(t) + \lambda(t)a(t), \tag{1}$$

so folgt für fast alle $t \in (t_0,T)$

$$a(t) \leqslant e^{\Lambda(t)}a(t_0) + \int_{t_0}^{t} e^{\Lambda(t)-\Lambda(s)} g(s)\,\mathrm{d}s$$

Theorem 0.2.1: Discrete Gronwall's Lemma

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, where $(b_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{\geqslant 0}$ and $c_n \leqslant c_{n+1}$ for all $n \in \mathbb{N}$. Let $\lambda, \tau > 0$ with $\lambda\tau < 1$. Then

$$a_n + b_n \leqslant \tau\lambda \sum_{j=1}^{n} a_j + c_n$$

implies

$$a_n + b_n \leqslant c_n(1-\lambda\tau)^{-n}.$$

Abstract functions

Bei vielen Prozessen unterscheiden sich raumliches und zeitliches Verhalten und insbesondere die Zeit nimmt eine besondere Stellung ein. Soll die zeitliche Entwicklung, die Evolution, beschrieben werden, so macht es Sinn, so genannte abstrakte Funktionen einzuführen: Unter einer abstrakten Funktion verstehen wir dabei eine Funktion $u = u(t)\colon [0,T] \to X$, die für jeden Zeitpunkt $t \in [0,T]$ Element eines linearen Raumes X ist.

Der Bildraum X kann auch selbst wieder aus Funktionen in x bestehen und etwa der uns schon bekannte Sobolev-Raum $H_0^1(a,b)$ sein. Für jedes $t \in [0,T]$ ist $u(t)$ dann noch eine Funktion in x. Über

$$[u(t)](x) = u(x,t) \tag{2}$$

stellen wir den Zusammenhang zu den reellwertigen Funktionen $u = u(x,t)\colon [a,b]\times[0,T] \to \mathbb{R}$ her. Wollen wir die Funktion u charakterisieren, so werden wir auf Funktionenräume zurückgreifen, die über $[a,b] \times [0,T]$ erklärt sind, zum Beispiel könnte $u \in \mathcal{C}([a,b] \times [0,T])$ gelten. Dagegen werden wir $u\colon [0,T] \to X$ bezüglich t charakterisieren und es konnte zum Beispiel $u \in \mathcal{C}([0,T];X)$ gelten. Geht die abstrakte Funktion via (2) aus einer reellwertigen Funktion hervor, so werden wir künftig in der Bezeichnung zwischen beiden nicht mehr unterscheiden, so dass $u(t) = u(\cdot,t)$ [Emm13, p. 147- 148].

Lemma 0.2.5 (Exercise 1.1)
Let $Q := \Omega \times (0,T)$ the space-time cylinder for the bounded domain $\Omega \subset \mathbb{R}^d$ and X a normed space. Then

$$\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X)) \cong \mathcal{C}(\overline{Q};X)$$

In particular we have $\mathcal{C}([0,T];\mathcal{C}([a,b];\mathbb{R})) \cong \mathcal{C}([a,b]\times[0,T];\mathbb{R})$.

Proof. Let

$$\Phi\colon\ \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X)) \to \mathcal{C}(\overline{\Omega}\times[0,T];X), \qquad (\Phi u)(t,x) := [u(t)](x).$$

Let $u \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X))$. To show that Φ is well defined, we show that $\Phi u \in \mathcal{C}(\overline{\Omega}\times[0,T];X)$ for $u \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X))$. For all $x \in \overline{\Omega}$ we have

$$\begin{aligned}\|(\Phi u)(t_n,x) - (\Phi u)(t,x)\|_X &= \|[u(t_n)-u(t)](x)\|_X \leqslant \max_{x\in\overline{\Omega}} \|[u(t_n)-u(t)](x)\|_X \\ &= \|u(t_n)-u(t)\|_{\mathcal{C}(\overline{\Omega};X)} \xrightarrow[u\in\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X))]{n\to\infty} 0.\end{aligned}$$

Furthermore, Φ is isometric and hence injective as for all $u \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X))$ we have

$$\begin{aligned}\|\Phi u\|_{\mathcal{C}(\overline{\Omega}\times[0,T];X)} &= \max_{(t,x)\in[0,T]\times\overline{\Omega}} |[u(t)](x)| = \max_{t\in[0,T]}\max_{x\in\overline{\Omega}} |[u(t)](x)| \\ &= \max_{t\in[0,T]} \|u(t)\|_{\mathcal{C}(\overline{\Omega};X)} = \|u\|_{\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X))}.\end{aligned}$$

Further, Φ is surjective: let $\tilde{u} \in \mathcal{C}(\overline{\Omega}\times[0,T];X)$. Now define the function u by $[u(t)](x) = \tilde{u}(t,x)$, which fulfills $\Phi u = \tilde{u}$. It remains to show that $u \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};X))$. Let $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ converge to $t \in [0,T]$. Then for all $x \in \overline{\Omega}$ we have

$$\|u(t_n)-u(t)\|_{\mathcal{C}(\overline{\Omega};X)} = \max_{x\in\overline{\Omega}} \|[u(t_n)-u(t)](x)\|_X = \max_{x\in\overline{\Omega}} \|\tilde{u}(t_n,x)-\tilde{u}(t,x)\|_X \xrightarrow{n\to\infty} 0$$

by the uniform continuity of $\tilde{u}$ (as $\overline{\Omega}\times[0,T] \subset \mathbb{R}^{d+1}$ is compact). $\square$

Lemma 0.2.6 (Partial derivatives of abstract functions (Exercise 1.2))
Let $u \in \mathcal{C}(\overline{\Omega} \times [0,T]; \mathbb{R})$ *and* $\tilde{u} \in \mathcal{C}([0,T]; \mathcal{C}(\overline{\Omega}; \mathbb{R}))$ *such that* $\tilde{u}(t) = u(t, \cdot)$*. Then the function* u *has a partial derivative* $\frac{\partial u}{\partial t} \in \mathcal{C}(\overline{\Omega} \times [0,T]; \mathbb{R})$ *if and only if* $\tilde{u} \in \mathcal{C}^1([0,T]; \mathcal{C}(\overline{\Omega}; \mathbb{R}))$*. Then we have* $\frac{\partial u}{\partial t}(t, \cdot) = \tilde{u}'(t)$*.*

Proof. TODO $\square$

1 The Bochner integral and the Bochner spaces

1.1 Bochner measurability

We will discuss Bochner measurability for functions taking values in Banach spaces, which is similar to Lebesgue measurability.

Definition 1.1.1 (Bochner integral of a simple function)
A function $u\colon [0,T] \to X$ is a simple function if there exist finitely many pairwise disjoint Lebesgue measurable sets $(E_i \subset [0,T])_{i=1}^m$ such that u takes constant values $u_i \in X$ on each of these sets, that is $u = \sum_{i=1}^m u_i \mathbb{1}_{E_i}$. The (Bochner) integral of u is

simple function

$$\boxed{\int_0^T u(t)\,\mathrm{d}t := \sum_{i=1}^m u_i |E_i| \in X.}$$

Definition 1.1.2 (Bochner measurability)
A function $u\colon [0,T] \to X$ is Bochner measurable (or strongly measurable) if there exists a sequence of simple functions $(u_n\colon [0,T] \to X)_{n\in\mathbb{N}}$ such that for almost all $t \in (0,T)$

Bochner measurable

$$\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0. \tag{3}$$

The following, non-reversible (Exercise!) implication holds.

Lemma 1.1.3 (Lebesgue-measurability of $\|u\|$)
Let $u\colon [0,T] \to X$ be Bochner measurable. Then $\|u\|$ is Lebesgue measurable on $[0,T]$.

Proof. As u is Bochner measurable, there exists a sequence of simple functions $(u_n\colon [0,T] \to X)_{n\in\mathbb{N}}$ such that (3) holds for almost all $t \in [0,T]$. For those $t \in [0,T]$ we thus have

$$\big|\|u_n(t)\| - \|u(t)\|\big| \overset{\triangle^{-1}}{\leqslant} \|u_n(t) - u(t)\| \xrightarrow[(3)]{n\to\infty} 0.$$

The functions $(\|u_n\|\colon [0,T] \to \mathbb{R})_{n\in\mathbb{N}}$ are simple functions (and hence measurable), because the functions $u_n = \sum_{i=1}^{m_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}}$ are simple:

$$\|u_n(t)\| = \left\|\sum_{i=1}^{m_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}}(t)\right\| \overset{(\star)}{=} \sum_{i=1}^{m_n} \|u_i^{(n)}\| \mathbb{1}_{E_i^{(n)}}(t),$$

where in $(\star)$ we use that the $(E_i^{(n)})_{i=1}^{m_n}$ are disjoint. Hence $\|u\|$ is measurable as the limit of the measurable functions $\|u_n\|$. $\square$

Fig. 2: The function u from (4).

Counterexample. 1.1.4 (Bochner measurability)
The function

$$u\colon [0,1]\times[0,1] \to \mathbb{R}, \qquad (x,t) \mapsto \begin{cases} 1, & \text{if } x < t, \\ 0, & \text{else} \end{cases} \tag{4}$$

considered as an abstract function

$$\tilde{u}\colon [0,1] \to L^\infty([0,1];\mathbb{R}), \qquad t \mapsto u(\cdot,t) = \mathbb{1}_{[0,t)} \tag{5}$$

is measurable but *not* Bochner measurable but continuous if X is equipped with the weak* topology (Homework 1.4). ⋄

Definition 1.1.5 (Weak Bochner measurability)
A function $u\colon [0,T] \to X$ is weakly Bochner measurable if for all $f \in X^*$ the map $t \mapsto \langle f, u(t) \rangle$ is Lebesgue measurable.

weakly Bochner measurable

Remark. 1.1.6 Since strong convergence implies weak convergence, Bochner measurability implies weak Bochner measurability: if $u_n(t) \to u(t)$ for almost all $t \in [0,T]$, then $\langle f, u(t) \rangle$ is Lebesgue-measurable as the limit of the Lebesgue-measurable functions $\langle f, u_n(t) \rangle$.

For the converse of the above implication to hold, an additional property is needed.

Definition 1.1.7 (Essentially separable valued)
A function $u\colon [0,T] \to X$ is (essentially) separable valued if it (up to a null set $N \subset [0,T]$) only takes values in a separable subset of X.

(essentially) separable valued

Theorem 1.1.1: Pettis' Measurability Theorem [DUJ78]

A function $u\colon [0,T] \to X$ is Bochner measurable if and only if u is weakly Bochner measurable and essentially separable valued.

Corollary 1.1.8 (Weak and strong Bochner measurability in separable spaces)
If X is separable, weak and strong Bochner measurability coincide.

Proof. Subsets of separable spaces are separable. □

1.2 The Bochner integral

19.04.2021

We want to integrate functions mapping into BANACH spaces X, in our case X often is a SOBOLEV spaces. This allows us to consider evolution PDEs, where the space derivative is in the SOBOLEV space X and the time derivative is considered in the abstract BOCHNER setting. Our approach will be similar to the construction of the LEBESGUE integral.

DEFINITION 1.2.1 (BOCHNER INTEGRAL)
Let $u\colon [0,T] \to X$ be BOCHNER measurable and $(u_n)_{n\in\mathbb{N}}$ a sequence of simple functions with $u_n(t) \to u(t)$ in X for almost all $t \in [0,T]$. Then u is BOCHNER integrable if $\int_0^T \|u_n(t) - u(t)\| \,\mathrm{d}t \to 0$, that is, for all $\varepsilon > 0$ there exists a $M_\varepsilon \in \mathbb{N}$ such that

$$\int_0^T \|u_n(t) - u(t)\| \,\mathrm{d}t < \varepsilon \qquad \forall n \geqslant M_\varepsilon. \tag{6}$$

This is a LEBESGUE integral, which is well-defined by lemma 1.1.3.

We set

$$\boxed{\int_0^T u(t) \,\mathrm{d}t := \lim_{n\to\infty} \int_0^T u_n(t) \,\mathrm{d}t \in X.} \tag{7}$$

For a measurable subset $B \subset [0,T]$, we set

$$\int_B u(t) \,\mathrm{d}t := \int_0^T u(t) \,\mathbb{1}_B(t) \,\mathrm{d}t.$$

Remark. 1.2.2 The limit in (7) is well defined as each u_n and u are BOCHNER measurable and hence the function $\|u_n - u\|$ is LEBESGUE measurable by lemma 1.1.3.

For $n, m \geqslant M_\varepsilon$ we have, as $u_n - u_m$ is again a simple function and the triangle equality is an equality for simple functions,

$$\begin{aligned}\left\| \int_0^T u_n(t) \,\mathrm{d}t - \int_0^T u_m(t) \,\mathrm{d}t \right\|_X &= \int_0^T \|u_n(t) - u_m(t)\| \,\mathrm{d}t \\ &\overset{\triangle\neq}{\leqslant} \int_0^T \|u_n(t) - u(t)\| + \|u(t) - u_m(t)\| \,\mathrm{d}t \overset{(6)}{\leqslant} 2\varepsilon.\end{aligned}$$

Hence $\left(\int_0^T u_n(t) \,\mathrm{d}t\right)_{n\in\mathbb{N}}$ is a CAUCHY sequence in X and thus converges for $n \to \infty$ as X is a BANACH space.

Remark. 1.2.3 (Independence of approximating sequence) The integral (7) is well defined, that is, independent of the approximating sequence of simple functions, as (6) holds for all such sequences and thus the procedure in the previous remark can be done with any such sequence.

Remark. 1.2.4 The BOCHNER integral is linear, which directly follows from the limit definition (7) and the fact that the integral of simple functions is linear.

Lemma 1.2.5
Every continuous function $\mathcal{C}([0,T];X)$ is BOCHNER *integrable.*

Proof. Homework 1.1. □

Theorem 1.2.1: Properties of the Bochner integral

Let $u \colon [0,T] \to X$ be a function.

1. Let u be Bochner measurable. Then u is Bochner integrable if and only if $t \mapsto \|u(t)\|$ is Lebesgue integrable.
2. Let u be Bochner integrable. For all measurable subsets $B \subset [0,T]$ and for all $f \in X^*$ we have
$$\left\| \int_B u(t)\,\mathrm{d}t \right\| \leqslant \int_B \|u(t)\|\,\mathrm{d}t \qquad \text{and} \qquad \left\langle f, \int_B u(t)\,\mathrm{d}t \right\rangle = \int_B \langle f, u(t)\rangle\,\mathrm{d}t.$$
3. Let $(Y, \|\cdot\|_Y)$ be a Banach space, $A \in L(X,Y)$ a linear bounded operator and u Bochner integrable. Then $Au \colon [0,T] \to Y$ is Bochner integrable with
$$\int_0^T (Au)(t)\,\mathrm{d}t = A\left(\int_0^T u(t)\,\mathrm{d}t\right). \tag{8}$$

Proof. 1. As u is Bochner measurable, there exists a sequence of simple function $(u_n \colon [0,T] \to X)_{n\in\mathbb{N}}$ such that $\|u_n(t) - u(t)\| \to 0$ for almost all $t \in [0,T]$ and by lemma 1.1.3, $\|u\|$ is Lebesgue measurable and $\|u_n(t)\| \to \|u(t)\|$ almost everywhere in $[0,T]$.

" $\Longrightarrow$ ": We want to show that $\left(\int_0^T \|u_n(t)\|\,\mathrm{d}t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence: for $n \in \mathbb{N}$ we have
$$\left|\int_0^T \|u_n(t)\| - \|u(t)\|\,\mathrm{d}t\right| \leqslant \int_0^T \big|\|u_n(t)\| - \|u(t)\|\big|\,\mathrm{d}t \leqslant \int_0^T \|u_n(t) - u(t)\|\,\mathrm{d}t \xrightarrow{n\to\infty} 0.$$

By Fatou's Lemma (FL) we conclude that
$$\int_0^T \|u(t)\|\,\mathrm{d}t = \int_0^T \lim_{n\to\infty} \|u_n(t)\|\,\mathrm{d}t \overset{\text{FL}}{\leqslant} \lim_{n\to\infty} \int_0^T \|u_n(t)\|\,\mathrm{d}t < \infty.$$

Fatou's Lemma states that $\int_0^T \liminf_{n\to\infty} f_n(t)\,\mathrm{d}t \leqslant \liminf_{n\to\infty} \int_0^T f_n(t)\,\mathrm{d}t$ if $f_n \geqslant 0$.

" $\Longleftarrow$ ": Define the cut-off function
$$v_n(t) := \begin{cases} u_n(t), & \text{if } \|u_n(t)\| \leqslant 2\|u(t)\|, \\ 0, & \text{else.} \end{cases}$$

First observe that $v_n(t) \to u(t)$ almost everywhere: let $M := \{t \in [0,T] : u_n(t) \to u(t)\}$. If $u(t) = 0$, then the statement is obvious. If $t \in M$ is such that $u(t) \neq 0$, then there exists a $\varepsilon > 0$ such that $\|u(t)\| > \varepsilon$. For ε choose $n_0 \in \mathbb{N}$ such that
$$\big|\|u_n(t)\| - \|u(t)\|\big| \leqslant |u_n(t) - u(t)| < \varepsilon$$
for all $n \geqslant n_0$. Then we have
$$\|u_n(t)\| \leqslant \varepsilon + \|u(t)\| < 2\|u(t)\|$$
and hence $v_n(t) = u_n(t) \to u(t)$.

We have $\|v_n(t) - u(t)\| \leqslant 3\|u(t)\|$, so we have found a dominating function and can apply Lebesgue's dominated convergence theorem:
$$\lim_{n\to\infty} \int_0^T \|v_n(t) - u(t)\|\,\mathrm{d}t = 0.$$

③ Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of simple functions with $u_n(t) \to u(t)$ in X for almost all $t \in [0,T]$. Consider the sequence of simple (due to the linearity of A) functions $(Au_n : [0,T] \to Y)_{n\in\mathbb{N}}$. We have

$$\int_0^T \|Au_n(t) - Au(t)\|_Y \, \mathrm{d}t \leqslant \|A\|_{L(X,Y)} \int_0^T \|u_n(t) - u(t)\|_X \, \mathrm{d}t \xrightarrow[(6)]{n\to\infty} 0$$

due to the Bochner measurability of u.

The assertion (8) is true for step functions. Due to the linearity of A, the continuity of A grants the property for all functions.

② For the second assertion take $Y = \mathbb{R}$ and $A = \langle f, \cdot \rangle_{X^* \times X}$. The first assertion is true for steps functions due to the linearity of the integral. By Fatou's Lemma, we can then take the limit. □

Absolutely continuous functions

We will see that absolutely continuous function are the weakest class of functions for which the weak derivative, which we define afterwards, makes sense. Sobolev already noticed in the forties that for absolutely continuous functions the almost everywhere derivative coincide with the weak derivative but if you weaken the function class, this is not true anymore.

The properties stated in the following theorem motivate the definition after it.

Theorem 1.2.2: Lebesgue points

Let $u: [0,T] \to X$ be Bochner integrable. Then almost everywhere in $[0,T]$ we have

① $\displaystyle\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} u(s)\,\mathrm{d}s = u(t)$, that is, almost all points are Lebesgue points,

② $\displaystyle\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \|u(s) - u(t)\|\,\mathrm{d}s = 0$,

where outside of $[0,T]$, u is continued by zero.

Proof. ① follows from ②: by the triangle inequality we have

$$\left\| \frac{1}{h} \int_t^{t+h} u(s)\,\mathrm{d}s - u(t) \right\| = \left| \frac{1}{h} \int_t^{t+h} u(s) - u(t)\,\mathrm{d}s \right| \leqslant \frac{1}{h} \int_t^{t+h} \|u(s) - u(t)\|\,\mathrm{d}s.$$

② We can't guarantee the measurability of $\|u(\cdot) - u(t)\|$, so we have to use an approximation step. By Pettis' theorem, u is essentially separable valued. Hence for almost all $t \in [0,T]$ there exists a sequence $(x_n^{(t)})_{n\in\mathbb{N}} \subset X$ converging to $u(t)$.

By the triangle inequality, we have

$$\frac{1}{h} \int_t^{t+h} \|u(s) - u(t)\|\,\mathrm{d}s \leqslant \frac{1}{h} \int_t^{t+h} \|u(s) - x_n^{(t)}\|\,\mathrm{d}s + \|u(t) - x_n^{(t)}\|.$$

Taking $n \to \infty$ and then $h \to 0$ guarantees the measurability $s \mapsto \|u(s) - x_n^{(t)}\|$. □

Definition 1.2.6 (Absolute continuity)

A function $u: [0,T] \to X$ is absolutely continuous if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $N \in \mathbb{N}$ and pairwise disjoint intervals $((a_i,b_i) \subset [0,T])_{i=1}^N$, $\sum_{i=1}^N |b_i - a_i| < \delta$ implies that $\sum_{i=1}^n \|u(b_i) - u(a_i)\| < \varepsilon$. absolutely continuous

Remark. 1.2.7 (Relationship to other kinds of continuity)
Absolute continuous functions are uniformly continuous and continuous differentiable functions are absolutely continuous.

The following fundamental theorem is analogous to the theorem for functions $u\colon [0,T] \to \mathbb{R}$.

Theorem 1.2.3: Fundamental theorem

Let $u\colon [0,T] \to X$ be a Bochner integrable function. Then $U\colon [0,T] \to X$, $t \mapsto \int_0^t u(s)\,\mathrm{d}s$ is absolutely continuous and almost everywhere differentiable with $U'(t) = u(t)$, in particular at all points of continuity.

Proof. (Idea) We show first differentiability. For $t \in [0,T]$ we have

$$\begin{aligned}\left\| \frac{U(t+h)-U(t)}{h} - u(t) \right\| &= \left\| \frac{1}{h}\left(\int_0^{t+h} u(s)\,\mathrm{d}s - \int_0^t u(s)\,\mathrm{d}s \right) - u(t) \right\| \\ &= \left\| \frac{1}{h} \int_t^{t+h} u(s)\,\mathrm{d}s - u(t) \right\| \xrightarrow{h\to 0} 0\end{aligned}$$

by Theorem 1.2.2 (1).

We now show absolute continuity. For $(a_i, b_i) \subset [0,T]$ as in the definition of absolute continuity, we have

$$\sum_{i=1}^N \|U(b_i) - U(a_i)\| = \sum_{i=1}^N \left\| \int_{a_i}^{b_i} u(s)\,\mathrm{d}s \right\| \leqslant \sum_{i=1}^N \int_{a_i}^{b_i} \|u(s)\|\,\mathrm{d}s$$

and the statement follows from the integrability of $\|u\|$ (cf. Theorem 1.2.1 (1)) as that implies the absolute continuity of $t \mapsto \int_0^t \|u(s)\|\,\mathrm{d}s$. $\square$

Theorem 1.2.4: Komura [DUJ78] [Bré73]

Let X be reflexive and $u\colon [0,T] \to X$ be absolutely continuous. Then u is classically differentiable in $(0,T)$, u' is Bochner integrable and

$$u(t) = u(t_0) + \int_{t_0}^t u'(s)\,\mathrm{d}s$$

for all $t, t_0 \in [0,T]$.

Counterexample. 1.2.8 (Komura: reflexivity of X is essential)
Consider again (5), which is absolutely continuous but not differentiable: let $((a_i,b_i) \subset [0,T])_{i=1}^N$, then

$$\|\tilde{u}(b_i) - \tilde{u}(a_i)\|_{L^1([0,T],X)} = \|\mathbb{1}_{[0,b_i)} - \mathbb{1}_{[0,a_i)}\|_{L^1([0,T],X)} = \|\mathbb{1}_{[a_i,b_i)}\|_{L^1([0,T],X)} = b_i - a_i, \tag{9}$$

and hence

$$\sum_{i=1}^N \|\tilde{u}(b_i) - \tilde{u}(a_i)\|_{L^1([0,T],X)} = \sum_{i=1}^N b_i - a_i < \delta =: \varepsilon,$$

so $\tilde{u}$ is absolutely continuous. (By (9) it is even Lipschitz continuous.)

But $\tilde{u}$ is not differentiable: for $h \neq 0$ we have by the previous calculations

$$\left\| \frac{\tilde{u}(t+h) - \tilde{u}(t)}{h} \right\|_{L^1([0,T],X)} = \frac{h}{h} = 1$$

hence if v should be a derivative of $\tilde{u}$, we must have

$$\left| \|v\|_{L^1([0,T],X)} - \left\| \frac{\tilde{u}(t+h) - \tilde{u}(t)}{h} \right\|_{L^1([0,T],X)} \right| \overset{\triangle\neq^{-1}}{\leqslant} \left\| \frac{\tilde{u}(t+h) - \tilde{u}(t)}{h} - v \right\|_{L^1([0,T],X)} \to 0$$

and hence $\|v\|_{L^1([0,T],X)} = 1$. But

$$\left\| \frac{\tilde{u}(t+h) - \tilde{u}}{h} - v \right\|_{L^1([0,T],X)} = \int_0^t |v(s)| \, \mathrm{d}s + \int_t^{t+h} \left| v(s) - \frac{1}{h} \right| \mathrm{d}s + \int_{t+h}^1 |v(s)| \, \mathrm{d}s$$

has to vanish for $h \to 0$, so v is zero on $(0,t) \cup (t+h, 1)$ for $h \to 0$, we thus have $v = 0$ almost everywhere in $(0,T)$, which is a contradiction to $\|v\|_{L^1([0,T],X)} = 1$. ◇

Example. 1.2.9 (Revisiting counterexample. 1.1.4 in L^2)
Consider the function

$$u \colon [0,1]^2 \to \mathbb{R}, \qquad (t,x) \mapsto \mathbb{1}_{[0,t]}(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant t, \\ 0, & \text{if } t < x \leqslant 1. \end{cases}$$

and the corresponding abstract function $\tilde{u} \colon [0,1] \to L^2([0,1];\mathbb{R})$, $t \mapsto u(t,\cdot)$.

Then $\tilde{u}$ is Bochner measurable, nowhere differentiable (and hence not absolutely continuous by Theorem 1.2.4) but continuous if $L^2([0,1];\mathbb{R})$ is equipped with the weak topology. (Exercise 1.3, TODO) ◇

The Bochner spaces

A compact subset $K \subset (0,T)$ is denoted by $K \subset\subset (0,T)$.

Definition 1.2.10 (Bochner space $L^p(0,T;X)$)
For $p \in [1,\infty)$, $L^p(0,T;X)$ is the linear space of equivalence classes (of functions only differing on a null set) of Bochner measurable functions $u \colon [0,T] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p \, \mathrm{d}t \right)^{\frac{1}{p}} < \infty$$

and $L^\infty(0,T;X)$ is the linear space of equivalence classes of bounded Bochner measurable functions $u \colon [0,T] \to X$ with

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\| < \infty.$$

The space $L^1_{\text{loc}}(0,T;X)$ can be used to model blow ups, for example at the endpoints of the interval.

Definition 1.2.11 (Bochner space $L^1_{\text{loc}}(0,T;X)$)
The space $L^1_{\text{loc}}(0,T;X)$ is the space of locally integrable functions

$$L^1_{\text{loc}}(0,T;X) := \left\{ u \colon [0,T] \to X \text{ such that } u\,\mathbb{1}_K \in L^1(0,T;X) \ \forall K \subset\subset (0,T) \right\}.$$

The Bochner spaces exhibit the usual properties we know from the Lebesgue spaces.

Theorem 1.2.5: Properties of the Bochner spaces

1. For $p \in [1, \infty]$, $L^p(0, T; X)$ is a Banach space.
2. For $p \in [1, \infty)$, the simple functions are dense in $L^p(0, T; X)$.
3. For $p \in [1, \infty)$, $\mathcal{C}([0, T]; X)$ is dense in $L^p(0, T; X)$.
4. For $p \in [1, \infty)$, $L^p(0, T; X)$ is separable if X is, too.
5. Let $u \in L^p(0, T; X)$ and $v \in L^q(0, T; X^*)$ where $p, q \in [1, \infty]$ are Hölder conjugates. Then $\langle v(\cdot), u(\cdot) \rangle \in L^1(0, T; \mathbb{R})$ and the Hölder inequality holds:
$$\left| \int_0^T \langle v(t), u(t) \rangle \,\mathrm{d}t \right| \leqslant \|v\|_{L^q(0,T;X^*)} \|u\|_{L^p(0,T;X)}.$$
6. For $p \in (1, \infty)$, $L^p(0, T; X)$ is reflexive if X is, too. If X is reflexive or X^* is separable, then $(L^p(0, T; X))^* \cong L^q(0, T; X^*)$ via the dual pairing
$$\langle v, u \rangle_{(L^p(0,T;X))^* \times L^p(0,T;X)} := \int_0^T \langle v(t), u(t) \rangle_{X^* \times X} \,\mathrm{d}t.$$
Furthermore, $(L^1(0, T; X))^* \cong L^\infty(0, T; X^*)$.
7. If $X = H$ is a Hilbert space, then $L^2(0, T; H)$ is Hilbert space with the inner product
$$\langle u, v \rangle_{L^2(0,T;H)} := \int_0^T \langle u(t), v(t) \rangle_H \,\mathrm{d}t.$$
8. If $X \hookrightarrow Y$ are Banach spaces, then $L^p(0, T; X) \hookrightarrow L^q(0, T; Y)$ for all $1 \leqslant q \leqslant p \leqslant \infty$.

As L^∞ is not separable, the simple functions cannot be dense in L^∞.

This only holds for bounded intervals.

Proof. Analogous to the standard case.

5. Homework 2.3
8. Homework 2.2. □

Remark. 1.2.12 (Not all properties can can be taken for granted)
A weakly continuous (that is, continuous with respect to the weak topology on X) function $u \in \mathcal{C}_w([0, T]; X)$ must *not* be Bochner integrable.

Lemma 1.2.13 (Abstract functions)
Let $\tilde{u}\colon [0, T] \to L^p((a, b); \mathbb{R})$ *with* $p \in [1, \infty]$ *be* Bochner *measurable. Then* $u\colon [a, b] \times [0, T] \to \mathbb{R}$, $(x, t) \mapsto [\tilde{u}(t)](x)$ *is* Lebesgue *measurable.*

Proof. Homework 1.2. □

Lemma 1.2.14 (The case $X = L^p((a, b); \mathbb{R})$, $p < \infty$)
For $p \in [1, \infty)$ *and* $a < b \in \mathbb{R}$ *we have*
$$L^p(0, T; L^p((a, b); \mathbb{R})) \cong L^p([a, b] \times (0, T); \mathbb{R})$$
but
$$L^\infty(0, T; L^p((a, b); \mathbb{R})) \subsetneq L^\infty([a, b] \times (0, T); \mathbb{R})$$

Proof. Homework 1.3.

The latter inclusion follows from lemma 1.2.13 and the standard example for nonequality is (5). □

Lemma 1.2.15 (Dominated convergence for abstract functions (Exercise 2.4))
Let $(u_n)_{n\in\mathbb{N}} \subset L^1(0,T;X)$, $u \in L^1(0,T;X)$ and $g \in L^1((0,T);\mathbb{R})$. If $u_n(t) \to u(t)$ in X and $\|u_n(t)\|_X \leqslant g(t)$ for almost every $t \in [0,T]$, then $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(0,T;X)$.

Proof. TODO □

2 Generalised time derivatives and the space $W(0,T)$

26.04.2021

In this section we will introduce the weak time derivative for abstract functions and obtain weak formulations for evolution equations, as before, by multiplying with test functions.

2.1 The generalised time derivative

Let X be a reflexive BANACH space.

DEFINITION 2.1.1 (WEAK TIME DERIVATIVE)
Let $u, v \in L^1_{\text{loc}}(0,T;X)$. Then v is the weak time derivative of u if

weak time derivative

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = -\int_0^T v(t)\varphi(t)\,\mathrm{d}t \qquad \forall \varphi \in \mathcal{C}_0^\infty([0,T];\mathbb{R}). \tag{10}$$

Remark. 2.1.2 (Dual characterisation of weak derivatives)
The function v is the weak time derivative of u if for all $f \in X^*$

$$\left\langle f, \frac{u(t+h)-u(t)}{h} - v \right\rangle \xrightarrow{h\to 0} 0.$$

THEOREM 2.1.1: FUNDAMENTAL THEOREM OF THE CALCULUS OF VARIATIONS

Let $u \in L^1_{\text{loc}}(0,T;X)$ with

$$\int_0^T u(t)\varphi(t)\,\mathrm{d}t = 0 \tag{11}$$

for all $\varphi \in \mathcal{C}_0^\infty(0,T)$. Then u vanishes almost everywhere.

Proof. Let $\varepsilon > 0$. We may define $\varphi \in \mathcal{C}_0^\infty(0,T)$ such that $\varphi|_{(\varepsilon,t-\varepsilon)} = 1$, $\varphi|_{(t,T)} = 0$ for some $t \in (0,T)$. Then

$$\begin{aligned}\left\|\int_0^t u(s)\,\mathrm{d}s\right\| &\overset{(11)}{=} \left\|\int_0^t u(s) - \varphi(s)u(s)\,\mathrm{d}s\right\| \leqslant \int_0^t (1-\varphi(s))\|u(s)\|\,\mathrm{d}s \\ &= \int_0^\varepsilon \underbrace{(1-\varphi(s))}_{\leqslant 1}\|u(s)\|\,\mathrm{d}s + \int_{t-\varepsilon}^t \underbrace{(1-\varphi(s))}_{\leqslant 1}\|u(s)\|\,\mathrm{d}s \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

Hence $\left\|\int_0^t u(s)\,\mathrm{d}s\right\| = 0$ for all $t \in (0,T)$ implies $u \equiv 0$ almost everywhere.

This proof only works for $u \in L^1(0,T) \subsetneq L^1_{\text{loc}}(0,T)$**??** This is a proof for the general case: for $f \in X^*$ we have

$$\int_0^T \langle f, u(t)\varphi(t)\rangle\,\mathrm{d}t = \left\langle f, \int_0^T u(t)\varphi(t)\,\mathrm{d}t \right\rangle = 0$$

and thus the Fundamental Theorem of the Calculus of Variations from DGL IIA implies that $\langle f, u(t)\rangle = 0$ almost everywhere and hence $u \equiv 0$ almost everywhere. $\square$

Corollary 2.1.3 (Testing with the derivative of the test function)
Let $u \in L^1_{\mathrm{loc}}(0,T;X)$ such that

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = 0$$

for all $\varphi \in \mathcal{C}_0^\infty(0,T;\mathbb{R})$. Then there is a constant $u_0 \in X$ such that $u \equiv u_0$ almost everywhere in $(0,T)$.

Theorem 2.1.2: Characterisation of weak derivatives

Let $u, v \in L^1_{\mathrm{loc}}(0,T;X)$. Then the following are equivalent

1. v is a weak derivative of u
2. there exists a $u_0 \in X$ such that

$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s$$

almost everywhere in $(0,T)$.

3. for all $f \in X^*$ the function $t \mapsto \langle f, u(t) \rangle$ has the weak derivative $t \mapsto \langle f, v(t) \rangle$.

Proof. 1 $\iff$ 3: If v is the weak time derivative of u, then

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = -\int_0^T v(t)\varphi(t)\,\mathrm{d}t$$

for all $\varphi \in \mathcal{C}_0^\infty(0,T)$. By a corollary of the Hahn-Banach theorem, the above equations are equivalent to

$$\left\langle f, \int_0^T u(t)\varphi'(t)\,\mathrm{d}t \right\rangle = \left\langle f, -\int_0^T v(t)\varphi(t)\,\mathrm{d}t \right\rangle$$

for all $f \in X^*$. By linearity and continuity of f, this is equivalent to

$$\int_0^T \langle f, u(t) \rangle \varphi'(t)\,\mathrm{d}t = -\int_0^T \langle f, v(t) \rangle \varphi(t)\,\mathrm{d}t$$

for all $f \in X^*$ due to Theorem 1.2.1 3 by choosing $A = \langle f, \cdot \rangle_{X^* \times X} \in L(X,\mathbb{R})$.

1 $\implies$ 2: By Fubini's theorem we have

$$\begin{aligned}\int_0^T \int_0^t v(s)\,\mathrm{d}s\varphi'(t)\,\mathrm{d}t &= \int_0^T v(s) \int_s^T \varphi'(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^T v(s)(\underbrace{\varphi(T)}_{=0} - \varphi(s))\,\mathrm{d}s = -\int_0^T v(s)\varphi(s)\,\mathrm{d}s \overset{(10)}{=} \int_0^T u(t)\varphi'(t),\end{aligned}$$

We can continue $\varphi \in \mathcal{C}_0^\infty(0,T)$ onto $[0,T]$ by zero due to φ being compactly supported, so $\varphi(T)$ makes sense.

using the fundamental theorem of calculus. Then

$$\int_0^T \left(u(t) - \int_0^t v(s)\,\mathrm{d}s \right) \varphi'(t)\,\mathrm{d}t = 0.$$

By corollary 2.1.3 we have, up to a constant, $u(t) = \int_0^t v(s)\,\mathrm{d}s$.

2 $\implies$ 1: Assume there exists a $u_0 \in X$ such that $u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s$. Then

$$0 = \int_0^T u_0\varphi'(t)\,\mathrm{d}t = \int_0^T \left(u(t) - \int_0^t v(s)\,\mathrm{d}s \right) \varphi'(t)\,\mathrm{d}t.$$

With the calculation from before we see that v is the weak derivative of u. $\square$

Definition 2.1.4 (The space $W^{1,1}(0,T;X)$)
Let

$$W^{1,1}(0,T;X) \coloneqq \{u \in L^1(0,T;X) : u \text{ has a weak derivative } u' \in L^1(0,T;X)\}$$

be equipped with

$$\|u\|_{1,1} \coloneqq \|u\|_1 + \|u'\|_1.$$

Theorem 2.1.3: $W^{1,1}$ functions are absolutely continuous

The space $W^{1,1}(0,T;X)$ is a Banach space. For every function $u \in W^{1,1}(0,T;X)$ we can find an absolutely continuous function, which is almost equal to u, that is, $W^{1,1}(0,T;X) \hookrightarrow \mathrm{AC}([0,T];X) \hookrightarrow \mathcal{C}([0,T];X)$.

Proof. 1 Completeness (sketch): Let $(u_n)_{n\in\mathbb{N}}$ converge to u in $W^{1,1}(0,T;X)$. Then there exists a v such that $u_n \to v$ in $L^1(0,T;X)$. We have

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T u_n(t)\varphi'(t)\,\mathrm{d}t = -\lim_{n\to\infty}\int_0^T u_n'(t)\varphi(t)\,\mathrm{d}t = -\int_0^T v(t)\varphi(t)\,\mathrm{d}t,$$

so v is the generalised derivative of u.

2 For an absolutely continuous function u we have the representation $u(t) = u_0 + \int_0^t u'(s)\,\mathrm{d}s$ by Theorem 1.2.4. By the integral mean value theorem there exists a $t_0 \in [0,T]$ such that $\|u(t_0)\| = \frac{1}{T}\int_0^T \|u(t)\|\,\mathrm{d}t$ and hence

$$\|u(t)\| = \left\| \|u(t_0)\| + \int_{t_0}^t u'(s)\,\mathrm{d}s \right\| \leqslant \frac{1}{T}\int_0^T \|u(t)\|\,\mathrm{d}t + \int_0^T \|u'(t)\|\,\mathrm{d}t \leqslant \frac{\max(1,T)}{T}\|u\|_{1,1}.$$

□

2.2 The space $W(0,T)$

We want to introduce the space $W(0,T)$, which is the standard space for handling evolution equations. The overall idea is that time derivative is a different space than the space derivative, so they have to be handled differently; the time derivative "lives in" another space than the function itself. The function can have spatial regularity, which is lost when differentiating in time.

Definition 2.2.1 (Gelfand triple, evolutionary triple)
Let $(V, \|\cdot\|)$ be a real reflexive separable Banach space, $(H, |\cdot|)$ a real Hilbert space and $V \overset{d}{\hookrightarrow} H$. We identify $H \cong H^*$. Since V is reflexive, we get $H^* \overset{d}{\hookrightarrow} V^*$. We call $V \subset H \subset V^*$ a Gelfand or evolutionary triple. evolutionary triple

The space H is called pivot space.

Remark. 2.2.2 (Notation of norms, dual pairing and scalar products)
The norm on V will be denoted by $\|\cdot\|$, the norm on H will be $|\cdot|$ and the norm on V^* will be $\|\cdot\|_*$. The dual pairing will be $\langle \cdot, \cdot \rangle_{V^* \times V}$ and the scalar product on H is $(\cdot,\cdot)$ such that we have $\langle g, v \rangle = (g, v)$ for all $g \in H$ and $v \in V$.

Proof. (Exercise 2.1) We show that $H \overset{d}{\hookrightarrow} V$ implies $H^* \overset{d}{\hookrightarrow} V^*$.

Let $j \colon V \to j(V) \overset{d}{\subset} H$ be the linear injective operator of the embedding $V \overset{d}{\hookrightarrow} H$ and $I \colon H \to H^*$ the isometric Riesz isomorphism satisfying $\langle I(w), v \rangle_{H^* \times H} = (w, v)_{H \times H}$ for all $v, w \in H$. Consider the Banach adjoint of j

$$j^* \colon H^* \to V^*, \qquad \langle j^*(w), v \rangle_{V^* \times V} = \langle w, j(v) \rangle_{H^* \times H}.$$

We want to show that $j^*(H^*) \subset V^*$ is dense.

TODO $\square$

Remark. 2.2.3 (Notation of embeddings) We have shown that we can understand the Gelfand triple $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$ as inclusions of sets: $V \subset H \subset V^*$. Therefore, it is common not to write the embedding operators and simply identify the elements. For example, for $f \in H$ and $v \in V$

$$\langle j^*(I(f)), v \rangle_{V^* \times V} = \langle I(f), j(v) \rangle_{H^* \times H} = (f, j(v)),$$

but we write simply

$$\langle f, v \rangle_{V^* \times V} = (f, v)_H.$$

Example. 2.2.4 (Gelfand triple)

- $V := W_0^{1,p}(\Omega)$, $H := L^2(\Omega)$, $V^* := W^{-1,q}(\Omega)$.
- $V := L^p(\Omega)$, $H := H^{-1}(\Omega) \cong H_0^1(\Omega)$, $V^* = L^q(\Omega)$ if $\frac{1}{p} + \frac{1}{q} = 1$.

  In order for $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, the Sobolev embedding theorem says that we need $\frac{1}{q} \leqslant \frac{1}{2} - \frac{1}{d}$, and together with the above condition this becomes $1 - \frac{1}{p} \leqslant \frac{1}{2} - \frac{1}{d}$, that is, $\frac{1}{p} \geqslant \frac{1}{2} + \frac{1}{d}$, that is, $p \leqslant \frac{2d}{d+2}$ where $\Omega \subset \mathbb{R}^d$. $\diamond$

Definition 2.2.5 (The spaces $W(0,T)$ and $W_p(0,T)$)
Let $V \subset H \subset V^*$ be a Gelfand triple. We define

$$\boxed{W(0,\ T) := \{u \in L^2(0,T;V) : \exists u' \in L^2(0,T;V^*)\}}$$

W(0, T)

and endow it with the norm

$$\|u\|_{W(0,T)} := \left(\|u\|^2_{L^2(0,T;V)} + \|u'\|^2_{L^2(0,T;V^*)}\right)^{\frac{1}{2}}.$$

Theorem 2.2.1: Properties of $W(0,T)$

1. The space $(W(0,T), \|\cdot\|_{W(0,T)})$ is a Banach space.
2. $\mathcal{C}^\infty([0,T];V) \subset W(0,T)$ is dense.
3. We have $W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$.
4. The integration-by-parts formula holds: for $u,v \in W(0,T)$ and $0 \leqslant s \leqslant t \leqslant T$

$$\int_s^t \langle u'(\tau), v(\tau)\rangle + \langle v'(\tau), u(\tau)\rangle \,\mathrm{d}\tau = (u(t),v(t)) - (u(s),v(s)).$$

Proof. 1 We only show completeness. Let $(u_n)_{n\in\mathbb{N}} \subset W(0,T)$ be a Cauchy sequence. Then $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2((0,T),V)$ and $(u_n')_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2((0,T),V^*)$. By Theorem 1.2.5 1, both of these spaces are complete, so there exist limits u,v with $u_n \to u$ and $u_n' \to v$. We want to show that $u' = v$. For all $\varphi \in \mathcal{C}_0^\infty(0,T)$ we have

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T u_n(t)\varphi'(t)\,\mathrm{d}t = -\lim_{n\to\infty}\int_0^T u_n'(t)\varphi(t)\,\mathrm{d}t = -\int_0^T v(t)\varphi(t)\,\mathrm{d}t,$$

where the (strong) convergence holds in V^*. Hence v is the weak derivative of u. The space $W(0,T)$ is well defined as $L^2(0,T;V) \subset L^1(0,T;V) \hookrightarrow L^1(0,T;V^*)$ and $u' \in L^2(0,T;V^*) \subset L^1(0,T;V^*)$, so $W(0,T) \subset W^{1,1}(0,T;V^*)$ is a Banach space as a closed subset of a Banach space.

2 This is proven, as usual, by mollifying (convolution with smoothing kernels $\rho_\varepsilon \in \mathcal{C}^\infty([0,T];V)$). Define the approximating sequence $(u_\varepsilon := \rho_\varepsilon * u)_{\varepsilon>0} \subset \mathcal{C}^\infty([0,T];V)$ with $u_\varepsilon \to u$ in $L^2((\varepsilon,T-\varepsilon);V)$. We have $(u_\varepsilon)' = u_\varepsilon'$ and hence $(u_\varepsilon)' \to u'$ in $L^2((\varepsilon,T-\varepsilon);V)$. Since $\varepsilon > 0$ was arbitrary, we deduce convergence.

3 Let $v \in \mathcal{C}^1([0,T];V)$, $\varphi \in \mathcal{C}^\infty([0,T];[0,1])$ with $\varphi(0) = 0$, $\varphi(T) = 1$. Let $v_1 := v\cdot\varphi$ and $v_2 := v\cdot(1-\varphi)$. Then $v_1 + v_2 = v$, $v_1(0) = v_2(T) = 0$, $v_1(T) = v(T)$, and $v_2(0) = v(0)$. By partial integration we have

$$\begin{aligned}(v_1(t),v(t)) &= (v_1(0),v(0)) + \int_0^t \langle v_1'(s), v(s)\rangle + \langle v'(s), v_1(s)\rangle \,\mathrm{d}s\\ &= \underbrace{(v_1(0),v(0))}_{=0} + \int_0^t \varphi'(s)|v(s)|^2 + 2\varphi(s)\langle v'(s), v(s)\rangle\,\mathrm{d}s\end{aligned}$$

and analogously

$$(v_2(t),v(t)) = \underbrace{(v_1(T),v_2(T))}_{=0} - \int_t^T (-\varphi'(s)|v(s)|^2) + 2(1-\varphi(s))\langle v'(s), v(s)\rangle \,\mathrm{d}s.$$

Hence

$$\begin{aligned}
|v(t)|^2 &= (v_1(t), v(t)) + (v_2(t), v(t)) \\
&= \int_0^T \varphi'(s)|v(s)|^2 + 2\varphi(s)\langle v'(s), v(s)\rangle \,\mathrm{d}s \\
&\quad - \int_t^T (-\varphi'(s)|v(s)|^2) + 2(1-\varphi(s))\langle v'(s), v(s)\rangle \,\mathrm{d}s \\
&= \int_0^T \underbrace{\varphi'(s)}_{\leqslant \|\varphi'\|_\infty} \underbrace{|v(s)|^2}_{\leqslant \alpha^2\|v(s)\|^2} \,\mathrm{d}s + 2\int_0^t \underbrace{\varphi(s)}_{\leqslant 1} \underbrace{\langle v'(s), v(s)\rangle}_{\leqslant \|v'(s)\|_*\|v(s)\|} \,\mathrm{d}s \\
&\quad - 2\int_t^T \underbrace{(1-\varphi(s))}_{\leqslant 1} \underbrace{\langle v'(s), v(s)\rangle}_{\leqslant \|v'(s)\|_*\|v(s)\|} \,\mathrm{d}s.
\end{aligned}$$

As $V \hookrightarrow H$, there exists a $\alpha > 0$ such that $|\cdot| \leqslant \alpha\|\cdot\|$. Thus by using the CAUCHY-SCHWARTZ inequality and YOUNGS's inequality, we obtain

$$\begin{aligned}
|v(t)|^2 &\leqslant \alpha^2\|\varphi'\|_\infty \|v\|^2_{L^2((0,T):V)} + 4\cdot\frac{1}{2}\left(\|v'\|^2_{L^2(0,T;V*)} + \|v\|^2_{L^2(0,T;V)}\right) \\
&\leqslant (\alpha^2\|\varphi'\|_\infty + 2)\|v\|^2_{W(0,T)} = \mathrm{const}\|v\|^2_{W(0,T)}
\end{aligned}$$

and thus $\|v\|_\infty \leqslant \mathrm{const}\|v\|_{W(0,T)}$. For every $u \in W(0,T)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^\infty([0,T];V)$ such that $u_n \to u$ in $W(0,T)$. By the above, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}([0,T];H)$, so there exists a $\tilde{u}$ such that $u_n \to \tilde{u}$ in $\mathcal{C}([0,T];H)$. We may identify the limit: we have

$$\begin{aligned}
\|\tilde{u}\|_{\mathcal{C}([0,T];H)} &\leqslant \|\tilde{u} - u_n\|_{\mathcal{C}([0,T];H)} + \|u_n\|_{\mathcal{C}([0,T];H)} \\
&\leqslant \|\tilde{u} - u_n\|_{\mathcal{C}([0,T];H)} + \|u_n - u\|_{W(0,T)} + \|u\|_{W(0,T)}.
\end{aligned}$$

Passing to the limit with $n \to \infty$, we obtain that the first two summands vanish.

4 The integration by parts rule holds for $\mathcal{C}^1$ functions (Exercise 2.2, TODO) and by density arguments it follows for functions in $W(0,T)$. □

Corollary 2.2.6 (Integration by parts: derivative of the squared norm)
For $u \in W(0,T)$ we have $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 = \langle u'(t), u(t)\rangle$ almost everywhere in $(0,T)$.

Proof. Let $\varphi \in \mathcal{C}_0^\infty(0,T)$. Then

$$\frac{1}{2}\int_0^T |u(t)|^2\varphi'(t)\,\mathrm{d}t = \int_0^T \langle u'(t), u(t)\rangle \varphi(t)\,\mathrm{d}t$$

by choosing $u = u$ and $v = \varphi u$ in Theorem 2.2.1 3. Since $L^2(0,T;V) \hookrightarrow L^2(0,T;H)$ by Theorem 1.2.5 8, we observe that $|u(\cdot)|^2 \in L^1(0,T)$ and $\langle u'(\cdot), u(\cdot)\rangle \in L^1(0,T;\mathbb{R})$. We observe that $t \mapsto |u(t)|^2$ is absolutely continuous and almost everywhere differentiable. □

3 Linear first order evolution equations

3.1 Introduction to Linear Operator-valued ODEs - Assumptions and weak formulation

04.05.2021

We will first look at linear first order PDEs. We want to prove well-posedness of certain evolutionary equations in the form of the Theorem of Lions (1962).

Let $V \subset H \subset V^*$ be a Gelfand triple and $a\colon [0,T] \times V \times V \to \mathbb{R}$ be such that

(A1) $a(\cdot, v, w)$ is Lebesgue measurable on $[0,T]$ for all $v, w \in V$,

(A2) $a(t, \cdot, \cdot)$ is bilinear for all $t \in [0,T]$,

(A3) the form a is uniformly bounded with respect to the first input variable, that is, there exists a $\beta > 0$ such that

$$|a(t,v,w)| \leqslant \beta \|v\| \|w\| \tag{12}$$

for all $t \in [0,T]$ and all $v, w \in V$.

(A4) the form a fulfills the Gårding inequality, that is, there exists a $\mu > 0$ and a $\kappa \geqslant 0$ such that

$$a(t,v,v) \geqslant \mu \|v\|^2 - \kappa |v|^2 \tag{13}$$

for all $t \in [0,T]$ and all $v \in V$. For $\kappa \in [0, \frac{\mu}{\alpha^2})$, $a(t,\cdot,\cdot)$ is uniformly strongly positive (that is, for all $t \in [0,T]$), where $\alpha > 0$ is the embedding constant: $|\cdot| \leqslant \alpha \|\cdot\|$ (Homework 3.2(a)).

Remark. 3.1.1 (Equivalent norm on V (Homework 3.2(b))) For any $t_0 \in [0,T]$ the form

$$((u,v)) := \frac{1}{2}\big(a(t_0,u,v) + a(t_0,v,u)\big) + \kappa(u,v)$$

defines an inner product on V which induces a norm equivalent to $\|\cdot\|$ on V.

We get the following implications.

(1) for all $t \in [0,T]$ and all $v \in V$, the map $a(t,v,\cdot)\colon V \to \mathbb{R}$ is linear and bounded. We define $A(t)v := a(t,v,\cdot) \in V^*$ which fulfils $\|A(t)v\|_* \leqslant \beta\|v\|$.

(2) for all $t \in [0,T]$, $A(t) \in L(V,V^*)$ with $\|A(t)\|_{L(V,V^*)} \leqslant \beta$.

Finally, define $A\colon [0,T] \to L(V,V^*)$, $t \mapsto A(t)$.

(3) the Gårding inequality now becomes

$$\langle (A(t) + \kappa I)v, v \rangle_{V^* \times V} \geqslant \mu \|v\|^2, \tag{14}$$

where $I\colon V \to V^*$ is the embedding via $(\cdot,\cdot)$: $\langle Iv, v\rangle_{V^* \times V} = (v,v) = |v|^2$. Hence A with a nonnegative shift is strongly positive.

Definition 3.1.2 (Nemitzkij operator)
The Nemitzkij operator of $u\colon [0,T] \to V$ is

$$\mathcal{A}u\colon [0,T] \to V^*, \qquad (\mathcal{A}u)(t) := [A(t)](u(t)) = A(t)u(t).$$

Lemma 3.1.3 (Properties of the Nemitzkij operator)
Let the assumptions A1 - A3 *be fulfilled. Then* $\mathcal{A}$ *maps*

1. *Bochner measurable functions to Bochner measurable functions and*
2. $L^2(0,T;V)$ *into* $L^2(0,T;V^*)$.

Proof. 1 As u is Bochner measurable, there exists a sequence

$$\left(u_n := \sum_{i=1}^{m_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}} : [0,T] \to V\right)_{n\in\mathbb{N}}$$

of simple functions such that $\|u(t) - u_n(t)\|_V \to 0$ for almost all $t \in [0,T]$. For $w \in V$ we have

$$\langle Au_n(t), w\rangle_{V^*\times V} = \sum_{i=1}^{m_n} \langle A(t)u_i^{(n)}, w\rangle \mathbb{1}_{E_i^{(n)}}(t) = \sum_{i=1}^{m_n} a(t, u_i^{(n)}, w) \mathbb{1}_{E_i^{(n)}}(t).$$

By assumption A1, $t \mapsto \langle Au_n(t), w\rangle_{V^*\times V}$ is Lebesgue measurable for all $n \in \mathbb{N}$. Since $A(t) \in L(V, V^*)$, we observe

$$\langle (Au_n)(t), w\rangle \to \langle (Au)(t), w\rangle$$

for all $w \in V$ and almost all $t \in [0,T]$. Hence $\langle (Au)(t), w\rangle$ is the pointwise limit of Lebesgue measurable functions $(\langle (Au_n)(t), w\rangle)_{n\in\mathbb{N}}$ and hence also Lebesgue measurable. Since V^* is separable, the statement follows by Theorem 1.1.1.

2 Let $u \in L^2(0,T;V)$, then Au is Bochner measurable in V^* by 1. By assumption A3 we have

$$\|(\mathcal{A}u)(t)\|_* = \|A(t)u(t)\|_* \leqslant \beta\|u(t)\|.$$

Integration yields

$$\|\mathcal{A}u\|^2_{L^2(0,T;V^*)} = \int_0^T \|(\mathcal{A}u)(t)\|_*^2 \,\mathrm{d}t = \int_0^T \|A(t)u(t)\|_*^2 \,\mathrm{d}t \leqslant \beta^2 \int_0^T \|u(t)\|^2 \,\mathrm{d}t = \beta^2\|u\|^2_{L^2(0,T;V)}.$$

□

Remark. 3.1.4 ($\mathcal{A} \in L(L^2(0,T;V), L^2((0,T);V^*))$) The proof of lemma 3.1.3 implies that $\mathcal{A} \in \mathcal{L}\big(L^2(0,T;V), L^2(0,T;V^*)\big)$ with norm bound β.

Later on we want to prove the existence of weak solutions to this linear problem by a time discretisation. Due to the discretisation of the time derivative, we have to deal with a shift of the operator A.

Lemma 3.1.5 (Strong positivity of $A + \kappa I$)
Additionally assuming the Gårding inequality A4, *the shifted operator*

$$\mathcal{A} + \kappa I : L^2(0,T;V) \to L^2(0,T;V^*)$$

is strongly positive:

$$\langle (\mathcal{A} + \kappa I)u, u\rangle \geqslant \mu\|u\|^2_{L^2(0,T;V)} \qquad \forall u \in L^2(0,T;V).$$

Proof. We have

$$\langle (\mathcal{A} + \kappa I)u, u\rangle = \int_0^T \langle (A(t) + \kappa I)u(t), u(t)\rangle_{V^*\times V} \,\mathrm{d}t \overset{(14)}{\geqslant} \mu \int_0^T \|u(t)\|^2 \,\mathrm{d}t = \mu\|u\|^2_{L^2(0,T;V)}.$$

□

In the following, we consider the problem

$$\boxed{\begin{cases} \text{To } u_0 \in H \text{ and } f \in L^2(0,T;V^*) \text{ find } u \in W(0,T) \text{ with} \\ u' + \mathcal{A}\,u = f \text{ in } L^2(0,T;V^*), & (\star) \\ u(0) = u_0. \end{cases}} \tag{P}$$

Remark. 3.1.6 (Well-definedness of the initial condition)
Since $u \in W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$ by Theorem 2.2.1 3, the initial condition has to be understood to be attained in H.

Remark. 3.1.7 For $u \in W(0,T)$ we find $u' = f - \mathcal{A}\,u \in L^2(0,T;V^*) \hookrightarrow L^1(0,T;V^*)$ by Theorem 1.2.5 8. Theorem 2.1.3 implies that u is an absolutely continuous function $u\colon [0,T] \to V^*$. Since V^* is reflexive, by Theorem 1.2.4 u is classically differentiable almost everywhere. Hence $(\star)$ is equivalent to $u'(t) + (\mathcal{A}\,u)(t) = f(t)$ in V^* almost everywhere in $(0,T)$.

Remark. 3.1.8 (Weak formulation) As $L^2(0,T;V^*) = (L^2(0,T;V))^*$, $(\star)$ is equivalent to

$$\int_0^T \langle\, u'(t), v(t)\,\rangle + \langle(\mathcal{A}\,u)(t), v(t)\,\rangle \,\mathrm{d}t = \int_0^T \langle\, f(t), v(t)\,\rangle \,\mathrm{d}t \qquad \forall v \in L^2(0,T;V). \tag{15}$$

Since $C_c^\infty(0,T) \otimes V$ is dense in $\mathcal{C}_c^\infty(0,T;V) \xhookrightarrow{\mathrm{d}} L^2(0,T;V)$ (Exercise!), we can restrict the test functions to $v(t) = \varphi(t)w$ with $\varphi \in \mathcal{C}_c^\infty(0,T)$ and $w \in V$. Hence (15) is equivalent to

$$\int_0^T \big(\langle\, u'(t), w\,\rangle + \langle(\mathcal{A}\,u)(t), w\,\rangle\big)\,\varphi(t)\,\mathrm{d}t = \int_0^T \langle\, f(t), w\,\rangle\,\varphi(t)\,\mathrm{d}t \qquad \forall \varphi \in \mathcal{C}_c^\infty(0,T),\ w \in V.$$

Theorem 2.1.1 now implies

$$\langle\, u'(t), w\,\rangle + a(t, u(t), w) = \langle\, f(t), w\,\rangle \qquad \forall w \in V \quad \text{almost everywhere in } (0,T).$$

Remark. 3.1.9 The function v being the generalised derivative of u is equivalent to the mapping $t \mapsto \langle\, u(t), w\,\rangle$ having the weak derivative $t \mapsto \langle\, v(t), w\,\rangle$ for every $t \in (0,T)$, where the derivative has to be interpreted in the weak sense.

Remark. 3.1.10 (Wlog a is uniformly bd., strongly pos. (Homework 3.2 (c)))
Using the transformation

$$\hat{u}(t) := e^{-\kappa t}u(t), \qquad \hat{f}(t) := e^{-\kappa t}f(t), \qquad \hat{a}(t,v,w) := a(t,v,w) + \kappa(v,w)$$

the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t), v) + a(t, u(t), v) = \langle\, f(t), v\,\rangle, \qquad v \in V,$$

where a fulfills the standard assumptions, is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{u}(t), v) + \hat{a}(t, \hat{u}(t), v) = \langle\, \hat{f}(t), v\,\rangle, \qquad v \in V,$$

where $\hat{a}(t,\cdot,\cdot)$ is a uniformly bounded, strongly positive bilinear form.

3.2 Existence and uniqueness of solutions

Theorem 3.2.1: Jacques-Louis Lions (1962)

Under the assumptions (A1) - (A4), the problem (P) is well-posed in the sense of Hadamard, that is, a unique solution exists and we have continuous dependence on the initial value and the right side.

Remark. 3.2.1 (Generalisation by Tartar/Temam)
In Theorem 3.2.1 we can allow $f \in L^2(0,T;V^*) \oplus L^1(0,T;H)$, i.e. $f = f_1 + f_2$ with $f_1 \in L^1(0,T;H)$ and $f_2 \in L^2(0,T;V^*)$.

Lemma 3.2.2 (A priori estimates: uniqueness and stability)
Under the assumptions (A1) - (A4) *the a priori estimate*

$$|u(t)|^2 + \mu \int_0^t \|u(s)\|^2 \,\mathrm{d}s \leqslant c\left(|u_0|^2 + \|f\|_{L^2(0,T;V^*)}\right) \tag{16}$$

holds for every solution $u \in W(0,T)$ of (P).

As usual, this is proven by testing the equation in an appropriate sense.

Proof. (1) We first show the estimate. Since $w \in W(0,T)$, we can test (P) by w:

$$\underbrace{\langle w'(t), w(t) \rangle}_{=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2} + \underbrace{\langle Aw(t), w(t) \rangle}_{\geqslant \mu\|w(t)\|^2 - \kappa|w(t)|^2} = \langle g(t), w(t) \rangle \leqslant \|g(t)\|_* \|w(t)\|.$$

For the first term we use Corollary 2.2.6 and for the second one we use Gårding's inequality as indicated above to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 + \mu\|w(t)\|^2 - \kappa|w(t)|^2 \leqslant \|g(t)\|_*\|w(t)\| \overset{(Y)}{\leqslant} \frac{1}{2\mu}\|g(t)\|_*^2 + \frac{\mu}{2}\|w(t)\|^2,$$

where (Y) is Young's inequality $ab \leqslant \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ for $\varepsilon > 0$. Combining alike terms yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 + \frac{\mu}{2}\|w(t)\|^2 - \kappa|w(t)|^2 \leqslant \frac{1}{2\mu}\|g(t)\|_*^2.$$

and this can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\kappa t}|w(t)|^2\right) + e^{-2\kappa t}\mu\|w(t)\|^2 \leqslant \frac{1}{\mu}e^{-2\kappa t}\|g(t)\|_*^2$$

by multiplying by $2e^{-2\kappa t}$ and absorbing the κ-term into the derivative in the first term.

Integrating with respect to time and multiplying with $e^{2\kappa t}$ we get

$$\begin{aligned} |w(t)|^2 + \int_0^t \underbrace{e^{2\kappa(t-s)}}_{\geqslant 1} \mu\|w(s)\|^2 \,\mathrm{d}s &\leqslant e^{2\kappa t}|w_0|^2 + \int_0^t e^{2\kappa(t-s)}\|g(s)\|_*^2 \,\mathrm{d}s \\ &\leqslant C\left(|w_0|^2 + \int_0^t \underbrace{e^{-2\kappa s}}_{\leqslant 1}\|g(s)\|_*^2 \,\mathrm{d}s\right), \end{aligned}$$

which proves the estimate, as we can upper bound the integral over $[0,t]$ on the RHS by the integral over $[0,T]$ and take $C := e^{2\kappa T}$.

(2) Assume that there exist two solutions of

$$\begin{cases} u' + Au = f_u, \\ u(0) = u_0. \end{cases} \quad \text{and} \quad \begin{cases} v' + Av = f_v, \\ v(0) = v_0. \end{cases}$$

Then $u - v$ solves (P) with $w := u - v$, $f := f_u - f_v$ and $w(0) = u_0 - v_0$. Via (16) we infer for almost every $t \in (0, T)$

$$|u(t) - v(t)|^2 + \mu \int_0^t \|u(s) - v(s)\|^2 \,\mathrm{d}s \leqslant C \left(|u_0 - v_0|^2 + \|f_u - f_v\|^2_{L^2(0,T;V*)} \right),$$

so the solution depends continuously on the initial values and the right hand side.

Thus if the initial conditions and the right hand sides coincide, then the solution is unique. □

Remark. 3.2.3 (Generalisation of the a-priori estimate (Exercise 3.1))
If instead $f \in L^2(0, T; V^*) \oplus L^1(0, T; H)$, then

$$|u(t)|^2 + \frac{\mu}{2} \int_0^t \|u(s)\| \,\mathrm{d}s \leqslant M(u_0, f)$$

holds for every solution $u \in W(0, T)$ of (P).

Proof. Testing the differential equation in (P) with the solution $u \in W(0, T)$ yields, as in the previous proof

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 + \mu\|u(t)\|^2 - \kappa|u(t)|^2 &\leqslant \|f_1(t)\|_*\|u(t)\| + |f_2(t)||u(t)| \\ &\overset{\text{(Y)}}{\leqslant} \frac{1}{2\mu}\|f_1(t)\|_*^2 + \frac{\mu}{2}\|u(t)\|^2 + |f_2(t)|(1 + |u(t)|^2)\end{aligned}$$

and thus by multiplying by 2 and combining like terms

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 + \mu\|u(t)\|^2 - 2\kappa|u(t)|^2 \leqslant \frac{1}{\mu}\|f_1(t)\|_*^2 + 2|f_2(t)|(1 + |u(t)|^2)$$

Integration in time provides

$$|u(t)|^2 + \mu \int_0^t \|u(s)\|^2 \,\mathrm{d}s \leqslant |u(0)|^2 + \frac{1}{\mu} \int_0^t \|f_1(s)\|_*^2 + |f_2(s)| \,\mathrm{d}s + 2 \int_0^t (|f_2(s)| + \kappa)|u(s)|^2 \,\mathrm{d}s.$$

By Gronwall's Lemma we have

$$|u(t)|^2 - \mu \int_0^t \|u(s)\|^2 \,\mathrm{d}s \leqslant \left(|u(0)|^2 + \frac{1}{\mu} \left(\|f_1\|^2_{L^2(0,T;V*)} + \|f_2\|_{L^1(0,T;H)} \right) \right) e^{2\|f_2\|_{L^1(0,T;H)} + 2T\kappa}.$$

□

We will now prove the existence part of Theorem 3.2.1 via time discretisation.

Proof. 1 Let $N \in \mathbb{N}$, $\tau := \frac{T}{N}$ be the step size and $t_n := n\tau$ be equidistant time step for $n \in \{1, \dots, N\}$. Then we consider the implicit Euler scheme: for $n \in \{1, \dots, N\}$ let

$$u^n := u(t_n), \qquad u'(t_n) \approx \frac{u^n - u^{n-1}}{\tau}$$

and for the right hand side use

$$f^n := \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(t) \,\mathrm{d}t \in V^*.$$

We consider the problem

$$\begin{cases} \text{To } u^{n-1} \text{ find } u^n \in V \text{ such that} \\ \frac{Iu^n - Iu^{n-1}}{\tau} + A(t_n)u^n = f^n, \qquad n \in \{1, \dots, N\}. \end{cases} \tag{17}$$

In the following we only consider $\kappa = 0$ and assume that A is independent of t (time), otherwise we would have to set $A(t_n) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} a(t, \cdot, \cdot) \,\mathrm{d}t$.

2 The approximate system (17) is well defined. For every $n \in \{1, \dots, N\}$ consider the problem in V^*

$$\left(\frac{1}{\tau} I + A\right) u^n = f^n + \frac{1}{\tau} I u^{n-1} \tag{18}$$

The operator $\frac{1}{\tau} I + A$ is a linear, bounded and strongly positive operator: for the last property observe

$$\left\langle \left(\frac{1}{\tau} I + A\right) v, v \right\rangle = \frac{1}{\tau}|v|^2 + \langle Av, v \rangle \geqslant \frac{1}{\tau}|v|^2 + \mu\|v\|^2 \geqslant \mu\|v\|^2.$$

For $\kappa > 0$, choose τ small enough, i.e. $\tau < \frac{1}{\kappa}$, then $\frac{1}{\tau} I + A$ is strongly positive.

We have $u_0 \in H$ and $u^{n-1} \in V$ and hence $u^{n-1} \in V^*$. By the Theorem of Lax-Milgram, there exists a unique u^n for every $n \in \{1, \dots, N\}$, that is, a solution to (18).

In the following we identify $Iu^n \leftrightarrow u^n$ and don't write the I anymore.

3 **A-priori estimates.** It holds that $\frac{1}{\tau}(u^n - u^{n-1}) + Au^n = f^n$. Testing with u^n implies

$$\frac{1}{\tau}(u^n - u^{n-1}, u^n) + \langle Au^n, u^n \rangle = \langle f^n, u^n \rangle \leqslant \|f^n\|_* \|u^n\|.$$

We use the following calculation rule:

$$(a - b, a) = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a - b|^2.$$

Applying this to the first term, using that $\langle Au^n, u^n \rangle \geqslant \mu\|u^n\|^2$ and using Young's inequality on the right hand side yields

$$\frac{1}{2\tau}\left(|u^n|^2 - |u^{n-1}|^2 + |u^n - u^{n-1}|^2\right) + \mu\|u^n\|^2 \overset{(Y)}{\leqslant} \frac{1}{2\mu}\|f^n\|_*^2 + \frac{\mu}{2}\|u^n\|^2.$$

Multiplying by 2τ and collecting alike terms yields

$$|u^n|^2 - |u^{n-1}|^2 + |u^n - u^{n-1}|^2 + \tau\mu\|u^n\|^2 \leqslant \frac{\tau}{\mu}\|f^n\|_*^2. \tag{19}$$

Summing from $i = 1$ to $m \in \{1, \dots, N\}$ and using a telescoping series in the first two summands we infer

$$|u^m|^2 - |u^0|^2 + \sum_{i=1}^m |u^i - u^{i-1}|^2 + \tau\mu \sum_{i=1}^m \|u^i\|^2 \leqslant \frac{\tau}{\mu} \sum_{i=1}^m \|f^i\|_*^2$$

By rearranging and estimating away the nonnegative terms this implies

$$|u^m|^2 \leqslant |u^0|^2 + \frac{\tau}{\mu} \sum_{i=1}^m \|f^i\|_*^2 \tag{20}$$

for any $m \in \{1, \dots, N\}$, which gives us a bound on the solution and also

$$\sum_{i=1}^N |u^i - u^{i-1}|^2 + \sum_{i=1}^N \tau\mu\|u^i\|^2 \leqslant |u^0|^2 + \frac{\tau}{\mu} \sum_{i=1}^N \|f^i\|_*^2, \tag{21}$$

which can be used to get a bound on the discrete derivative: additionally, we have

$$\left\|\frac{u^n - u^{n-1}}{\tau}\right\|_* \overset{(18)}{=} \|f^n - Au^n\|_* \leqslant \|f^n\|_* + \|Au^n\|_* \leqslant \|f^n\|_* + \beta\|u^n\|$$

and hence

$$\left\|\frac{u^n - u^{n-1}}{\tau}\right\|_*^2 \leqslant 2\|f^n\|_*^2 + 2\beta\|u^n\|^2, \tag{22}$$

as $(a+b)^2 \leqslant 2(a^2+b^2)$. Hence

$$\begin{aligned}\tau \sum_{i=1}^{N} \left\| \frac{u^i - u^{i-1}}{\tau} \right\|_*^2 &\overset{(22)}{\leqslant} 2\tau \sum_{i=1}^{N} \|f^i\|_*^2 + 2\beta\tau \sum_{i=1}^{N} \|u^i\|^2 \\ &\overset{(21)}{\leqslant} 2\tau \sum_{i=1}^{N} \|f^i\|_*^2 + 2\beta \left(\frac{1}{\mu}|u^0|^2 + \frac{\tau}{\mu^2} \sum_{i=1}^{N} \|f^i\|_*^2 \right) \\ &= \frac{2\beta}{\mu}|u^0|^2 + 2\tau \left(1 + \frac{\beta}{\mu^2}\right) \sum_{i=1}^{N} \|f^i\|_*^2. \end{aligned} \tag{23}$$

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4 **Constructing the approximate solutions.** In this step we will construct an approximate solution to the problem (P) using the solutions of the time-discretised problems. For $t \in (t_{n-1}, t_n]$ define

$$u_\tau(t) := u^n$$

and $u_\tau(0) = u^0$. Hence u_τ is piecewise constant. Let

$$\hat{u}_\tau(t) := u^{n-1} + (t - t_{n-1})\frac{u^n - u^{n-1}}{\tau} \qquad \text{and} \qquad f_\tau(t) := f^n \tag{24}$$

for $t \in (t_{n-1}, t_n]$ (f_τ is only defined almost everywhere, so we can neglect the value for $t = 0$).

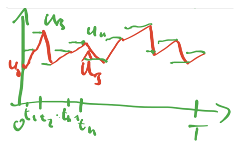

Fig. 3: $\hat{u}_\tau$ gives a (continuous) piecewise linear prolongation of u_τ.

As $\hat{u}_\tau$ is piecewise linear, it is LIPSCHITZ continuous and hence weakly differentiable almost everywhere with derivative

$$\hat{u}_\tau'(t) = \frac{u^n - u^{n-1}}{\tau} \tag{25}$$

for $t \in (t_{n-1}, t_n]$. Hence we can interpret the implicit EULER scheme via these functions and write

$$\boxed{\hat{u}_\tau'(t) + Au_\tau = f_\tau.}$$

5 **A priori estimate for the approximate solutions.** We now have to translate the estimates from before to the previous functions living in BOCHNER spaces. Let $N_\ell \to \infty$ for $\ell \to \infty$ with $N_\ell \in \mathbb{N}$ and $\tau_\ell := \frac{T}{N_\ell}$. Additionally, let the sequences $(u_{\tau_\ell})_{\ell\in\mathbb{N}}$, $(\hat{u}_{\tau_\ell})_{\ell\in\mathbb{N}}$ and $(f_{\tau_\ell})_{\ell\in\mathbb{N}}$ be constructed as above. For $\ell \in \mathbb{N}$, we choose a sequence $(u_\ell^0)_{\ell\in\mathbb{N}} \subset H$ such that $u_\ell^0 \to u_0$ as $\ell \to \infty$.

We have $u_{\tau_\ell}^0 \to u_0$ as $\ell \to \infty$ in H and $u_{\tau_\ell}(0) = u_\ell^0$. We want to show that $(f_{\tau_\ell})_{\ell\in\mathbb{N}}$ converges to f in $L^2(0,T;V^*)$. We have

$$\begin{aligned}\|f_{\tau_\ell}\|_{L^2(0,T;V*)}^2 &= \int_0^T \|f_{\tau_\ell}(t)\|_*^2 \, dt \overset{(24)}{=} \sum_{i=1}^{N_\ell} \int_{t_{i-1}}^{t_i} \|f^i\|_*^2 \, dt \\ &= \tau_\ell \sum_{i=1}^{N_\ell} \left\| \frac{1}{\tau_\ell} \int_{t_{i-1}}^{t_i} f(t) \, dt \right\|_*^2 dt \leqslant \sum_{i=1}^{N_\ell} \int_{t_{i-1}}^{t_i} \|f(t)\|_*^2 \, dt = \|f\|_{L^2(0,T;V*)}^2.\end{aligned}$$

The a priori estimates are independent of ℓ and we may deduce

$$\begin{aligned}\|u_{\tau_\ell}\|_{L^\infty(0,T;H)}^2 &= \max_{i=1}^{N_\ell} |u^i|^2 \overset{(20)}{\leqslant} |u_\ell^0|^2 + \frac{\tau_\ell}{\mu} \sum_{i=1}^{N} \|f^i\|_*^2, \\ \|u_{\tau_\ell}\|_{L^2(0,T;V)}^2 &= \tau_\ell \sum_{i=1}^{N_\ell} \|u_{\tau_\ell}\|^2 \overset{(20)}{\leqslant} \mu \left(|u_\ell^0|^2 + \frac{\tau_\ell}{\mu} \sum_{i=1}^{N} \|f^i\|_*^2 \right), \\ \|\widehat{u_{\tau_\ell}}\|_{L^\infty(0,T;H)}^2 &= \max_{i=1}^{N_\ell} |u^i|^2 \overset{(20)}{\leqslant} |u_\ell^0|^2 + \frac{\tau_\ell}{\mu} \sum_{i=1}^{N} \|f^i\|_*^2.\end{aligned}$$

6 **Extracting subsequences.** As $(u_{\tau_\ell})_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$ and $(\hat{u}_{\tau_\ell})_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)$, there exists a $u \in L^\infty(0,T;H)\cap L^2(0,T;V)$ such that

$$u_{\tau_\ell} \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(0,T;H)\cap L^2(0,T;V)$$

and there exists a $\hat{u}\in L^\infty(0,T;H)$ such that

$$\hat{u}_{\tau_\ell} \overset{*}{\rightharpoonup} \hat{u}.$$

One has to observe that one can do this steps one after the other to infer that this holds afterwards for *one* subsequence. Since $L^2(0,T;V)$ is reflexive and $L^\infty(0,T;H)\cong (L^1(0,T;H))^*$ is the dual of a separable Banach space, this follows from the Theorem of Banach-Alaoglu.

7 **Identify the limits u and $\hat{u}$.** For $t\in(t_{n-1},t_n]$ we have

$$\begin{aligned} |\hat{u}_{\tau_\ell}(t)-u_{\tau_\ell}(t)| &\overset{(24)}{=} \left|u^{n-1}+(t-t_{n-1})\frac{u^n-u^{n-1}}{\tau_\ell}-u^n\right| \\ &\leqslant \left|u^n-u^{n-1}\right| + \underbrace{\frac{|t-t_{n-1}|}{\tau_\ell}}_{\leqslant 1}|u^n-u^{n-1}| \leqslant 2|u^n-u^{n-1}|. \end{aligned}$$

Hence

$$\begin{aligned} \|\hat{u}_{\tau_\ell}-u_{\tau_\ell}\|^2_{L^2(0,T;H)} &= \sum_{k=1}^{N_\ell}\int_{t_{k-1}}^{t_k} |\hat{u}_{\tau_\ell}(t)-u_{\tau_\ell}(t)|^2\,\mathrm{d}t \leqslant 4\tau_\ell\sum_{i=1}^{N_\ell}|u^i-u^{i-1}|^2 \\ &\overset{(21)}{\leqslant} \tau_\ell\text{const} \xrightarrow{\ell\to\infty} 0. \end{aligned}$$

Hence $\hat{u}=u$ in $L^2(0,T;H)$.

8 **Time derivative.** For $t\in(t_{n-1},t_n]$ we have (25) and thus

$$\|\hat{u}'_{\tau_\ell}\|^2_{L^2(0,T;V^*)} = \tau_\ell\sum_{i=1}^{N}\left\|\frac{u^i-u^{i-1}}{\tau_\ell}\right\|_*^2 = \frac{1}{\tau_\ell}\sum_{i=1}^{N}\|u^i-u^{i-1}\|_*^2 \overset{(23)}{\leqslant} \text{const}.$$

We deduce that $(\hat{u}_{\tau_\ell})_\ell$ is bounded in $L^2(0,T;V^*)$ such that we may extract another subsequence such that $\hat{u}'_{\tau_\ell}\rightharpoonup v$ in $L^2(0,T;V^*)$.

We have to identify that $\hat{u}'=v$ in the weak sense. Let $\varphi\in\mathcal{C}_0^\infty(0,T)$ and $w\in V$ such that $\varphi w,\varphi' w\in L^2(0,T;V)$. By the weak convergence and the linearity of the weak derivative we have

$$\begin{aligned} \int_0^T \langle v(t),w\rangle\varphi(t)\,\mathrm{d}t + \int_0^T\langle \hat{u}(t),w\rangle\varphi'(t)\,\mathrm{d}t &= \int_0^T\langle \underbrace{v(t)-\hat{u}'_{\tau_\ell}(t)}_{\xrightarrow{\ell\to\infty}0},w\rangle\varphi(t)\,\mathrm{d}t \\ &\quad + \int_0^T\langle\underbrace{\hat{u}(t)-\hat{u}_{\tau_\ell}(t)}_{\xrightarrow{\ell\to\infty}0},w\rangle\varphi'(t)\,\mathrm{d}t. \end{aligned}$$

Hence $t\mapsto\langle v(t),w\rangle$ is the weak derivative of $t\mapsto\langle\hat{u},w\rangle$ for all $w\in V$. Hence $v=\hat{u}$ in $W(0,T)$.

9 **Passing to the limit.** We have $f_{\tau_\ell}\to f$ in $L^2(0,T;V^*)$ (Homework 3.1). We observe that $A\colon L^2(0,T;V)\to L^2(0,T;V^*)$ is linear and continuous. Hence A is weak-weak-continuous and thus $Au_{\tau_\ell}\rightharpoonup Au$ in $L^2(0,T;V^*)$. We find in $L^2(0,T;V^*)$

$$\hat{u}'_{\tau_\ell}+Au_{\tau_\ell}=f_{\tau_\ell} \qquad \text{in } L^2(0,T;V^*).$$

The three terms above converge weakly to u', Au and f in $L^2(0,T;V^*)$, respectively. This implies that u is a solution to the abstract equation.

10 **Identify the initial condition.** We have to show that $u(0) = u_0$ in H. We have $\hat{u}_{\tau_\ell} \rightharpoonup u$ in $W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$. The embedding is linear and continuous and hence weak-weak-continuous, so the weak convergence is translated to a pointwise weak convergence on H. There exists a linear continuous trace operator $\Gamma\colon W(0,T) \to H$, $\Gamma(u) := \Gamma_u := u(0)$ in H. Hence $\hat{u}_{\tau_\ell} \rightharpoonup u$ in $W(0,T)$ and so $\hat{u}_{\tau_\ell}(0) \rightharpoonup u(0)$ in H. We had the condition that $\hat{u}_{\tau_\ell}(0) = u_\ell^0 \to u_0$ in H. Hence the weak and strong limits have to coincide.

One can also show this more directly without the trace operator: we have $\hat{u}_{\tau_\ell}(T) = u^N$ and by an a priori estimate $|u^N| \leqslant$const. Hence we can extract another subsequence such that $\hat{u}_{\tau_\ell}(T) \rightharpoonup V_T$ in H. For any $v \in V$ and $\varphi \in C^1([0,T])$ we have (using the integration by parts rule for $W(0,T)$ functions)

$$\begin{aligned}
\underbrace{(\hat{u}_{\tau_\ell}(T), v)\varphi(T)}_{\rightharpoonup (v_T, v)\varphi(T)} - \underbrace{(\hat{u}_{\tau_\ell}(0), v)\varphi(0)}_{\rightharpoonup (u_0, v)\varphi(0)} &= \int_0^T \langle \hat{u}_{\tau_\ell}, v \rangle \varphi + \langle \hat{u}_{\tau_\ell}, v \rangle \varphi' \, \mathrm{d}t \\
&\to \int_0^T \langle u', v \rangle \varphi + \langle u, v \rangle \varphi' \, \mathrm{d}t \\
&= (u(T), v)\varphi(T) - (u(0), v)\varphi(0).
\end{aligned}$$

As v and φ are arbitrary and $V \overset{\mathrm{d}}{\hookrightarrow} H$, we deduce (we choose φ such that $\varphi(T) = 0$) $u(0) = u_0$ and $v_T = u(T)$. □

3.3 Error estimates

THEOREM 3.3.1: ERROR ESTIMATES

Let $u \in W(0,T)$ be the solution of (P) with $f \in L^2(0,T;V^*)$ and let additionally $(f - u')' \in L^2((0,T);V^*)$. Then the error estimate

$$|u(t_n) - u^n|^2 - \mu\tau \sum_{j=1}^{n} \|u(t_j) - u^j\|^2 \leqslant |u_0 - u^0|^2 + \frac{\tau^2}{3\mu} \|(f - u')'\|^2_{L^2(0,T;V^*)}$$

for all $n \in \{1, \dots, N\}$ holds for the implicit EULER time discretisation given in the previous proof.

Remark. 3.3.1 Error estimates provide convergence rates for time discretisation under additional regularity assumptions.

Proof. From $(f - u')' \in L^2(0,T;V^*)$ and $f - u' \in L^2(0,T;V^*)$ we conclude that $f - u' \in \mathrm{AC}([0,T];V^*)$ by Theorem 2.1.3. Let $e^n := u(t_n) - u^n$. From $\frac{u^n - u^{n-1}}{\tau} + Au^n = f^n$ we obtain

$$\begin{aligned}
\frac{e^n - e^{n-1}}{\tau} + Ae^n &= \frac{u(t_n) - u(t_{n-1})}{\tau} + Au(t_n) - f^n \\
&= \frac{u(t_n) - u(t_{n-1})}{\tau} - f^n + f(t_n) - u'(t_n) \\
&= \frac{1}{\tau} \int_{t_{n-1}}^{t_n} u'(s) \, \mathrm{d}s - \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(s) \, \mathrm{d}s + (f - u')(t_n).
\end{aligned}$$

For some BOCHNER integrable function g we have by the Fundamental Theorem of Calculus and FUBINI's theorem

$$\int_{t_0}^{t} g(t) - g(s) \, \mathrm{d}s = \int_{t_0}^{t} \int_{s}^{t} g'(r) \, \mathrm{d}r \, \mathrm{d}s = \int_{t_0}^{t} \int_{t_0}^{r} g'(r) \, \mathrm{d}s \, \mathrm{d}r = \int_{t_0}^{t} g'(r)(r - t_0) \, \mathrm{d}r.$$

Hence

$$\frac{e^n - e^{n-1}}{\tau} + Ae^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (f - u')'(s)(s - t_{n-1})\,\mathrm{d}s =: \rho^n \in V^* \tag{26}$$

We call ρ^n the consistency error, the error one makes when inserting the real solution in the discretised scheme. Since e^n and e^{n-1} are solutions to the discrete scheme, we can deduce a priori estimates as beforehand:

$$|e^n|^2 + \mu\tau \sum_{i=1}^m \|e^i\|^2 \leqslant |e^0|^2 + \frac{\tau}{\mu}\sum_{i=1}^m \|\rho^i\|_*^2.$$

We have, using HÖLDER's inequality,

$$\begin{aligned}
\tau \sum_{n=1}^m \|\rho^n\|_*^2 &\overset{(26)}{=} \tau \sum_{n=1}^m \left\| \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (f-u')'(s)(s-t_{n-1})\,\mathrm{d}s \right\|_*^2 \\
&\overset{\triangle\neq}{\leqslant} \tau \sum_{n=1}^m \frac{1}{\tau^2}\left(\int_{t_{n-1}}^{t_n} \|(f-u')'\|_*(s-t_{n-1})\,\mathrm{d}s \right)^2 \\
&\overset{(H)}{\leqslant} \frac{1}{\tau}\sum_{n=1}^m \underbrace{\left(\int_{t_{n-1}}^{t_n} (s-t_{n-1})^2\,\mathrm{d}s \right)}_{=\frac{1}{3}(t_n-t_{n-1})^3=\frac{1}{3}\tau^3} \left(\int_{t_{n-1}}^{t_n} \|(f-u')'\|_*^2\,\mathrm{d}s \right) \\
&= \frac{\tau^2}{3}\|(f-u')'\|_{L^2(0,T;V^*)}^2.
\end{aligned}$$

□

Remark. 3.3.2 (Motivation to deduce additional regularity.)
Differentiating $u' = f - \mathcal{A}\,u$ with respect to time yields

$$u'' = f - (\mathcal{A}\,u)' = f' - \mathcal{A}'\,u - \mathcal{A}\,u' = f' - \mathcal{A}'\,u - \mathcal{A}\,f + \mathcal{A}^2\,u. \tag{27}$$

With the compatibility condition $u'(0) = f(0) - (\mathcal{A}\,u)(0)$ and

$$f \in W^{1,2}(0,T;V^*) := \{u \in L^2(0,T;V^*) : \exists u' \in L^2(0,T;V^*)\}$$

and $\mathcal{A}' \colon L^2(0,T;V) \to L^2(0,T;V^*)$ we may rewrite the equation above using $u' = v$: v solves the linear first order differential equation

$$v' + \mathcal{A}\,v = f' - \mathcal{A}'\,u \in L^2(0,T;V^*).$$

By Theorem 3.2.1, v is the unique solution in $W(0,T)$ of this equation if the initial condition coincides, which is the case by assumption. The compatibility condition is needed to infer that $u' = v$ in $W(0,T)$. Without this condition, we deduce only regularity for u away from 0. One can deduce regularity for u multiplied with a function that vanishes in zero and this is called instant smoothing property.

3.4 Regularity by Rothe's method

17.05.2021

We will show regularity via a time approximation scheme which is called ROTHE's method (which can also be used to prove existence). In the last section we had the assertion that we can provide better (that is, higher order) error estimates when assuming additional regularity of the solution. In a sense, we are trying to get better estimates using the structure of the PDE (e.g. linearity).

Lemma 3.4.1 (Additional regularity $\boldsymbol{u \in W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;V)}$)
Let $u_0 \in V$ and $f \in W^{1,2}(0,T;V^*)$ and the operator $A'(t)\colon V \to V^*$ be continuous and linear for all $t \in (0,T)$. If the compatibility condition $A(u_0) - f(0) \in H$ holds, then the solution $u \in W(0,T)$ to (P) admits the additional regularity $u \in W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;V)$. Especially, it holds

$$\hat{u}_n' \rightharpoonup u' \qquad \text{in } L^\infty(0,T;H), \tag{28}$$

$$\hat{u}_n' \rightharpoonup u' \qquad \text{in } L^2(0,T;V). \tag{29}$$

Remark. 3.4.2 If the compatibility conditions is not assumed, we only infer regularity for the function multiplied by a weight exploding at zero (so-called instant smoothing property).

General idea:

By corollary 2.2.6 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u'|^2 = \langle u'', u' \rangle \overset{(27)}{=} \underbrace{-\langle A'u + Au', u' \rangle}_{\leqslant \|A'u\|_* \|u'\| - \langle Au', u' \rangle} + \underbrace{\langle f', u' \rangle}_{\leqslant \|f'\|_* \|u'\|}$$

and hence by the Cauchy-Schwarz and Goarding's inequality

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u'|^2 + \mu\|u'\|^2 &\overset{(14)}{\leqslant} \kappa|u'|^2 + \underbrace{\|A'u\|_*}_{\leqslant \tilde{\beta}\|u\|}\|u'\| + \|f'\|_*\|u'\| \\
&\leqslant \kappa|u'|^2 + (\tilde{\beta}\|u\| + \|f'\|_*)\|u'\| \\
&\overset{(Y)}{\leqslant} \kappa|u'|^2 + \frac{\mu}{2}\|u'\|^2 + \frac{1}{2\mu}(\tilde{\beta}\|u\| + \|f'\|_*)^2 \\
&\overset{(\star)}{\leqslant} \kappa|u'|^2 + \frac{\mu}{2}\|u'\|^2 + \frac{1}{\mu}\left(\tilde{\beta}^2\|u\|^2 + \|f'\|_*^2\right),
\end{aligned}$$

where in the second inequality we use the continuity and linearity of A' with some constant $\tilde{\beta} > 0$ which is independent of t:

$$\|A'(t)\|_{L(V;V^*)} \leqslant \tilde{\beta}$$

for all $t \in (0,T)$ and in the last inequality $(\star)$ we use $(a+b)^2 \leqslant 2(a^2+b^2)$. Collecting alike terms yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u'|^2 + \frac{\mu}{2}\|u'\|^2 - \kappa|u'|^2 \leqslant \frac{1}{\mu}\left(\tilde{\beta}^2\|u\|^2 + \|f'\|_*^2\right)$$

Lemma 3.4.3
Let $u_0 \in V$, $f \in L^2(0,T;H)$ and for all $t \in [0,T]$ let the operator $A(t)\colon V \to V^*$ be continuous, linear and self-adjoint, that is, $\langle A(t)v, w \rangle = \langle A(t)w, v \rangle$ for all $v, w \in V$. Then $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ and (28) and (29) hold.

Proof. Exercise. □

Proof. (of lemma 3.4.1) Consider the backward (or: implicit) Euler scheme and its solution $\{u^{(n)}\}_{n=0}^{N_\tau}$ which solves

$$\frac{u^{(n)} - u^{(n-1)}}{\tau} + A^{(n)}u^{(n)} - \left(\frac{u^{(n-1)} - u^{(n-2)}}{\tau} + A^{(n-1)}u^{(n-1)}\right) = f^{(n)} - f^{(n-1)}.$$

Multiplying by the discrete time derivative $\frac{u^{(n)}-u^{(n-1)}}{\tau^2}$ (up to a factor of $\frac{1}{\tau}$) yields

$$\frac{1}{2\tau}\left|\frac{u^{(n)}-u^{(n-1)}}{\tau}\right|^2-\frac{1}{2\tau}\left|\frac{u^{(n-1)}-u^{(n-2)}}{\tau}\right|^2+\frac{1}{2\tau}\left|\frac{u^{(n)}-2u^{(n-1)}+u^{(n-2)}}{\tau}\right|^2$$
$$+\left\langle A^{(n)}\frac{u^{(n)}-u^{(n-1)}}{\tau},\frac{u^{(n)}-u^{(n-1)}}{\tau}\right\rangle+\left\langle (A^{(n)}-A^{(n-1)})u^{(n-1)},\frac{u^{(n)}-u^{(n-1)}}{\tau^2}\right\rangle$$
$$\leqslant\frac{1}{\mu}\left\|\frac{f^{(n)}-f^{(n-1)}}{\tau}\right\|^2+\frac{\mu}{4}\left\|\frac{u^{(n)}-u^{(n-1)}}{\tau}\right\|^2$$

Applying Goardings inequality yields (we set $u^{(-1)}:=u^{(0)}$)

$$\left|\frac{u^{(n)}-u^{(n-1)}}{\tau}\right|^2+\sum_{i=1}^{n}\left|\frac{u^{(i)}-2u^{(i-1)}+u^{(i-2)}}{\tau}\right|^2$$
$$+\mu\tau\sum_{i=1}^{n}\left\|\frac{u^{(i)}-u^{(i-1)}}{\tau}\right\|^2-\kappa\tau\sum_{i=1}^{n}\left|\frac{u^{(i)}-u^{(i-1)}}{\tau}\right|^2$$
$$\leqslant\left|\frac{u^{(1)}-u^{(0)}}{\tau}\right|^2+\tau\sum_{i=1}^{n}\frac{2}{\mu}\left(\left\|\frac{f^{(i)}-f^{(i-1)}}{\tau}\right\|_*^2+\left\|\frac{f^{(i)}-f^{(i-1)}}{\tau}\right\|_{L(V,V*)}^2\|u^{(i-1)}\|_V^2\right). \tag{30}$$

Here we already see that this very much looks as beforehand: in the previous proof we didn't incorporate the κ term but now we can use the discrete GRONWALL lemma. First, observe (this is given by the scheme, when we define $f^{(0)}=f(0)$)

$$\frac{u^{(0)}-u^{(1)}}{\tau}=f(0)-A(u_0)\in H$$

We can show as before that the RHS of (30) is bounded. The last term from (30) can be estimated as follows:

$$\tau\sum_{i=1}^{N}\frac{2}{\mu}\left\|\frac{f^{(i)}-f^{(i-1)}}{\tau}\right\|_*^2+\left\|\frac{A^{(i)}-A^{(i-1)}}{\tau}\right\|_{L(V,V*)}\|u^{(i-1)}\|_V^2$$
$$\leqslant C_\mu\left(\|f'\|^2_{L^2(0,T;V*)}+\sup_{t\in[0,T]}\|A'(t)\|_{L(V,V*)}\|u\|_{L^2(0,T;V)}\right)=:\tilde{C}$$

We get

$$\max_{n\in\{1,\dots,N\}}\left\|\frac{u^{(n)}-u^{(n-1)}}{\tau}\right\|^2\leqslant C_1|f(0)-A(u_0)|^2+\tilde{C}+\kappa\tau\sum_{i=1}^{n}\left\|\frac{u^{(i)}-u^{(i-1)}}{\tau}\right\|^2$$

and

$$\tau\sum_{i=1}^{N}\left\|\frac{\|u^{(i)}-u^{(i-1)}\|}{\tau}\right\|^2\leqslant C_1|f(0)-A(u_0)|^2+\tilde{C}+\kappa\tau\sum_{i=1}^{n}\left\|\frac{u^{(i)}-u^{(i-1)}}{\tau}\right\|^2$$

by a discrete GRONWALL argument.

Summary: We only differentiated the equation, which, one this discrete level amounts to dividing by τ and subtracting the previous step from the current one. We tested the equation with u' and deduced the same upper estimates as one would for the continuous equation. We actually need to show this on the discrete level and then pass to the limit to make this rigorous, otherwise we don't know that these calculations are allowed. □

3.5 Examples for linear PDEs

Consider the heat equation, which models the distribution of heat in a material. The rough modelling idea is the following: consider a small volume $\omega\in\mathbb{R}^d$. Let u be the thermal heat equation

energy. Then
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega u\,\mathrm{d}x = \int_\omega \partial_t u\,\mathrm{d}x - \int_{\partial\omega} n\nabla u\,\mathrm{d}\mathbb{S} = \int_\omega \partial_t u - \Delta u\,\mathrm{d}x$$
where n is the outer normal and the second term represents what flows out of ω and in the second step we use Gauss' Theorem (Divergence Theorem).

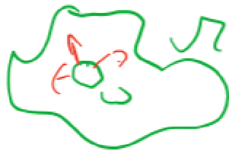

Fig. 4: We look at how u behaves in ω. The red arrows visualise $\int_{\partial\omega} n\nabla u\,\mathrm{d}x$.

Integrating yields
$$\begin{cases} \partial_t u - \Delta u = f, & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ n \cdot \nabla u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases}$$

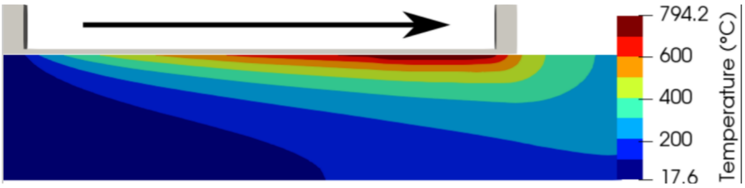

Fig. 5: The solution gives the distribution of a heat profile.

Example. 3.5.1 (Magnetoquasistatic approximation of Maxwell's equation)
The quasi-magneto-static (QMS) approximation to the macroscopic Maxwell equations is sensible when we are considering good conductors and slowly varying external magnetic fields. These lead to induced electric fields, which in turn stir up so-called eddy currents, inside the conductors. For example, these induced currents might be used to heat up the material.

In the considered setting the displacement current $\partial_t D$ and the charge density ρ are negligible and the Maxwell equations in differential/local form become
$$\begin{aligned} \nabla \cdot D &= 0, & \nabla \times E &= -\partial_t B, \\ \nabla \cdot B &= 0, & \nabla \times H &= J, \end{aligned} \tag{31}$$
where
$$\begin{aligned} E &= (\text{electric field}), & H &= (\text{magnetic field}), \\ D &= (\text{electric flux density}), & B &= (\text{magnetic flux density}), \\ J &= (\text{current density}) \end{aligned}$$
and we assume the linear material relations
$$D = \varepsilon E, \qquad B = \mu H \qquad J = \sigma E.$$

To get a better understanding of the physics, we write (31) in integral form. We have
$$\begin{aligned} \int_{\partial\Omega} D \cdot \mathrm{d}S &= 0, & \int_{\partial A} E \cdot \mathrm{d}\ell &= -\int_A \partial_t B\,\mathrm{d}x, \\ \int_{\partial\Omega} B \cdot \mathrm{d}S &= 0, & \int_{\partial A} H \cdot \mathrm{d}\ell &= \int_A J\,\mathrm{d}x \end{aligned}$$
in a considered volume Ω or on a surface A. The interpretations are
$$\begin{aligned} &(\text{electric flux out of } \Omega) = 0, \\ &\qquad (\text{electric field integral around } A) = -(\text{change of } B \text{ over time thru } A), \\ &(\text{magnetic flux out of } \Omega) = 0, \\ &\qquad (\text{magnetic field integral around } A) = (\text{current thru } A). \end{aligned}$$

The QMS approximation breaks the symmetry in the Maxwell equations; in the full equations we have

$\nabla \times H = J + \partial_t D,$

(magnetic field integral around A) = (current thru A) + (change of D over time thru A),

hence change of electric field implies change of magnetic field, which implies change of electric field implying change of magnetic field and so on. In QMS the scheme is that change of the external magnetic field implies change of the electric field in the material, but this electric field does not noticeably change the outer magnetic field again.

The relation $\nabla \cdot B = 0$ motivates the introduction of the vector potential A with

$$B = \nabla \times A \quad \text{and} \quad \nabla \cdot A = 0.$$

Then

$$\nabla \times (E + \partial_t A) = 0, \qquad \text{hence} \qquad E = -\nabla\varphi - \partial_t A,$$

where φ is a scalar potential.

In fact, when considering an induction coil creating an outer magnetic field to heat up a piece of metal as in [ADG⁺19], we find $J_{\text{ext}} = -\sigma\nabla\varphi$, where J_{ext} is the induced source current, which only lives in the induction coil. It can be precomputed and thus serve as a given right-hand side.

We may then derive a set of equations for A inside the conductor Ω. Assuming sufficient regularity we find

$$\nabla \times \frac{1}{\mu}\nabla \times A = \nabla \times \frac{1}{\mu}B = \nabla \times H = J = \sigma E = -\sigma(\nabla\varphi + \partial_t A) \qquad \text{in} \qquad \Omega \times (0,T),$$

thus

$$\sigma\partial_t A + \nabla \times \frac{1}{\mu}\nabla \times A = J_{\text{ext}} \quad \text{and} \quad \nabla \cdot A = 0 \qquad \text{in} \qquad \Omega \times (0,T).$$

These equations are then to be supplemented by boundary conditions, e.g.

$$A \times n = 0 \qquad \text{on} \qquad \partial\Omega \times (0,T),$$

as in [ADG⁺19]. ◇

Example. 3.5.2 (Poisson equation (Exercise 4.1 (i)))
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. The weak formulation of the Poisson equation with Neumann boundary conditions

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g, & \text{on } \partial\Omega \end{cases}$$

for suitable functions f, g is

$$\langle Au, v \rangle = \langle \tilde{f}, v \rangle \qquad \forall v \in V := H^1(\Omega),$$

where

$$A\colon V \to V^*, \qquad \langle Au, v \rangle := \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x$$
$$\tilde{f}\colon V \to \mathbb{R}, \qquad \langle \tilde{f}, v \rangle := \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_{\partial\Omega} g(x)v(x)\,\mathrm{d}\sigma$$

Then A is bounded as $\langle Au, v \rangle \leqslant \|\nabla u\|_{L^2(\Omega)^d}\|\nabla v\|_{L^2(\Omega)^d} \leqslant \|u\|_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}$, but A is not strongly positive (choose $u \equiv C$ and $f = g = 0$) and hence there is no unique solvability.

We require that $f \in H^{-1}(\Omega)$ and $g \in (H^{\frac{1}{2}}(\partial\Omega))^*$, so that $\tilde{f}$ is well-defined.

However, we can choose the solution space

$$V := \left\{ v \in H^1(\Omega) : \int_\Omega v(x)\,\mathrm{d}x = 0 \right\},$$

on which A is coercive by the Poincaré-Wirtinger inequality. ◇

Example. 3.5.3 (Heat equation (Exercise 4.1 (ii)))
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. The weak formulation of the initial value problem of the heat equation with Neumann boundary conditions

$$\begin{cases} u_t - \Delta u = f, & \text{on } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = g, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, & \text{in } \Omega \end{cases}$$

for $u_0 \in H$, $f \in L^2(0,T;V^*) \oplus L^1(0,T;H)$ and $g \in L^2(0,T;(H^{\frac{1}{2}}(\partial\Omega))^*)$ is: find $u \in W(0,T)$ such that

$$\begin{aligned} \int_0^T \langle u'(t), v(t) \rangle \,\mathrm{d}t + \int_0^T \langle Au(t), v(t) \rangle \,\mathrm{d}t &= \int_0^T \int_{\partial\Omega} v(x,t) g(x,t)\,\mathrm{d}\sigma(x)\,\mathrm{d}t \\ &\quad + \int_0^T \langle f(t), v(t) \rangle \,\mathrm{d}t \qquad \forall v \in L^2(0,T;V). \end{aligned}$$

We can define

$$\tilde{f} \colon [0,T] \to V^*, \qquad \langle \tilde{f}(t), v \rangle = \int_{\partial\Omega} v(x) g(x,t)\,\mathrm{d}\sigma(x) + \langle f(t), v \rangle_{V^* \times V}, \qquad v \in H^1(\Omega),$$

then $\tilde{f} \in L^2(0,T;V^*) \oplus L^1(0,T;H)$. Lastly, $\mathcal{A}$ is continuous and satisfies Garding's inequality since

$$\langle Au, u \rangle = \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x = \int_\Omega |\nabla u(x)|^2 + u^2(x)\,\mathrm{d}x - \int_\Omega u^2(x)\,\mathrm{d}x = \|u\|^2 - |u|^2.$$

By the Theorem of Lions, has a unique solution $u \in W(0,T)$.

We note that thanks to the Goarding inequality, we obtain uniqueness of a solution with values in $H^1(\Omega)$ which we failed to accomplish in the stationary problem in the previous example due to the missing coercivity of a. ◇

Example. 3.5.4 (Biharmonic equation (Homework 4.2))
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. We consider the initial-boundary value problem

$$\begin{cases} \partial_t u(t,x) = -\Delta^2 u(t,x), & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ \frac{\partial}{\partial \nu} u(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{32}$$

where $\Delta^2 = \Delta\Delta$ denotes the bi-Laplace operator.

The bi-Laplace operator appears in various problems of linear elasticity, for example when looking at small displacements of a plate (whereas the Laplacian describes the behaviour of a membrane).

We first consider the stationary problem

$$\begin{cases} \Delta^2 u(x) = 0, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \\ \frac{\partial}{\partial \nu} u(t) = 0, & \text{on } \partial\Omega. \end{cases}$$

We choose the spaces $V := H_0^2(\Omega)$ and $H := L^2(\Omega)$ and recall that $|v|_{2,2}^2 := \int_\Omega |\Delta v(x)|^2 \,\mathrm{d}x$ is an equivalent norm on V, that is, there exists $c, C > 0$ such that for all $v \in V$ we have $c\|v\| \leqslant |v| \leqslant C\|v\|$. Next, let

$$A\colon V \to V^*, \qquad \langle Au, v \rangle := \int_\Omega \Delta u(x) \Delta v(x) \,\mathrm{d}x,$$

which is linear and bounded as by Cauchy-Schwarz we have

$$\langle Au, v \rangle \leqslant \|\Delta u\|_{L^2(\Omega)}^2 \|\Delta v\|_{L^2(\Omega)}^2 = |u|_{2,2}^2 |v|_{2,2}^2 \leqslant C^4 \|u\|^2 \|v\|^2$$

Furthermore, it is coercive:

$$\langle Au, u \rangle = \int_\Omega |\Delta u(x)|^2 \,\mathrm{d}x = |u|_{2,2}^2 \geqslant c^2 \|u\|^2.$$

Hence there exists a weak solution to the stationary problem by the Theorem of Lax-Milgram, as to obtain the weak formulation we have to integrate by parts twice and use that the boundary terms vanish.

Now considering the instationary formulation, the definitions don't change and by the coerciveness of A we also get that $\mathcal{A}$ fulfills Goarding's inequality. By Lion's theorem, we get wellposedness if $u_0 \in H = L^2(\Omega)$. $\diamond$

4 Nonlinear first order evolution equations with monotone operator

4.1 Preliminaries

Nonlinear equations will be the main focus of this course. In the beginning, the nonlinearities will be monotone and hence not so severe.

Introduction to nonlinear PDEs

Example. 4.1.1 (p-Laplacian)
Consider

$$\begin{cases} \partial_t u - \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = f, & \text{in } \Omega \times (0,T) \\ u = 0, & \text{on } \partial\Omega \times (0,T) \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{33}$$

with $u_0 \in H := L^2(\Omega)$, $V := W_0^{1,p}(\Omega) \overset{d}{\hookrightarrow} H \hookrightarrow V^*$ and right hand side $f \in (L^p(0,T;V))^* \cong L^{p'}(0,T;V^*)$ with $\frac{1}{p'} + \frac{1}{p} = 1$. Then the problem (33) can be formulated as

$$\text{find } u \in W_p(0,T) \text{ such that } \begin{cases} u' + \mathcal{A}(u) = f, & \text{in } L^{p'}(0,T;V^*), \\ u(0) = u_0, & \text{in } H. \end{cases}$$

with $A =$TODO. ◇

Remark. 4.1.2 (Assumptions for the nonlinear case) Let $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $V \subset H \subset V^*$ a Gelfand triple.

Let $A_0, B \colon V \to V^*$ and $A := A_0 + B$ with $(\mathcal{A}_0 v)(t) := A_0 v(t)$, $(\mathcal{B} v)(t) := Bv(t)$ for $v \colon [0,T] \to V$ and $\mathcal{A} := \mathcal{A}_0 + \mathcal{B}$ with

$\mathcal{A}_0 \colon L^p(0,T;V) \to L^{p'}(0,T;V^*)$ being monotone and hemi-continuous

$\mathcal{B} \colon L^p(0,T;V) \to L^{p'}(0,T;V^*)$ being strongly (or totally) continuous

$\mathcal{A} \colon L^p(0,T;V) \to L^{p'}(0,T;V^*)$ being coercive with $\mu > 0$ and $\lambda \geqslant 0$ such that

$$\langle \mathcal{A} v, v \rangle = \int_0^T \langle \mathcal{A} v(t), v(t) \rangle \, \mathrm{d}t \geqslant \mu \|v\|^p_{L^p(0,T;V)} - \lambda$$

for all $v \in L^p(0,T;V)$ and bounded with $\beta \geqslant 0$ such that

$$\|\mathcal{A} v\|_{L^{p'}(0,T;V^*)} \leqslant \beta(1 + \|v\|^{p-1}_{L^p(0,T;V)}) \quad \forall v \in L^p(0,T;V).$$

Lemma 4.1.3 (Properties of $W_p(0,T)$: completeness, IBP rule, embedding)
Let $p \in (1,\infty)$. Then

$$W_p(0,T) := \{u \in L^p(0,T;V) : \exists \in u' \in L^{p'}(0,T;V^*)\}$$

equipped with the norm

$$\|u\|_{W_p(0,T)} := \|u\|_{L^p(0,T;V)} + \|u'\|_{L^{p'}(0,T;V^*)}$$

is a Banach space. We have

$$W_p(0,T) \hookrightarrow \mathcal{C}([0,T];H)$$

and the rule of integration by parts:

$$\int_s^t \langle u'(\tau), v(\tau) \rangle + \langle v'(\tau), u(\tau) \rangle \mathrm{d}\tau = (u(t), v(t)) - (u(s), v(s))$$

for all $v, w \in W_p(0,T)$ and all $s, t \in [0,T]$. Finally, it also holds

$$\mathcal{C}^\infty([0,T]; V) \overset{d}{\hookrightarrow} W_p(0,T).$$

Proof. Analogous to the proof of Theorem 2.2.1. □

Remark. 4.1.4 Another norm of $W_p(0,T)$ is

$$\|u\|_{W_p(0,T)} := \left(\|u\|^r_{L^p(0,T;V)} + \|u'\|^r_{L^q(0,T;V^*)} \right)^{\frac{1}{r}}$$

for any $r \geqslant 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. This norm is equivalent since it is the r-norm of the vector $(\|u\|, \|u'\|) \in \mathbb{R}^2$.

Lemma 4.1.5 (Properties generalise to abstract operators)
Let $A_0 \colon V \to V^*$ be monotone, hemi-continuous and coercive with $\tilde{\mu} > 0$, $\tilde{\lambda} \geqslant 0$ such that

$$\langle A_0 v, v \rangle \geqslant \tilde{\mu}\|v\|^p - \tilde{\lambda} \qquad \forall v \in V$$

and bounded with $\tilde{\beta} \geqslant 0$ such that $\|A_0 u\|_{V^*} \leqslant \tilde{\beta}(1 + \|u\|^{p-1})$ for all $u \in V$. Then $\mathcal{A}_0 \colon L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is monotone, hemi-continuous and bounded with $\beta \geqslant 0$ such that

$$\| \mathcal{A}_0 u\|_{L^{p'}(0,T;V^*)} \leqslant \beta(1 + \|u\|^{p-1}_{L^p(0,T;V)})$$

and coercive with $\mu > 0$ and $\lambda \geqslant 0$ such that

$$\langle \mathcal{A}_0 u, u \rangle \geqslant \mu\|u\|^p_{L^p(0,T;V)} - T\lambda \qquad \forall u \in L^p(0,T;V).$$

Proof. 1 First we show that A_0 maps Bochner measurable functions to Bochner measurable functions. Let u be Bochner measurable, then there exists a sequence of simple functions $(u_n = \sum_{i=1}^{N_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}})_{n \in \mathbb{N}}$ such that $u_n(t) \to u(t)$ pointwise almost everywhere in $(0,T)$. Then

$$\begin{aligned}(\mathcal{A}_0 u_n)(t) = A_0 \left(\sum_{i=1}^{N_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}}(t) \right) &= \sum_{i=1}^{N_n} A_0(u_i^{(n)}) \mathbb{1}_{E_i^{(n)}}(t) + A_0(0) \mathbb{1}_{(E_i^{(n)})^\complement}(t) \\ &= \sum_{i=1}^{N_n} A_0(u_i^{(n)}) \mathbb{1}_{E_i^{(n)}}(t) + A_0(0) \mathbb{1}_{\left(\bigcup_{i=1}^{N_n} E_i^{(n)}\right)^\complement}(t)\end{aligned}$$

where the Lebesgue measurable sets $(E_i^{(n)})_{i=1}^{N_n}$ are pairwisely disjoint for every $n \in \mathbb{N}$. As E_i is measurable, so is $(E_i^{(n)})^\complement$ for all $i \in \{1, \ldots, N_n\}$. Hence $(A_0 u_n)_{n \in \mathbb{N}}$ is a sequence of simple functions.

A_0 being monotone and hemi-continuous implies A_0 being demicontinuous. Hence

$$\langle (A_0 u_n)(t), w \rangle \to \langle (A_0 u)(t), w \rangle$$

for all $w \in V$ and almost every $t \in (0,T)$. Hence $(A_0 u)(t)$ is weakly Bochner measurable and by Pettis' Theorem $(A_0 u)(t)$ is Bochner measurable.

2 TODO: show that A_0 is monotone and hemi-continuous (Exercise 5.2)

Now we show the estimates. We have

$$\begin{aligned}\|A_0 u\|^{p'}_{L^{p'}(0,T;V*)} &= \int_0^T \|A_0 u(t)\|^{p'}_{V*}\,\mathrm{d}t \leqslant \int_0^T \left(\tilde{\beta}(1+\|u(t)\|^{p-1})\right)^{p'}\mathrm{d}t\\ &= \tilde{\beta}^{p'}\int_0^T (1+\|u(t)\|^{p-1})^{p'}\,\mathrm{d}t \leqslant 2^{p'-1}\tilde{\beta}^{p'}\left(T+\int_0^T \|u(t)\|^p\,\mathrm{d}t\right)\\ &= \frac{1}{2}(2\tilde{\beta})^{p'}\left(T+\|u\|^p_{L^p(0,T;V)}\right),\end{aligned}$$

as for $p>1$ we have $|a+b|^p \leqslant 2^{p-1}(|a|^p+|b|^p)$ (this is because $x\mapsto |x|^p$ is convex and thus $\left|\frac{a+b}{2}\right|^p \leqslant \frac{|a|^p+|b|^p}{2}$).

Hence

$$A_0\colon L^p(0,T;V)\to L^{p'}(0,T;V^*)$$

is well defined and bounded such that for all $v\in L^p(0,T;V)$ we have

$$\|A_0 v\|_{L^{p'}(0,T;V*)} \leqslant \beta\left(1+\|v\|^{p-1}_{L^p(0,T;V)}\right)$$

as $p\frac{1}{p'} = \frac{p}{\frac{p}{p-1}} = p-1$ and by choosing $\beta := \tilde{\beta}\max(1,T^{\frac{1}{p'}})$.

TODO: show coercivity (Exercise 5.2) □

Remark. 4.1.6 We can show a similar statement for $B\colon V\to V^*$, but it is more involved. Furthermore, this Lemma can be generalised to the case where A, A_0 and B are time-dependent, if the estimates hold uniformly in t.

Lemma 4.1.7 (Properties transfer from $\mathcal{A}$ to A (Homework 5.1))
Let $V\hookrightarrow H\hookrightarrow V^*$ be a Gelfand *triple and let $p,q\in(1,\infty)$ be* Hölder *conjugates. Let $\mathcal{A}\colon L^p(0,T;V)\to L^q(0,T;V^*)$ be given by $\mathcal{A}u(t)=Au(t)$ for an operator $A\colon V\to V^*$. If $\mathcal{A}$ is bounded / monotone / coercive / hemicontinuous / strongly continuous, then A is, too.*

4.2 Existence

Remark. 4.2.1 (Generalisation to pseudomonotone Operators)
This can be generalised to pseudomonotone operators $A\colon L^p(0,T;V)\to L^{p'}(0,T;V^*)$, that is, to operators A that are bounded and fulfil pseudomonotone

$$\begin{cases} u_k \rightharpoonup u \text{ in } L^p(0,T;V)\\ \limsup\limits_{k\to\infty}\langle A(u_k),u_k-u\rangle \leqslant 0\end{cases} \implies \begin{cases}\langle A(u),u-v\rangle \leqslant \liminf\limits_{k\to\infty}\langle A(u_k),u_k-v\rangle\\ \forall v\in L^p(0,T;V).\end{cases}$$

Theorem 4.2.1: Main Theorem on Monotone nonlinear PDEs

Assuming the standard assumptions, for every $u_0\in H$ and $f\in L^{p'}(0,T;V^*)$, there exists a solution $u\in W_p(0,T)$ with

$$\begin{cases} u'+\mathcal{A}(u)=f & \text{in } L^{p'}(0,T;V^*),\\ u(0)=u_0, & \text{in } H.\end{cases} \tag{34}$$

Proof. Previously we used time discretisation but now we want to use a Galerkin space discretisation to prove existence.

1. **Galerkin scheme.** As V is separable, there exists a Galerkin basis $\{\varphi_i\}_{i\in\mathbb{N}} \subset V$ such that any finite selection of the φ_i is linearly independent. Let

$$V_m := \operatorname{span}(\varphi_1, \ldots, \varphi_m).$$

We also have

$$\overline{\bigcup_{m=1}^{\infty} V_m}^{\|\cdot\|} = V. \qquad \text{(Completeness in the limit)}$$

2. **Approximate scheme.** Consider

$$(u_m', v) + \langle Au_m, v \rangle = \langle f, v \rangle \qquad \forall v \in V_m. \tag{P_m}$$

We write $u_m(t) := \sum_{i=1}^m u_i^m(t)\varphi_i$. Then we can write $u_m'(t) := \sum_{i=1}^m (u_i^m)'(t)\varphi_i$, where $u_i^m\colon [0,T] \to \mathbb{R}$ for all $m \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$.

To solve (P_m) it suffices to solve a system of ODEs

$$\begin{cases} (u_m'(t), \varphi_i) + \langle (Au_m)(t), \varphi_i \rangle = \langle f(t), \varphi_i \rangle & \forall i \in \{1, \ldots, m\} \\ u_m(0) = u_0^m, \end{cases}$$

where u_0^m is such that $u_0^m \in V_m$ for all $m \in \mathbb{N}$ and $u_0^m \to u_0$ in H with $u_0^m = \sum_{i=1}^m u_{0,i}^m \varphi_i^m$. (For instance we can choose $u_0^m = P_m u_0$, where $P_m\colon H \to V_m$ is the projection from H onto V_m.)

Let $U_m(t) := (u_1^m(t), \ldots, u_m^m(t))^\mathsf{T} \in \mathbb{R}^m$, $U_m^0 := (u_{0,1}^m, \ldots, u_{0,m}^m)^\mathsf{T} \in \mathbb{R}^m$ and

$$(M_m)_{i,j=1}^m := \big((\varphi_i, \varphi_j)\big)_{i,j=1}^m,$$

where $(\cdot,\cdot)$ is the scalar product in H. Then M_m is invertible as the $\{\varphi_i\}_{i\in\mathbb{N}}$ are linearly independent in the sense described in step 1. Lastly, let

$$\big(F_m(t, u_m)\big)_j := \langle f(t), \varphi_j \rangle - \langle Au_m, \varphi_j \rangle = \langle f(t), \varphi_j \rangle - \left\langle A\left(\sum_{i=1}^m u_i^m \varphi_i\right), \varphi_j \right\rangle.$$

Hence (P_m) is equivalent to

$$\boxed{\begin{cases} U_m'(t) = M_m^{-1} F_m(t, U_m(t)), \\ U_m(0) = U_m^0. \end{cases}} \tag{35}$$

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3. **Existence of approximate solutions.** We show that F_m is a Carathéodory function and has a dominating function. For the former observe that since A is monotone and bounded, it is demicontinuous, so $\langle A\left(\sum_{j=1}^m y_j\varphi_j\right), \varphi_k \rangle$ converges strongly in $\mathbb{R}$. For the dominating function, we find for $y \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$

$$\begin{aligned} F_m(t, y)_k &= \langle f(t), \varphi_k \rangle - \left\langle A\left(\sum_{j=1}^m y_j\varphi_j\right), \varphi_k \right\rangle \\ &\leqslant \|\varphi_k\|_V \left(\|f\|_{L^{p'}(0,T;V*)} + \left\| A\left(\sum_{j=1}^m y_j\varphi_j\right)\right\|_* \right) \\ &\leqslant \|\varphi_k\|_V \left(\|f\|_{L^{p'}(0,T;V*)} + C(1 + \|y\|_{\mathbb{R}^m}^{p-1}) \right) \end{aligned}$$

such that for any compact subset $K \subset \mathbb{R}^m \times [0,T]$ there exists a dominating function $\ell_K\colon [0,T] \to [0,\infty)$ with

$$|M_m^{-1} F_m(t,y)| \leqslant \ell_K(t) \qquad \forall (t,y) \in K.$$

By the Theorem of Carathéodory, there exists a solution U_m to (35). Without loss of generality, we may assume U_m to be the maximal prolongation on $[0,T_m)$. Hence U_m is absolutely continuous on every compact subinterval of $[0,T_m)$. Hence U_m is almost everywhere classically differentiable and thus also in the weak sense. Theorem of Carathéodory

4 **A-priori estimates.** We may choose $v_m = u_m$ in (P_m) and obtain

$$\langle u_m'(t), u_m(t) \rangle + \langle Au_m(t), u_m(t) \rangle = \langle f(t), u_m(t) \rangle.$$

With the product rule, coercivity and Young's inequality we find

Young's inequality states $|xy| \leqslant \varepsilon|x|^p + \frac{1}{p'}(p\varepsilon)^{1-p'}|y|^{p'}$ if $\frac{1}{p} + \frac{1}{p'} = 1$ and $\varepsilon > 0$.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m(t)|^2 + \mu\|u_m(t)\|^p \leqslant \|f(t)\|_*\|u_m(t)\| \overset{(Y)}{\leqslant} \frac{\mu}{2}\|u_m(t)\|^p + C\|f(t)\|_*^{p'} + \lambda.$$

Integrating in time and multiplying by 2 yields

$$|u_m(t)|^2 + \mu\int_0^t \|u_m(s)\|^p \,\mathrm{d}s \leqslant |u_0^m|^2 + 2\int_0^t C\|f(s)\|_* \,\mathrm{d}s + 2t\lambda, \tag{36}$$

implying that

$$\|u_m\|^2_{L^\infty(0,T;H)} \leqslant |u_0^m|^2 + 2C_1\left(\|f\|^{p'}_{L^p(0,T;V*)} + 1\right)$$

and

$$\mu\|u_m\|^p_{L^p(0,T;V)} \leqslant |u_0^m|^2 + 2C_1\left(\|f\|^{p'}_{L^p(0,T;V*)} + 1\right).$$

We would like to deduce some estimate of u_m' in (P_m) but this is rather difficult for any general Galerkin approximation since (P_m) is a rather weak form: we don't have any estimate for the time derivative unless we assume any regularity of the Galerkin space. We would need that the H-projection onto the Galerkin spaces is V-stable, which gives a estimate for the projection in the V-norm. Otherwise we don't get any information on the time derivative because from (P_m) - we only have information in V_m^*, which is strictly larger than V^*. That is why we can't deduce any bound on the time derivative. If we would do a time discretisation, this might be easier, because we test with the whole space.

5 **Extracting subsequences** We know

- $u_m \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;H)$,
- $u_m \rightharpoonup u$ in $L^p(0,T;V)$,
- $u_m(t) \rightharpoonup \theta_T$ in H,
- $\|Au_m\|_{L^{p'}(0,T;V*)} \leqslant \beta\left(1 + \|u_m\|^{p-1}_{L^p(0,T;V)}\right) \leqslant C$ and thus

$$Au_m \rightharpoonup a \qquad \text{in } L^{p'}(0,T;V^*).$$

6 **Weak derivative.** We use the structure of the Galerkin scheme to identify the weak time derivative. We want to to show that $u' = f - a$ and thus also has the same regularity in $L^{p'}(0,T;V^*)$. To this end consider a test function $\varphi \in \mathcal{C}_c^\infty$ and $v_n \in V_n$ with $n \leqslant m$ such that $V_n \subset V_m$ (this embedding allows a decoupling of the index of the test function and the index of the solution). Then

$$\int_0^T (u_m, v_n)\varphi' \,\mathrm{d}t + \int_0^T \langle Au_m, v_n \rangle \varphi \,\mathrm{d}t = \int_0^T \langle f, v_n \rangle \varphi \,\mathrm{d}t.$$

Passing to the limit with $m \to \infty$, we observe

$$-\int_0^T (u, v_n)\varphi' \,\mathrm{d}t + \int_0^T \langle a, v_n \rangle \varphi \,\mathrm{d}t = \int_0^T \langle f, v_n \rangle \varphi \,\mathrm{d}t.$$

Since $m \to \infty$ this holds for all $v_n \in \bigcup_{i=1}^\infty V_i$. The set $\bigcup_{i=1}^\infty V_i$ is dense in U such that

$$-\int_0^T (u, v)\varphi' \,\mathrm{d}t = \int_0^T \langle f - a, v \rangle \varphi \,\mathrm{d}t$$

and hence $u' = f - a$ in $L^{p'}(0, T; V^*)$ and thus $u \in W_p(0, T)$ and hence we may use the integration by parts formula $(\star)$.

7 **Identifying the initial value.** Let $\varphi \in \mathcal{C}^1([0, T])$ and $v \in V_n$ with $n \leqslant m$. Then

$$\begin{aligned}(u(t) - u_m(T), v)\varphi(T) - (u(0) - u_m(0), v)\varphi(0) &\overset{(\star)}{=} \int_0^T \langle u'(t) - u_m'(t), v \rangle \varphi(t) \\ &\qquad + (u(t) - u_m(t), v)\varphi'(t) \,\mathrm{d}t \\ &= \int_0^T \langle f - a - (f - Au_m), v \rangle \varphi(t) \\ &\qquad + (u(t) - u_m(t), v)\varphi'(t) \,\mathrm{d}t \\ &= \int_0^T \langle Au_m - a, v \rangle \varphi(t) \\ &\qquad + (u(t) - u_m(t), v)\varphi'(t) \,\mathrm{d}t \\ &\xrightarrow{m \to \infty} 0\end{aligned}$$

as $Au_m \rightharpoonup a$ in $L^{p'}(0, T; V^*)$ and $u_m \rightharpoonup u$ in $L^p(0, T; V)$. Thus

$$(u(t) - \theta_T, V)\varphi(T) - (u(0) - u_0, v)\varphi(0)$$

for all $v \in V$ via a density argument, as $u_m(t) \rightharpoonup \theta$ and $u_m(0) \to u_0$ in H. Thus for all $v \in V$ and $\varphi \in \mathcal{C}^1([0, T])$ we have $u(T) = \theta_T$ and $u(0) = u_0$.

(For the identification $u_m \rightharpoonup u$ in $L^1(0, T; V^*)$ would suffice to identify $u(0)$ with u_0 in V^*. But by $u_m(0) \to u_0$ in H it is known that $u(0) \in H$.)

8 **Identifying $a = Au$ by Minty's trick.** We use the assumption that $A = A_0 + B$, where $B\colon L^p(0, T; V) \to L^{p'}(0, T; V^*)$ is strongly continuous. From $u_m \rightharpoonup u$ in $L^p(0, T; V)$ we deduce that $Bu_m \to Bu$ in $L^{p'}(0, T; V^*)$. Additionally, we observe that (by the integration by parts formula)

$$\int_0^T \langle u_m'(t), u_m(t) \rangle \,\mathrm{d}t = \frac{1}{2}|u_m(T)|^2 - \frac{1}{2}|u_0^m|^2.$$

Due to the weak convergence $u_m(T) \rightharpoonup \theta_T = u(T)$ in H we have (by a corollary of Hahn-Banach)

$$\liminf_{m \to \infty} \frac{1}{2}|u_m(T)|^2 \geqslant \frac{1}{2}|u(T)|^2.$$

Now we have to apply Minty's Trick. We have

$$\begin{aligned}\int_0^T \langle A_0 u_m(t), u_m(t) \rangle \,\mathrm{d}t &= \int_0^T \langle f(t), u_m(t) \rangle - \langle Bu_m(t), u_m(t) \rangle - \langle u_m'(t), u_m(t) \rangle \,\mathrm{d}t \\ &= \int_0^T \langle f(t), u_m(t) \rangle - \langle Bu_m(t), u_m(t) \rangle \,\mathrm{d}t \\ &\qquad - \frac{1}{2}|u_m(t)|^2 + \frac{1}{2}|u_0^m|^2\end{aligned}$$

as u_m solves the discretised problem. For $m \to \infty$ we observe (as we identified the initial condition)

$$\begin{aligned}\limsup_{m\to\infty} \int_0^T \langle A_0 u_m(t), u_m(t) \rangle \mathrm{d}t &\leqslant \int_0^T \langle f(t), u(t) \rangle - \langle Bu(t), u(t) \rangle \mathrm{d}t \\ &\quad - \frac{1}{2}|u(t)|^2 + \frac{1}{2}|u(0)|^2 \\ &= \int_0^T \langle f(t), u(t) \rangle - \langle Bu(t), u(t) \rangle \mathrm{d}t - \langle u'(t), u(t) \rangle \\ &= \int_0^T \langle a(t), u(t) \rangle - \langle Bu(t), u(t) \rangle \mathrm{d}t.\end{aligned}$$

As A_0 is monotone, we have for any $w \in L^p(0,T;V)$

$$\begin{aligned}\int_0^T \langle A_0 u_m(t), u_m(t) \rangle \mathrm{d}t &= \int_0^T \langle A_0 u_m(t) - A_0 w(t), u_m(t) - w(t) \rangle \mathrm{d}t \\ &\quad + \int_0^T \langle A_0 w(t), u_m(t) - w(t) \rangle + \langle A_0 u_m(t), w(t) \rangle \mathrm{d}t \\ &\geqslant \int_0^T \langle A_0 w(t), u_m(t) - w(t) \rangle + \langle A_0 u_m(t), w(t) \rangle \mathrm{d}t \\ &\xrightarrow{m\to\infty} \int_0^T \langle A_0 w(t), u(t) - w(t) \rangle - \langle a - Bu(t), w(t) \rangle \mathrm{d}t\end{aligned}$$

and hence

$$\int_0^T \langle A_0 w(t), u_m(t) - w(t) \rangle \mathrm{d}t \leqslant \int_0^T \langle a(t) - Bu(t), u(t) - w(t) \rangle \mathrm{d}t.$$

We continue with MINTY's trick. Choosing $w(t) := u(t) \pm \alpha v(t)$ with $v \in L^p(0,T;V)$ and $\alpha > 0$ yields

$$\frac{1}{\alpha} \int_0^T \langle A_0(u(t) \pm \alpha v(t)), \mp\alpha v(t) \rangle \mathrm{d}t \leqslant \frac{1}{\alpha} \int_0^T \langle a(t) - Bu(t), \mp\alpha v(t) \rangle \mathrm{d}t.$$

As $\alpha \to 0$ we conclude by demicontinuity that

$$\int_0^T \langle A_0 u(t), v(t) \rangle \mathrm{d}t = \int_0^T \langle a(t) - Bu(t), v(t) \rangle \mathrm{d}t$$

for all $v \in L^p(0,T;V)$. This implies

$$a(t) = (A_0 + B)(u(t)) = A(u(t))$$

in $L^{p'}(0,T;V^*)$. □

Remark. 4.2.2 (Hemi-continuity and radial continuity) The hemi-continuity can be generalised to radial continuity, one only needs that the mapping

$$s \mapsto \langle A_0(u + sv), v \rangle$$

is continuous on $[0,1]$ for all $u, v \in L^p(0,T;V)$.

Remark. 4.2.3 (Monotonicity and pseudomonotonicity)
As usual, the monotone operator A_0 with strongly continuous perturbation B, $A = A_0 + B$ can be replaced by a pseudomonotone operator. In this scenario, we have as beforehand

$$\begin{aligned}\limsup_{k\to\infty} \langle A(u_k), u_k - v \rangle &= \limsup_{k\to\infty} \langle A(u_k), u_k - v_k \rangle + \lim_{k\to\infty} \langle A(u_k), v_k - v \rangle \\ &\leqslant \langle f - u', u - v \rangle\end{aligned}$$

for a sequence $(v_k)_{k\in\mathbb{N}} \subset L^p(0,T;V)$ with $v_k \to v$ in $L^p(0,T;V)$ (exists because the Galerkin scheme is complete in the limit) such that

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u \rangle \leqslant 0.$$

By pseudomonotonicity we have

$$\langle A(u), u - v \rangle \leqslant \liminf_{k\to\infty} \langle A(u_k), u_k - v \rangle \leqslant \limsup_{k\to\infty} \langle A(u_k), u_k - v \rangle \leqslant \langle f - u', u - v \rangle$$

for all $v \in L^p(0,T;V)$, implying $A(u) = f - u'$.

4.3 Uniqueness and continuous dependence

Lemma 4.3.1 (Uniqueness)
Assuming the standard assumptions and $B = 0$, the solution of (34) is unique and the whole sequence of approximate solutions converges to u.

Proof. Let $u, v \in W_p(0,T)$ be two solutions to the problems

$$\begin{cases} u' + \mathcal{A}(u) = f, & \text{in } L^{p'}(0,T;V^*), \\ u(0) = u_0 & \text{in } H \end{cases} \quad \text{and} \quad \begin{cases} v' + \mathcal{A}(v) = f, & \text{in } L^{p'}(0,T;V^*), \\ v(0) = u_0 & \text{in } H \end{cases}$$

As $\mathcal{A}$ is monotone we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u-v|^2 &= \langle u' - v', u - v \rangle \\ &\leqslant \langle u' - v', u - v \rangle + \langle \mathcal{A}\,u - \mathcal{A}\,v, u - v \rangle = \langle f - f, u - v \rangle = 0 \end{aligned}$$

and hence (by integration)

$$|u(t) - v(t)|^2 \leqslant |u_0 - u_0|^2 = 0$$

for all $t \in [0,T]$. □

Lemma 4.3.2 (Continuous dependence)
Assuming the standard assumptions and $B = 0$, the solution operator of the problem (34)

$$L^{p'}(0,T;V^*) \times H \to \mathcal{C}([0,T];H), \qquad (f, u_0) \mapsto u$$

is continuous on bounded sets.

Proof. Let $u, v \in W_p(0,T)$ be the two solutions to the problems

$$\begin{cases} u' + \mathcal{A}(u) = f, & \text{in } L^{p'}(0,T;V^*), \\ u(0) = u_0 & \text{in } H \end{cases} \quad \text{and} \quad \begin{cases} v' + \mathcal{A}(v) = g, & \text{in } L^{p'}(0,T;V^*), \\ v(0) = v_0 & \text{in } H \end{cases} \tag{37}$$

for $f, g \in L^{p'}(0,T;V^*)$ and $u_0, v_0 \in H$.

We have, as before,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u-v|^2 &\leqslant \langle u' - v', u - v \rangle + \langle Au - Av, u - v \rangle = \langle f - g, u - v \rangle \\ &\leqslant \|f-g\|_*\|u-v\| \leqslant \|f-g\|_*(\|u\| + \|v\|). \end{aligned}$$

Integrating we find

$$\frac{1}{2}|u(t)-v(t)|^2 \leqslant \frac{1}{2}|u_0-v_0|^2 + \|f-g\|_{L^{p'}(0,T;V*)}(\|u\|_{L^p(0,T;V)} + \|v\|_{L^p(0,T;V)}).$$

From the a-priori estimate (36) we observe

$$\mu\|u\|^p_{L^p(0,T;V)} \leqslant |u_0|^2 + \frac{C}{\mu}\|f\|^{p'}_{L^{p'}(0,T;V)} + \lambda T$$

and similar for v and hence

$$\|u\|_{L^p(0,T;V)} \leqslant M(u_0,f) \qquad \text{and} \qquad \|v\|_{L^p(0,T;V)} \leqslant M(v_0,g).$$

Thus

$$\frac{1}{2}|u(t)-v(t)|^2 \leqslant \frac{1}{2}|u_0-v_0|^2 + \|f-g\|_{L^{p'}(0,T;V*)}\left(M(u_0,f)+M(v_0,g)\right)$$

for all $t \in [0,T]$. We may take the supremum over $t \in [0,T]$, implying the assertion. The solution operator is even Lipschitz-continuous in the initial values and $\mathcal{C}^{0,\frac{1}{2}}$-Hölder-continuous in the right-hand side. □

We now prove two different results about continuous dependence.

If $\mathcal{B} \neq 0$ but we require a condition similar to monotonicity on $\mathcal{A}$, we have Lipschitz-continuous dependence on the data.

Theorem 4.3.1: Lipschitz continuous dependence

Let the standard assumptions be fulfilled. Additionally, we require that $A\colon [0,T]\times V \to V^*$ fulfills

$$\langle \mathcal{A}(t)v - \mathcal{A}(t)w, v-w\rangle \geqslant -g(t)|v-w|^2$$

for $v,w \in V$ and $g \in L^1(0,T)$. The operator $\mathcal{A}\colon L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is then given by $(\mathcal{A}u)(t) = Au(t)$. Then the solution operator of the problem (34)

$$L^2(0,T;H)\times H \to \mathcal{C}([0,T];H), \qquad (f,u_0)\mapsto u$$

is Lipschitz-continuous.

Proof. Exercise. □

Theorem 4.3.2: Continuous dependence (p-monotone setting, $B=0$)

Let the standard assumptions be fulfilled with $\boldsymbol{A} = \boldsymbol{A_0}\colon V \to V^*$ being p-monotone, that is, there exists a $\tilde{\mu} > 0$ such that

$$\langle \mathcal{A}v - \mathcal{A}w, v-w\rangle \geqslant \tilde{\mu}\|v-w\|^p \qquad \forall v,w \in V.$$

Then the solution operator of the problem (34)

$$L^{p'}(0,T;V^*)\times H \to \mathcal{C}([0,T];H)\cap L^p(0,T;V), \qquad (f,u_0)\mapsto u$$

is continuous.

Proof. First we observe that

$$\begin{aligned}\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle_{L^{p'}(0,T;V*)\times L^{p'}(0,T;V*)} &= \int_0^T \langle Au - Av, u - v \rangle_{V*\times V} \,\mathrm{d}t \\ &\geqslant \tilde{\mu} \int_0^T \|u - v\|^p \,\mathrm{d}t = \tilde{\mu}\|u - v\|^p_{L^p(0,T;V)}.\end{aligned}$$

Assume that $u, v \in W_p(0,T)$ solve the problems (37). Then

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u - v|^2 + \tilde{\mu}\|u - v\|^p_{L^p(0,T;V)} &\leqslant \langle u' - v', u - v \rangle + \langle Au - Av, u - v \rangle = \langle f - g, u - v \rangle \\ &\leqslant \|f - g\|_* \|u - v\|.\end{aligned}$$

Integration and applying Young's inequality implies

$$\begin{aligned}\frac{1}{2}|u(t) - v(t)|^2 + \tilde{\mu}\int_0^t \|u - v\|^p \,\mathrm{d}s &\leqslant \int_0^t \|f - g\|_* \|u - v\| \,\mathrm{d}s + \frac{1}{2}|u_0 - v_0|^2 \\ &\leqslant C\int_0^t \|f - g\|_*^{p'} \,\mathrm{d}s + \frac{2}{\tilde{\mu}}\int_0^t \|u - v\|^2 \,\mathrm{d}s + \frac{1}{2}|u_0 - v_0|^2\end{aligned}$$

and thus (by taking suprema)

$$|u(t) - v(t)|^2 + \mu\int_0^T \|u - v\|^2 \,\mathrm{d}s \leqslant \frac{1}{2}|u_0 - v_0|^2 + C\int_0^T \|f - g\|_*^{p'} \,\mathrm{d}s$$

for all $t \in [0,T]$ and thus

$$|u(t) - v(t)|^2_{\mathcal{C}([0,T],H)} + \|u - v\|^p_{L^p(0,T;V)} \leqslant C\left(|u_0 - v_0|^2 + \|f - g\|^{p'}_{L^{p'}(0,T;V*)}\right)$$

and thus the continuous dependence is shown. We are Lipschitz-continuous with respect to the initial values again and some Hölder-continuity. $\square$

Example. 4.3.3 We consider the initial value (and boundary value ?) problem

$$\begin{cases} u_t - (\rho(u_x))_x = f, & \text{on } (a,b)\times(0,T), \\ u(a,\cdot) = u(b,\cdot) = 0 & \text{on } (0,T), \\ u(\cdot,0) = u_0 & \text{on } (a,b) \end{cases}$$

with

$$\rho\colon\ \mathbb{R}\to\mathbb{R}, \qquad z \mapsto \begin{cases} \frac{z}{\sqrt{|z|}}, & \text{if } |z| \in (0,1), \\ z & \text{else.} \end{cases}$$

We show that for a suitable choice of u_0 and f there exists a unique solution $u \in W(0,T)$. It is lengthy, but not difficult to show that

$$\big(\rho(x) - \rho(y)\big)\big(x - y\big) \geqslant \frac{1}{2}|x - y|^2 \qquad \forall x, y \in \mathbb{R}\,. \tag{38}$$

By the arithmetic mean inequality we have for $z \in (-1,1)$

$$|\rho(z)| = \sqrt{|z|} \leqslant \frac{1}{2} + \frac{1}{2}|z|$$

and hence $|\rho(z)| \leqslant 1 + |z|$ for all $z \in \mathbb{R}$. By putting $y = 0$ in (38), we obtain $x\rho(x) \geqslant \frac{1}{2}|x|^2 \geqslant 0$ for all $x \in \mathbb{R}$.

We choose the space $V := H^1_0(a,b)$ and $H := L^2(a,b)$ and define

$$A\colon V \to V^*, \qquad \langle Av, w \rangle_{V*\times V} := \int_a^b \rho(v'(x))w'(x)\,\mathrm{d}x,$$

which comes from testing the differential equation

$$(\rho(u_x(x)))_x = f(x), \qquad x \in (a,b) \qquad \diamond$$

with w and integrating by parts, where the boundary term vanishes due to the homogeneous NEUMANN boundary conditions.

4.4 Strong Convergence by embedding and monotonicity

In this section we want to deduce more information about the approximate sequence u_m. In this existence proof we deduced weak convergence $u_m \rightharpoonup u$ in $L^p(0,T;V)$, but we want to deduce strong convergence. The strong convergence can then be used to deduce better properties of the approximation. If this PDE was coupled with another differential equation, we might need strong convergence in order to pass to the limit. 31.05.2021

We either deduce strong convergence in a weaker topology, like in $L^p(0,T;H)$ (achieved by a compact embedding) or we assume additional monotonicity.

The following Theorem is omnipresent in the field of nonlinear evolutionary equations, since every time we want to pass to the limit in some nonlinear term, which has to be of lower order (in some sense) and we don't have any monotonicity, we need the Theorem of LIONS-AUBIN to deduce additional strong convergence in time.

THEOREM 4.4.1: LIONS-AUBIN [Rou13]

Let $T > 0$ and $1 < r, s < \infty$ and $V_1 \overset{c}{\hookrightarrow} V_0 \hookrightarrow V_{-1}$ BANACH spaces such that $V_{\pm 1}$ are reflexive. Then

$$\{u \in L^r(0,T;V_1) : \exists u' \in L^s(0,T;V_{-1})\} \overset{c}{\hookrightarrow} L^r(0,T;V_0),$$

where the first space is equipped with the norm $\|\cdot\|_{L^r(0,T;V_1)} + \|\cdot'\|_{L^s(0,T;V_{-1})}$.

Remark. 4.4.1 There are many different generalisations of this result. We may consider $s = 1$ or even further v' just being a measure. It also suffices if V_{-1} is a locally convex topological space.

Corollary 4.4.2 ($V \overset{c}{\hookrightarrow} H \implies W(0,T) \overset{c}{\hookrightarrow} L^2(0,T;H)$)
If $V \overset{c}{\hookrightarrow} H \subset V^*$ is a GELFAND triple, then $W(0,T) \overset{c}{\hookrightarrow} L^2(0,T;H)$.

Lemma 4.4.3 (V-stability of H-projection grants strong convergence)
Let the standard assumptions be fulfilled. If the H-projection onto the GALERKIN spaces is V-stable, that is, there exists a $C > 0$ such that $\|P_m v\| \leqslant C\|v\|$ for all $v \in V$, where $P_m \colon H \to V_m$ is the orthogonal projection onto V_m, then there exists a subsequence such that $u_{m'} \to u$ in $L^q(0,T;H)$ for all $q \in [1,\infty)$.

Lemma 4.4.4 (From interpolation inequality)
Let $1 \leqslant p < q \leqslant \infty$, $\theta \in (0,1)$ and $r \in [p,q]$ with

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}. \tag{39}$$

Then for all $u \in L^q(0,T;H)$ we have

$$\|u\|_{L^r(0,T;H)} \leqslant \|u\|^{\theta}_{L^q(0,T;H)} \|u\|^{1-\theta}_{L^p(0,T;H)}$$

Proof. We have by Hölder's inequality with (due to (39) Hölder conjugate) exponents $\tilde{p} := \frac{r\theta}{q}$ and $\tilde{q} := \frac{r(1-\theta)}{q}$

$$\begin{aligned}\|u\|^r_{L^r(0,T;H)} &= \int_0^T |u(t)|^r \,\mathrm{d}t = \||u|^{r\theta}|u|^{r(1-\theta)}\|_{L^1(0,T;\mathbb{R})} \\ &\overset{\text{(H)}}{\leqslant} \||u|^{r\theta}\|_{L^{\frac{q}{r\theta}}(0,T;\mathbb{R})} \||u|^{r(1-\theta)}\|_{L^{\frac{p}{(1-\theta)r}}(0,T;\mathbb{R})} = \||u|\|^{r\theta}_{L^q(0,T;\mathbb{R})} \||u|\|^{r(1-\theta)}_{L^p(0,T;\mathbb{R})}.\end{aligned}$$ □

Hence if we have some boundedness in the L^q space and we have some convergence in the L^p space, we get convergence in L^r by lemma 4.4.4.

Lemma 4.4.5 (Second interpolation inequality (Exercise 5.1 (ii)))
Let $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ be a Gelfand triple. Then for $1 \leqslant p, q \leqslant \infty$ and $\theta \in (0,1)$ we have

$$\|u\|_{L^r(0,T;H)} \leqslant \|u\|^{\frac{1}{2}}_{L^p(0,T;V)} \|u\|^{\frac{1}{2}}_{L^q(0,T;V^*)} \qquad \text{for all } u \in L^p(0,T;V) \cap L^q(0,T;V^*)$$

with $\frac{2}{r} = \frac{1}{p} + \frac{1}{q}$.

Proof. (of lemma 4.4.3) Recall the approximate scheme from the existence proof

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_m(t), v) + \langle Au_m(t), v \rangle = \langle f(t), v \rangle \qquad \forall v \in V_m$$

with the condition

$$u_m(0) = u^m \qquad \text{in } V_m.$$

Using the stability, we can now circumvent the difficulty in the existence proof stemming from the fact that we didn't know enough about the time derivative.

Via stability of the projection, we deduce for any $v \in V$ that $P_m v \in V_m$ and thus

$$\langle u_m'(t), P_m v \rangle + \langle Au_m(t), P_m v \rangle = \langle f(t), P_m v \rangle \qquad \forall v \in V.$$

Hence by the stability there exists a $C > 0$ independent of m such that

$$|\langle u_m'(t), P_m v \rangle| \leqslant \|f(t)\|_* \|P_m v\| + \|Au_m(t)\|_* \|P_m v\| \leqslant C\left(\|f(t)\|_* + \|Au_m(t)\|_*\right)\|v\|.$$

Hence

$$\begin{aligned}\|u_m'\|_{L^{p'}(0,T;V^*)} = \|u_m'\|_{(L^p(0,T;V))^*} &= \sup_{\substack{v \in L^p(0,T;V) \\ \|v\|_{L^p(0,T;V)}=1}} \left| \int_0^T \langle u_m'(t), P_m v \rangle \mathrm{d}t \right| \\ &\overset{\triangle\neq}{\leqslant} \sup_{\substack{v \in L^p(0,T;V) \\ \|v\|_{L^p(0,T;V)}=1}} C \int_0^T \left(\|f(t)\|_* + \|Au_m(t)\|_*\right)\|v(t)\| \,\mathrm{d}t \\ &\leqslant C\left(\|f\|_{L^{p'}(0,T;V^*)} + \|Au_m\|_{L^{p'}(0,T;V^*)}\right)\end{aligned}$$

which is bounded as A is bounded (shown in the existence proof).

Hence $\|u_m'\|_{L^{p'}(0,T;V*)} \leqslant \tilde{C}$ independent of m. Since we already know that $\|u_m\|_{L^p(0,T;V)} \leqslant C_1$ independent of m, $(u_m)_{m\in\mathbb{N}}$ is bounded in $W_p(0,T) \overset{c}{\hookrightarrow} L^2(0,T;H)$, where the compactness comes from corollary 4.4.2. Hence there exists a subsequence such that $u_m' \to u$ in $L^p(0,T;H)$. Since $(u_m)_{m\in\mathbb{N}}$ is also bounded in $L^\infty(0,T;H)$, the convergence follows for every $L^q(0,T;H)$ with $q \in [1,\infty)$ by lemma 4.4.4. □

Strong convergence by monotonicity

Using compact embeddings, we could deduce strong convergence in the H-norm but now we want to deduce strong convergence in the V-norm, which we get by assuming additional monotonicity of the operator.

Lemma 4.4.6 (Strong convergence for *d*-monotone operators)
Let the standard assumption be fulfilled. Additionally assume that $A\colon V \to V^*$ is d-monotone, that is, there exists a $\tilde{\mu} \geqslant 0$ such that

$$\langle Av - Aw, v - w \rangle \geqslant \tilde{\mu}(\|v\|^{p-1} - \|w\|^{p-1})(\|v\| - \|w\|) \geqslant 0 \qquad \forall v, w \in L^p(0,T;V).$$

Then we have $u_m \to u$ in $L^p(0,T;V)$.

Remark. 4.4.7 (*d*-monotony and uniform *p*-monotony) The assumption of d-monotony is weaker than the assumption of p-monotony. An operator $A\colon V \to V^*$ is uniformly p-monotone if there exists a $\tilde{\mu} > 0$ such that

$$\langle Av - Aw, v - w \rangle \geqslant \tilde{\mu}\|v - w\|^p \qquad \forall v, w \in L^p(0,T;V).$$

Both conditions hold for the p-Laplacian.

Idea. We would like to use that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t) - u_m(t)|^2 + \langle Au(t) - Au_m(t), u(t) - u_m(t) \rangle = \langle f(t) - f(t), u(t) - u_m(t) \rangle = 0,$$

but the second equality is not true because we cannot test with $u - u_m$.

Instead, since $(V_n)_{n\in\mathbb{N}}$ is a Galerkin scheme, we know that $\bigcup_{n\in\mathbb{N}} V_m \subset V$ is dense. We take a sequence $(v_m)_{m\in\mathbb{N}} \subset L^p(0,T;V)$ such that $v_m \to u$ in $W_p(0,T)$. Then

$$\begin{aligned}
&|u_m - u|^2\Big|_0^T + \int_0^T \langle Au_m(t) - Au(t), u_m(t) - u(t) \rangle_{V^* \times V}\,\mathrm{d}t \\
&= \langle u_m' - u', u_m - u \rangle + \langle Au_m - Au, u_m - u \rangle_{L^{p'}(0,T;V^*)\times L^p(0,T;V)} \\
&= \langle \underbrace{u_m' + Au_m}_{=f}, u_m - v_m \rangle + \langle u_m' + Au_m, v_m - u \rangle - \langle Au + u', u_m - u \rangle \\
&= \langle f, u_m - v_m \rangle - \langle f, u_m - u \rangle - \langle u_m, \underbrace{v_m' - u'}_{\substack{\xrightarrow{m\to\infty} 0 \text{ as} \\ v_m' \to u' \text{ in } L^{p'}(0,T;V^*)}} \rangle + \langle \underbrace{u_m(t)}_{\substack{\text{bounded in} \\ L^p(0,T;V)}}, \underbrace{v_m(t) - u(t)}_{\substack{\xrightarrow{m\to\infty} 0 \text{ as} \\ W_p \hookrightarrow C([0,T];H)}} \rangle \\
&\qquad - (u_m^0, v_m(0) - u(0)) + \langle \underbrace{Au_m}_{\substack{\text{bounded in} \\ L^{p'}(0,T;V^*)}}, \underbrace{v_m - u}_{\substack{\to 0 \text{ strongly} \\ \text{in } L^p(0,T;V)}} \rangle \xrightarrow{m\to\infty} 0.
\end{aligned}$$

We infer that $\langle Au_m - Au, u_m - u \rangle \xrightarrow{m\to\infty} 0$. The d-monotonicity implies that $\int_0^T (\|u_m(t)\|^{p-1} -$

$\|u(t)\|^{p-1})(\|u_m(t)\| - \|u(t)\|)\,\mathrm{d}t \xrightarrow{m\to\infty} 0$. We have by HÖLDER's inequality

$$
\begin{aligned}
&\int_0^T (\|u_m(t)\|^{p-1} - \|u(t)\|^{p-1})(\|u_m(t)\| - \|u(t)\|)\,\mathrm{d}t \\
&= \int_0^T \|u_m(t)\|^p + \|u(t)\|^p \,\mathrm{d}t - \int_0^T \|u_m(t)\|^{p-1}\|u(t)\|\,\mathrm{d}t - \int_0^T \|u(t)\|^{p-1}\|u_m(t)\|\,\mathrm{d}t \\
&\overset{\text{(H)}}{\geqslant} \int_0^T \|u_m(t)\|^p + \|u(t)\|^p \,\mathrm{d}t - \left(\int_0^T \|u_m(t)\|^p\right)^{\frac{p-1}{p}} \left(\int_0^T \|u(t)\|^p\right)^{\frac{1}{p}} \\
&\quad - \left(\int_0^T \|u_m(t)\|^p\right)^{\frac{1}{p}} \left(\int_0^T \|u(t)\|^p\right)^{\frac{p-1}{p}} \\
&= \left(\|u_m\|^{p-1}_{L^p(0,T;V)} - \|u\|^{p-1}_{L^p(0,T;V)}\right)\left(\|u_m\|_{L^p(0,T;V)} - \|u\|_{L^p(0,T;V)}\right) \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}
$$

We conclude from $u_m \rightharpoonup u$ in $L^p(0,T;V)$ and $\|u_m\|_{L^p(0,T;V)} \to \|u\|_{L^p(0,T;V)}$ in $\mathbb{R}$ and $L^p(0,T;V)$ being uniformly convex, that indeed $u_m \to u$ in $L^p(0,T;V)$.

5 Instationary Navier-Stokes problem

This chapter differs considerably from the previous ones since the Navier-Stokes equation does not fit in our framework, so we will not be able to show existence and uniqueness in the same spaces, at least for weak solutions.

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ and the incompressible Navier-Stokes equation

$$\boxed{\begin{cases} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega, \end{cases}}$$

where $u \colon \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ is the velocity field, $p \colon \overline{\Omega} \times [0,T] \to \mathbb{R}$ is the pressure and ν is the viscosity. The time derivative of the velocity is the acceleration, the second (dissipative) term $\nu \Delta \boldsymbol{u}$ describes how friction behaves in the fluid.

This equation is difficult to solve due to the convection term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ (which is due to the flow of the material) and also the pressure term ∇p and the additional constraint $\nabla \cdot u = 0$.

We recall that $\frac{1}{\nu} = \mathrm{Re}$, where Re is the Reynolds number, a dimensionless parameter in fluid dynamics which provides a ratio between the inertial and viscous forces. A small Reynolds number implies a viscous fluid with little or no turbulence, whereas a large Reynolds number implies a turbulent flow.

5.1 Modelling and Applications

Before we do some rigorous mathematical analysis, we talk about the modelling - where these equations come from.

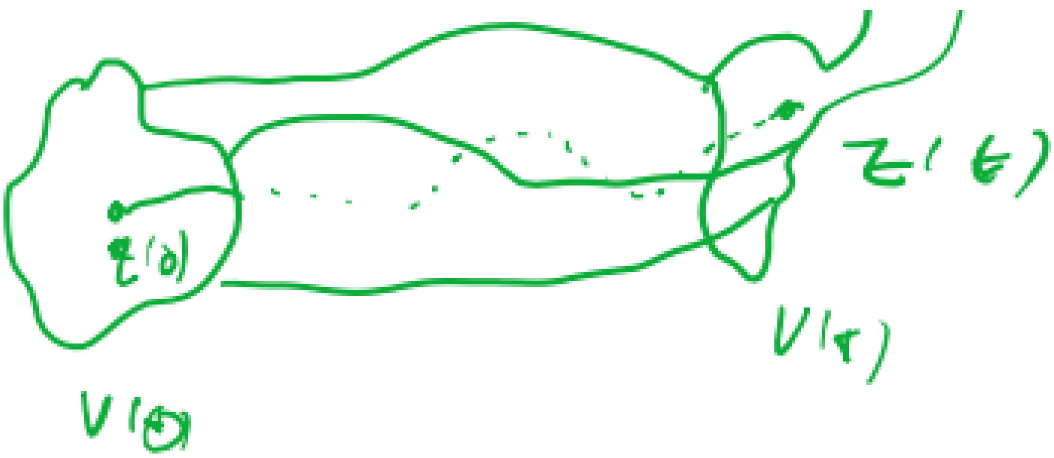

Fig. 6: A fluid part $v(0)$ moves around with time and is deformed to some other $v(t)$. This is what the Navier-Stokes equations describe. Consider a point $z(0)$ in $v(0)$, which is transformed to $z(t)$.

In the figure, z is the displacement field. Hence $z'(t) = u(t)$ is the velocity field. Let $\rho(t)$ be the mass density - how many molecules are in a certain region of the flow.

The Lagrangian perspective is imagining oneself sitting on such as stream line (like in the figure above) and describing the evolution this way, while in the Eulerian perspective one sits at a reference point and observes how the material flows along that point.

Mathematically speaking we usually work with the Eulerian perspective, because in the Lagrangian framework we need that the evolution of the flow must be smooth, but not in the Eulerian flow.

Let us look at the physical principle involved. The mass conservation is

$$\int_V \rho(t_1, x)\,\mathrm{d}x = \int_V \rho(t_2, x)\,\mathrm{d}x - \int_{t_1}^{t_2} \int_{\partial V} \rho(t,x)u(t,x) \cdot n(t,x)\,\mathrm{d}S\,\mathrm{d}t,$$

where V is a time-independent volume, $t_1, t_2 \in (0,T)$, and the inflow/outflow is described by the rightmost term. By Gauss' Theorem, the right hand side is equal to

$$\int_V \rho(t_2, x)\,\mathrm{d}x - \int_{t_1}^{t_2} \int_V \nabla \cdot (\rho(t,x)u(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

A pointwise relation holds:

$$\partial_t \rho + \nabla(\rho u) = 0.$$

The second principle is the conservation of momentum, similar to Newton's second law - $a = mf$, where a is the acceleration, m is the mass and f is the force applied. In order to "press" this into the PDE framework we consider $u(x,t) = m(z(t),t)$, where z is the displacement field which gives x at a certain time t. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} m(z(t),t) = \partial_t m(z(t),t) + \nabla m(z(t),t)\partial_t z(t) = \partial_t m(z(t),t) + \nabla m(z(t),t)u(t).$$

This is the material derivative for the momentum $m = \rho u$

5.2 Solenoidal function space

We want to treat the Navier-Stokes equation more rigorously, so we start by defining the appropriate function spaces.

As we saw, the conditions

$$u = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \quad \text{and} \quad \nabla \cdot u = 0 \quad \text{in} \quad (0,T) \times \Omega \tag{40}$$

are deeply entrenched in the problem formulation for an incompressible and viscous fluid with fixed boundary. Thus, the definition of the following, so-called solenoidal, spaces is motivated. Usually, we work with a pivot space in L^2, but now we want incorporate the incompressibility of the fluid.

As test functions we take

$$\boxed{\mathcal{V} := \left\{ \varphi \in \mathcal{C}_c^\infty\left(\overline{\Omega}; \mathbb{R}^d\right) \mid \nabla \cdot \varphi \equiv 0 \quad \text{in} \quad \Omega \right\}.}$$

Since this is to regular for our purposes, we will take the closure with respect to the H^1-norm. An abstract function on $[0,T]$ having values in $\mathcal{V}$ clearly fulfills (40). Now the spaces

$$\boxed{V := \operatorname{clos}_{\|\cdot\|_{H^1}} \mathcal{V} \quad \text{and} \quad H := \operatorname{clos}_{\|\cdot\|_{L^2}} \mathcal{V}}$$

form a Gelfand-triple. The space V is reflexive as the closed subspace of the reflexive space $H_0^1(\Omega)^d$. Furthermore, $V \overset{\mathrm{d}}{\hookrightarrow} H$, since for every $h \in H$ by definition there exists a sequence

$(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{V} \subset V$ with $\varphi_n \to h$ in H. The compact embedding follows from RELLICH-KONDRACHOV, as we have $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ (if $\{0, -\frac{1}{2}\} \ni k - \frac{d}{p} > \ell - \frac{d}{q} \in \{-1, -\frac{3}{2}\}$, then we have $W^{k,p} \overset{c}{\hookrightarrow} W^{\ell,q}$).

$$V \quad \overset{c}{\hookrightarrow} \quad H \cong H^* \quad \hookrightarrow \quad V^*,$$

where V is equipped with $\|\cdot\| := \|\cdot\|_{H_0^1(\Omega)}$ and the scalar product

$$((\boldsymbol{u}(t), \boldsymbol{v}(t))) := \int_\Omega \nabla \boldsymbol{u}(t) : \nabla \boldsymbol{v}(t) \, \mathrm{d}x$$

and H with $|\cdot| := \|\cdot\|_{L^2(\Omega)}$ and the scalar-product

$$(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x.$$

One can show the characterisations

$$\boxed{V = \left\{\boldsymbol{u} \in H_0^1(\Omega)^d \mid \nabla \cdot \boldsymbol{u} \equiv 0 \quad \text{in} \quad \Omega \right\}}$$

and

$$\boxed{H = \left\{\boldsymbol{u} \in L^2(\Omega)^d \mid \nabla \cdot \boldsymbol{u} \equiv 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{n} \cdot \nabla \boldsymbol{u} \equiv 0 \quad \text{on} \quad \partial\Omega \right\},}$$

where the condition of zero divergence means

$$\int_\Omega \boldsymbol{u} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all} \quad \varphi \in \mathcal{C}_c^\infty(\overline{\Omega}; \mathbb{R})$$

and the vanishing on the boundary is to be understood in the sense of a certain trace.

We now will develop the mentioned trace. First we define the auxiliary space (also called anisotropic SOBOLEV space)

$$\boxed{E := \left\{\boldsymbol{u} \in L^2(\Omega)^d \mid \nabla \cdot \boldsymbol{u} \in L^2(\Omega)\right\}}$$

(hence $V \hookrightarrow H \subset E$) and equip it with the norm

$$\|\boldsymbol{u}\|_E^2 := \|\boldsymbol{u}\|_{L^2(\Omega)^d}^2 + \|\nabla \cdot \boldsymbol{u}\|_{L^2(\Omega)}^2.$$

Then E is a Banach space (due to the fact that $L^2(\Omega)$ is a BANACH space and that the divergence is a linear differential operator) with the density

$$\operatorname{clos}_{\|\cdot\|_E} \mathcal{C}_c^\infty(\overline{\Omega}; \mathbb{R}^d) = E,$$

which is shown in [TC78, Thm 1.1] (for the boundary terms, one uses some intermediate contraction step).

Lemma 5.2.1 (Linear normal trace operator)
Let $\Omega \subset \mathbb{R}^d$ be a bounded LIPSCHITZ domain. We define the linear normal-trace operator $\gamma_n \colon E \to H^{-1/2}(\partial\Omega)$ by

$$\gamma_{\boldsymbol{n}}(\boldsymbol{e}) = \boldsymbol{e} \cdot \boldsymbol{n} \quad \text{on} \quad \partial\Omega \qquad \forall \boldsymbol{e} \in \mathcal{C}(\overline{\Omega}; \mathbb{R}^d) \cap E.$$

Then γ_n is well-defined on E.

Proof. The idea is to use the integration by parts formula:

$$\int_\Omega \nabla \cdot \boldsymbol{v} \cdot \varphi \,\mathrm{d}x + \int_\Omega \boldsymbol{v} \cdot \nabla\varphi \,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{v}\varphi \,\mathrm{d}S \qquad \forall \boldsymbol{v} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^d), \varphi \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}). \tag{41}$$

From DGL IIB we know that the Dirichlet trace

$$\Gamma_0 \colon H^1(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is a well-defined surjective linear operator with $\ker(\Gamma_0) = H_0^1(\Omega)$.

For $\boldsymbol{e} \in E$ and $\varphi \in H_0^1(\Omega)$, we observe that

$$\int_\Omega \nabla \cdot \boldsymbol{e} \cdot \varphi \,\mathrm{d}x + \int_\Omega \boldsymbol{e} \cdot \nabla\varphi \,\mathrm{d}x \overset{(41)}{=} 0,$$

since it holds for $\mathcal{C}_c^\infty(\overline{\Omega}; \mathbb{R}^d)$ functions and this set is dense in E and $H_0^1(\Omega)$.

We define an operator

$$L_e(\Gamma_0(\varphi)) := \int_\Omega \nabla \cdot \boldsymbol{e} \cdot \varphi \,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{e} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x}.$$

For all $\varphi \in H^1(\Omega)$, this is well defined: let $\varphi_1, \varphi_2 \in H^1(\Omega)$ with $\Gamma_0(\varphi_1) = \Gamma_0(\varphi_2)$. Then $\varphi_1 - \varphi_2 \in H_0^1(\Omega)$ and $L_e(\gamma_0(\Phi)) = 0$ for all $\Phi \in H_0^1(\Omega)$. Hence $H_0^1(\Omega)$ is the kernel of L_e and $L_e \colon H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is a linear bounded surjective operator (follows from the surjectivity of Γ_0).

For all $\psi \in \mathcal{C}^1(\overline{\Omega})$ we identify

$$L_{\boldsymbol{e}}(\psi) = \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{e}\psi \,\mathrm{d}S$$

for all $\psi \in H^{\frac{1}{2}}(\partial\Omega)$ (since $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$, the integral is well defined). Thus

$$L_{\boldsymbol{e}}(\psi) = \int_\Omega \nabla \cdot \boldsymbol{e}\varphi \,\mathrm{d}x + \int_\Omega \boldsymbol{e} \cdot \nabla\varphi \,\mathrm{d}x$$

for all $\varphi \in H^1(\Omega)$ with $\Gamma_0(\varphi) = \psi$.

We may argue by density that for all $e \in E$, $L_e \in (H^{\frac{1}{2}}(\partial\Omega))^* \cong H^{-\frac{1}{2}}(\partial\Omega)$ so $L_e = \gamma_n(\varphi)$. □

Remark. 5.2.2 (Helmholtz decomposition) Let

$$\boxed{Y := \{\nabla p : p \in L^2(\Omega)\} \subset H^{-1}(\Omega)^d.}$$

Then $Y \perp V$ with respect to $L^2(\Omega)$: for $\boldsymbol{y} = \nabla p$ for some $p \in L^2(\Omega)$ and $\boldsymbol{v} \in V$ we have by integration by parts and by $V \hookrightarrow H$

$$(\boldsymbol{y}, \boldsymbol{v}) = \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \nabla p(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = -\int_\Omega p(\boldsymbol{x}) \underbrace{(\nabla \cdot \boldsymbol{v}(\boldsymbol{x}))}_{=0} \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \underbrace{\boldsymbol{n} \cdot \boldsymbol{v}(\boldsymbol{x})}_{=0 \text{ as } v \in H} p(\boldsymbol{x}) \,\mathrm{d}S(x) = 0.$$

5.3 Weak formulation

The abstract operators and their properties

We define the bilinear form

$$\boxed{a \colon V \times V \to \mathbb{R}, \qquad (\boldsymbol{v}, \boldsymbol{w}) \mapsto \nu \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w} \,\mathrm{d}x}$$

to deal with the diffusive part and the trilinear form

$$\boxed{b\colon V\times V\times V\to\mathbb{R},\qquad (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\mapsto\int_\Omega\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\big)\cdot\boldsymbol{w}\,\mathrm{d}x=\int_\Omega\big((\nabla\boldsymbol{v})\boldsymbol{u}\big)\cdot\boldsymbol{w}\,\mathrm{d}x}$$

for the convective part. We denote $(\nabla\boldsymbol{v})_{i,j}=\partial_{x_j}v_i$ and $\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\big)_i=\sum_{i=1}^d u_j\partial_{x_j}v_i$ and thus

$$b(u,v,w)=\sum_{i=1}^d\int_\Omega\big((u\cdot\nabla)v\big)_i w_i\,\mathrm{d}x=\sum_{i=1}^d\int_\Omega\left(\sum_{j=1}^d u_j\partial_j v_i\right)w_i\,\mathrm{d}x=\sum_{i,j=1}^d\int_\Omega u_j\partial_j v_i w_i\,\mathrm{d}x.$$

The form a is bounded, symmetric and strongly positive (by Poincaré's inequality) and b is bounded and skew-symmetric with respect to the second and third argument: due to the fact that u is divergence-free, the integration-by-parts formula together with the product rule yields

$$\begin{aligned}b(\boldsymbol{u},\boldsymbol{v},w)&=\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}x\\&\overset{\text{IBP}}{=}-\int_\Omega(\underbrace{\nabla\cdot\boldsymbol{u}}_{=0})\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}x-\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{v}\,\mathrm{d}x+\int_{\partial\Omega}\underbrace{\gamma_n(\boldsymbol{u})}_{=0}\,\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}S\\&=-\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{v}\,\mathrm{d}x=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\end{aligned}$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V$. In particular we have $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0$ for all $\boldsymbol{u},\boldsymbol{v}\in V$.

We consider the Gelfand triple $(V,\|\cdot\|,((\cdot,\cdot))),(H,|\cdot|,(\cdot,\cdot)),(V^*,\|\cdot\|_*)$.

Remark. 5.3.1 We will see later that V^* is a very *weak* space, e.g. if $H=L^2(\Omega)$, then V^* is weaker than $H^{-1}(\Omega)$ and we can't even interpret its elements in the distributional sense [Sim99].

We define the operators

$$\boxed{A\colon V\to V^*,\quad\langle A(\boldsymbol{u}),\boldsymbol{v}\rangle:=a(\boldsymbol{u},\boldsymbol{v})\qquad\text{and}\qquad B\colon V\to V^*,\quad\langle B(\boldsymbol{u}),\boldsymbol{v}\rangle:=b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}),}$$

where A is linear and B is nonlinear. We now define

$$\boxed{(\mathcal{A}\,\boldsymbol{v})(t):=A\boldsymbol{v}(t)\qquad\text{and}\qquad(\mathcal{B}\,\boldsymbol{v})(t):=B(\boldsymbol{v}(t))}\tag{42}$$

Then the instationary Navier-Stokes equation is equivalent to the operator equation

$$\begin{cases}\boldsymbol{u}'+\mathcal{A}\,\boldsymbol{u}+\mathcal{B}(\boldsymbol{u})=\boldsymbol{f}&\text{in }L^1(0,T;V^*),\\\boldsymbol{u}(0)=\boldsymbol{u}_0&\text{in }H.\end{cases}\tag{43}$$

We wrote the Navier-Stokes equation as

$$\partial_t\boldsymbol{u}-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p=\boldsymbol{f}.$$

If we test by $v\in V$, then remark 5.2.2 implies that p vanishes.

We consider the associated Nemytskii operators.

Lemma 5.3.2 (Range of the Nemytskii operator of the linear operator A)
The linear operator $\mathcal{A}\colon L^2(0,T;V)\to L^2(0,T;V^*)$ is well defined and continuous.

Proof. To show that $\mathcal{A}$ is continuous, we just have to show that it is bounded because it is linear. For $u \in L^2(0,T;V)$ we have

$$\| \mathcal{A} u\|^2_{L^2(0,T;V^*)} \overset{(42)}{=} \int_0^T \|Au(t)\|^2_* \,\mathrm{d}t = \nu^2 \|u\|^2_{L^2(0,T;V)}.$$

We use the definition of the dual norm, the Cauchy-Schwarz inequality and plug in $\boldsymbol{v} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$ to obtain

$$\nu\|\boldsymbol{u}\| = \sup_{\boldsymbol{v}\in V, \|\boldsymbol{v}\|=1} \|\boldsymbol{v}\|\nu\|\boldsymbol{u}\| \geqslant \|A\boldsymbol{u}\|_* \geqslant \frac{\nu\|\boldsymbol{u}\|^2}{\|\boldsymbol{u}\|} = \nu\|\boldsymbol{u}\|$$

and thus $\|A\boldsymbol{u}\|_* = \nu\|\boldsymbol{u}\|$. Hence $\mathcal{A}$ is continuous and thus maps Bochner measurable functions to Bochner measurable functions (by lemma 3.1.3 1?). □

Lemma 5.3.3 (Range of the Nemytskii operator of the nonlinear operator B) *The nonlinear operator $\mathcal{B}\colon L^2(0,T;V) \to L^1(0,T;V^*)$ is well defined. More precisely, the nonlinear operator $\mathcal{B}\colon L^\infty(0,T;H) \cap L^2(0,T;V) \to L^p(0,T;V^*)$ with*

$$p = \begin{cases} 2, & \text{if } d = 2, \\ \frac{4}{3}, & \text{if } d = 3. \end{cases} \tag{44}$$

is well defined.

Proof. 1 From DGL II B we know that there exists a $C > 0$ such that $b(u,\boldsymbol{v},w) \leqslant C\|u\|\|\boldsymbol{v}\|\|w\|$, showing the first statement, as thus

$$|b(u,u,\boldsymbol{v}) - b(\bar{u},\bar{u},\boldsymbol{v})| \leqslant |b(u,u-\bar{u},\boldsymbol{v})| + |b(u-\bar{u},\bar{u},\boldsymbol{v})| \leqslant C\|u-\bar{u}\|\|\boldsymbol{v}\|(\|u\|+\|\bar{u}\|)$$

and hence

$$\|B(u) - B(\bar{u})\|_* \leqslant C\|u-\bar{u}\|(\|u\|+\|\bar{u}\|).$$

Consider a sequence of simple functions $u_n = \sum_{i=1}^{m_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}}$ such that $u_n(t) \to u(t)$ almost everywhere in $(0,T)$. Then by the trilinearity of b we have

$$\langle B(u_n), \boldsymbol{v} \rangle = b(u_n,u_n,\boldsymbol{v}) = \sum_{i,j=1}^{m_n} b(u_i^{(n)}, u_i^{(n)}, \boldsymbol{v}) \mathbb{1}_{E_i^{(n)} \cap E_j^{(n)}} = \sum_{i=1}^{m_n} \underbrace{b(u_i^{(n)}, u_i^{(n)}, \boldsymbol{v})}_{\in V^*} \mathbb{1}_{E_i^{(n)}}$$

as the $E_i^{(n)}$ are pairwisely disjoint. Hence $\langle B(u_n), \boldsymbol{v} \rangle$ is a simple function in V^*.

Hence by continuity,

$$\|B(u_n(t)) - B(u(t))\|_* = C(\|u_n(t)\| + \|u(t)\|)\|u_n(t) - u(t)\|.$$

Since $(u_n)_{n\in\mathbb{N}}$ converges, it is also bounded. The pointwise convergence follows for $B(u_n)$, so $B(u)$ is also Bochner measurable. We have

$$\| \mathcal{B} u\|_{L^1(0,T,V^*)} = \int_0^T \|Bu(t)\|_* \,\mathrm{d}t \leqslant C \int_0^T \|u(t)\|^2 \,\mathrm{d}t = C\|u\|^2_{L^2(0,T;V)}.$$

2 Now let us prove the more precise statement. By the generalised Hölder inequality we have

$$b(u,\boldsymbol{v},w) = \int_\Omega (\nabla \boldsymbol{v} u) \cdot w \,\mathrm{d}x \leqslant \|\nabla \boldsymbol{v}\|_{L^\alpha} \|u\|_{L^\beta} \|w\|_{L^\gamma},$$

where $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$. We choose $\alpha = \gamma = 4$ and $\beta = 2$ to infer $b(u,\boldsymbol{v},w) \leqslant \|u\|_{L^4}\|\boldsymbol{v}\|\|w\|_{L^4}$.

The GAGLIARDO–NIRENBERG inequality states that

$$\|u\|_{L^4} \leqslant \begin{cases} C\|u\|^{\frac{1}{2}}|u|^{\frac{1}{2}}, & \text{for } d=2, \\ \tilde{C}\|u\|^{\frac{3}{4}}|u|^{\frac{1}{4}}, & \text{for } d=3. \end{cases}$$

Hence for $d = 2$ we have (using that $|b(u, \boldsymbol{v}, w)| = |b(u, w, \boldsymbol{v})|$)

$$\|\mathcal{B}u\|^2_{L^2(0,T;V^*)} \leqslant \int_0^T \|u(t)\|^4_{L^4}\,\mathrm{d}t \leqslant C\int_0^T \|u(t)\|^2|u(t)|^2\,\mathrm{d}t \leqslant C\|u\|^2_{L^2(0,T;V)}\|u\|^2_{L^\infty(0,T;H)}$$

and for $d = 3$ we have

$$\|\mathcal{B}u\|^{\frac{4}{3}}_{L^2(0,T;V^*)} \leqslant \int_0^T \|u(t)\|^{\frac{8}{3}}_{L^4}\,\mathrm{d}t \leqslant \tilde{C}\int_0^T \|u(t)\|^2|u|^{\frac{2}{3}}\,\mathrm{d}t \leqslant \tilde{C}\|u\|^2_{L^2(0,T;V)}\|u\|^{\frac{2}{3}}_{L^\infty(0,T;H)}.$$

□

We observe that $\mathcal{A}$ and $\mathcal{B}$ are not maps from $L^p(0,T;V)$ to $L^{p'}(0,T;V^*)$ (as before), so the previous theory is not applicable, and hence we can't test with the function itself (this showed upper estimates and uniqueness), since we have less regularity of the range of the operator and thus the solutions' time derivative has less regularity.

Weak formulation

The problem may be formulated as

$$\text{to } \boldsymbol{u}_0 \in H \text{ and } f \in L^2(0,T;V^*) \text{ find } \boldsymbol{u} \in L^2(0,T;V) \text{ with}$$
$$\langle \boldsymbol{u}'(t), \boldsymbol{v}\rangle + \nu((\boldsymbol{u}(t), \boldsymbol{v})) + b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}) = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle \,\forall \boldsymbol{v} \in V, \text{ for almost all } t \in (0,T).$$

Remark. 5.3.4 From $f \in L^2(0,T;V^*)$ and $\boldsymbol{u} \in L^2(0,T;V) \hookrightarrow L^1(0,T;V^*)$ as well as $\mathcal{A}\boldsymbol{u}+\mathcal{B}(\boldsymbol{u}) \in L^1(0,T;V^*)$ (by lemma 5.3.2 and lemma 5.3.3) we infer that $\boldsymbol{u}' \in L^1(0,T;V^*) \oplus L^2(0,T;V^*) = L^1(0,T;V^*)$, which implies that $\boldsymbol{u} \in W^{1,1}(0,T;V^*) \hookrightarrow \mathrm{AC}([0,T];V^*)$ and that the initial condition is attained in V^*, that is, $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in V^*$

Hence the weak formulation can be written as

$$\text{to } \boldsymbol{u}_0 \in H \text{ and } \boldsymbol{f} \in L^2(0,T;V^*) \text{ find } \boldsymbol{u} \in L^2(0,T;V^*) \text{ with } \boldsymbol{u}' \in L^1(0,T;V^*) \text{ such that}$$
$$\begin{cases} \boldsymbol{u}' + \mathcal{A}\boldsymbol{u} + \mathcal{B}(\boldsymbol{u}) = \boldsymbol{f} & \text{almost everywhere in } V^* \text{ or } L^1(0,T;V^*) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } V^*. \end{cases}$$

If $d = 2$ we find $\boldsymbol{u} \in L^2(0,T;V^*) \cap L^\infty(0,T;H)$ such that

$$\boldsymbol{u}' = \boldsymbol{f} - \mathcal{A}\boldsymbol{u} - \mathcal{B}(\boldsymbol{u}) \qquad \text{in } L^2(0,T;V^*).$$

We infer that $\boldsymbol{u} \in W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$ such that $\boldsymbol{u}(t) \xrightarrow{t\searrow 0} \boldsymbol{u}_0 \in H$.

If $d = 3$, we have $\boldsymbol{u} \in L^2(0,T;V^*) \cap L^\infty(0,T;H)$ such that $\boldsymbol{u}' \in L^{\frac{4}{3}}(0,T;V^*)$ and we again infer that $\boldsymbol{u} \in \mathrm{AC}([0,T];V^*) \hookrightarrow \mathcal{C}_w([0,T];V^*)$, which is the space of continuous functions with respect to V^* equipped with the weak topology, that is

$$\mathcal{C}_w([0,T];X) := \{f : [0,T] \to X \mid f \text{ is demicontinuous}\}$$

for any BANACH space X.

Lemma 5.3.5 (Weak continuity)
Let H be a reflexive BANACH *space and $H \hookrightarrow V^*$, where V is also a* BANACH *space. Then*

$$\mathcal{C}_w([0,T];V^*) \cap L^\infty(0,T;H) \subset \mathcal{C}_w([0,T];H).$$

This doesn't work for $H = L^1(\Omega)$, which is not reflexive.

Proof. Exercise!

The idea is that one has a pointwise bound in H on every sequence of functions and this gives weak convergence in H of a subsequence by reflexivity. In this lower space one is able to identify the weak limit via the weak continuity in V^*. □

Hence $\boldsymbol{u}(t) \rightharpoonup \boldsymbol{u}_0$ in H as $t \to 0$.

5.4 Existence

THEOREM 5.4.1: GLOBAL EXISTENCE

Given a right-hand side $\boldsymbol{f} \in L^2(0,T;V^*)$, initial data $\boldsymbol{u_0} \in H$ and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, where $d \in \{2,3\}$, a solution

$$\boldsymbol{u} \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap W^{1,p}(0,T;V^*)$$

to (43) exists, where (44).

Proof. In this proof we use a approximate system of combined linearisation and SCHAUDER arguments. One could also use time or GALERKIN discretisation.

1. Regularisation. Consider

$$\boxed{\tilde{\mathcal{B}}_\varepsilon \colon L^2(0,T;V)^2 \to L^2(0,T;V^*), \quad \langle \tilde{\mathcal{B}}_\varepsilon(\boldsymbol{u},\boldsymbol{v}), \boldsymbol{w} \rangle := \int_\Omega \big((\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}) \cdot \nabla\big)\boldsymbol{v} \cdot \boldsymbol{w} \,\mathrm{d}x,}$$

where $(\boldsymbol{\rho}_\varepsilon)_\varepsilon \subset \mathcal{C}_c^\infty(\overline{\Omega};\mathbb{R}^d)$ are d-dimensional mollifiers, where

$$(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u})(\boldsymbol{x}) = \int_\Omega \boldsymbol{\rho}_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.$$

Formally, one has to integrate over $\mathbb{R}^d$ and introduce some extension operator for SOBOLEV functions from Ω to $\mathbb{R}^d$, but since we deal with zero boundary conditions, the extension is done by zero.

Let

$$\boxed{\mathcal{B}_\varepsilon \colon L^2(0,T;V) \to L^2(0,T;V^*), \qquad \boldsymbol{u} \mapsto \tilde{\mathcal{B}}_\varepsilon(\boldsymbol{u},\boldsymbol{u}).}$$

Recall from DGL IIA that for non-abstract functions $\boldsymbol{u}$

- $\boldsymbol{\rho}_\varepsilon * \boldsymbol{u} \to \boldsymbol{u}$ in $L^p(\Omega)$ with $p \in [1,\infty)$
- $\boldsymbol{\rho}_\varepsilon * \boldsymbol{u} \to \boldsymbol{u}$ almost everywhere in Ω,
- $\|\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}\|_{L^p(\Omega)} \leqslant \|\boldsymbol{u}\|_{L^p(\Omega)}$.

Tim's alternative Smoothing. Let ρ_ε^T for $\varepsilon > 0$ be a smoothing kernel on $(0,T)$ and ρ_ε^Ω a smoothing kernel on Ω. For

$$\bar{u} \in L^2(0,T;H) \subset L^2((0,T)\times\Omega)$$

we obtain

$$\rho_\varepsilon * \bar{u} \in \mathcal{C}_c^\infty((0,T)\times\Omega),$$

where

$$(\rho_\varepsilon * \bar{u})(t,x) := (\rho_\varepsilon^\Omega * (\rho_\varepsilon^T * \bar{u}))(t,x) = \int_\Omega \rho_\varepsilon^\Omega(x-y) \int_0^T \rho_\varepsilon^T(t-s)\bar{u}(s,y)\,\mathrm{d}s\,\mathrm{d}y.$$

We set

$$\mathcal{B}_\varepsilon \colon L^2(0,T;H) \times L^2(0,T;V) \to L^2(0,T;V^*),$$
$$\langle \mathcal{B}_\varepsilon(\bar{u},u), w\rangle = \int_0^T \langle B(\rho_\varepsilon * \bar{u}, u), w\rangle\,\mathrm{d}t = \int_0^T ((\rho_\varepsilon * \bar{u} \cdot \nabla)u, w)\,\mathrm{d}t,$$

then measurability follows similarly to the proof earlier.

2 Approximate problem. For $\varepsilon > 0$ we consider

$$\begin{cases} \boldsymbol{u}_\varepsilon' + \mathcal{A}\,\boldsymbol{u}_\varepsilon + B(\boldsymbol{u}_\varepsilon) = f, \\ \boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0 \end{cases} \tag{P_ε}$$

We have

$$\langle \mathcal{B}_\varepsilon(\boldsymbol{u}), v\rangle = \langle \tilde{\mathcal{B}}_\varepsilon(\boldsymbol{u},\boldsymbol{u}), v\rangle \leqslant C_\varepsilon \|\boldsymbol{u}\|_{L^2(0,T;H)}\|\boldsymbol{u}\|_{L^2(0,T;V)}\|\boldsymbol{v}\|_{L^2(0,T;V)}.$$

3 Existence of solutions to (P_ε). For every $\bar{\boldsymbol{u}} \in L^2(0,T;H)$ we solve the abstract regularised problem

$$\begin{cases} \partial_t \boldsymbol{u}_\varepsilon + \mathcal{A}\,\boldsymbol{u}_\varepsilon + \tilde{\mathcal{B}}_\varepsilon(\bar{\boldsymbol{u}}, \boldsymbol{u}_\varepsilon) = \boldsymbol{f}, & \text{in } L^2(0,T;V^*) \\ \boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0 & \text{in } H. \end{cases} \tag{P_S}$$

As argued beforehand, $\mathcal{A} + \tilde{\mathcal{B}}_\varepsilon(\bar{\boldsymbol{u}},\cdot) \colon L^2(0,T;V) \to L^2(0,T;V^*)$ is a bounded linear operator. In order to apply the Theorem of Lions, we need some kind of coerciveness. We have

$$\begin{aligned}\langle \mathcal{A}\,\boldsymbol{u} + \tilde{\mathcal{B}}_\varepsilon(\bar{\boldsymbol{u}},\boldsymbol{u}), \boldsymbol{u}\rangle &= \nu((\boldsymbol{u},\boldsymbol{u})) + \int_\Omega \big((\boldsymbol{\rho}_\varepsilon * \bar{\boldsymbol{u}})\cdot\nabla\big)\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \\ &\overset{(\star)}{=} \nu\|\boldsymbol{u}\|_V^2 - \int_\Omega \nabla\cdot(\boldsymbol{\rho}_\varepsilon * \bar{\boldsymbol{u}})\frac{1}{2}|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} = \nu\|\boldsymbol{u}\|^2,\end{aligned}$$

where in $(\star)$ we used integration by parts and $(u'(t),u(t)) = \frac{1}{2}\partial_t|u(t)|^2$. The last step is due to, by writing out the convolution, we have

$$\nabla\cdot(\boldsymbol{\rho}_\varepsilon * \bar{\boldsymbol{u}}) = \int_\Omega \nabla\cdot\rho(\boldsymbol{x}-\boldsymbol{y})\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho(\boldsymbol{x}-\boldsymbol{y})\nabla\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x} = 0.$$

By Lions' Theorem of linear abstract ODEs, to every $\bar{\boldsymbol{u}} \in L^2(0,T;H)$ there exists a unique solution to (P_S).

4 Schauder fixed-point argument. Consider the set

$$\mathcal{M}_R^p := \{v \in L^p(0,T;H) : \|v\|_{L^p(0,T;H)} \leqslant R\}$$

for $p < \infty$ and some $R \geqslant 0$ determined later. Then, we define the solution operator

$$J \colon \mathcal{M}_R^p \to \mathcal{M}_R^p, \qquad v \mapsto u,$$

where u is the solution of (P_S) in the sense of Lions' Theorem. (Tim's interjection:

$$R^2 := \frac{1}{c\nu}\big(|u_0|^2 + \frac{1}{\nu}\|f\|_{L^2(0,T;V^*)}^2\big) > 0.$$

The operator is well-defined, since a solution always exists and moreover

$$\|u\|_{L^2(0,T;H)}^2 \leqslant \frac{1}{c}\|u\|_{L^2(0,T;V)}^2 \leqslant R^2$$

holds for any $u = J\bar{u}$ with $\bar{u} \in \mathcal{M}_R^p$.)

In order to apply the Schauder fixed-point theorem, we have to show that J maps $\mathcal{M}_R$ to $\mathcal{M}_R$, is continuous and compact and that $\mathcal{M}_R$ is bounded, non-empty and convex.

1 Clearly, $\mathcal{M}_R^p$ is non-empty, bounded, closed and convex.

2 Compactness of J. We know that $W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$ by Theorem 2.2.1 3. By the theorem of Lions-Aubin, we observe that $W(0,T) \overset{c}{\hookrightarrow} L^2(0,T;H)$ since $V \overset{c}{\hookrightarrow} H$ (by choosing the Hilbert spaces $V_1 := V$, $V_0 := H$, $V_1 := V^*$ and $r = s = 2$). By the interpolation lemma we deduce that $W(0,T) \overset{c}{\hookrightarrow} L^p(0,T;H)$ for all $p \in [1,\infty)$ (HOW???).

3 Welldefinedness of J. We show an a-priori estimate: we test (P_S) by $\boldsymbol{u}$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}(t)|^2 + \nu\|\boldsymbol{u}\|^2 + \underbrace{\tilde{\mathcal{B}}_\varepsilon(\boldsymbol{v},\boldsymbol{u},\boldsymbol{u})}_{=0} = \langle \boldsymbol{f}, \boldsymbol{u} \rangle$$
$$\leqslant \frac{1}{2\nu}\|\boldsymbol{f}\|_*^2 + \frac{\nu}{2}\|\boldsymbol{u}\|^2$$

by Young's inequality. Hence

$$\|\boldsymbol{u}\|^2_{L^\infty(0,T;H)} + \nu\|\boldsymbol{u}\|^2_{L^2(0,T;V)} \leqslant 4\left(|\boldsymbol{u}_0|^2 + \frac{1}{\nu}\|f\|^2_{L^2(0,T;V^*)}\right) =: \frac{R}{C},$$

where C is chosen such that $\|\boldsymbol{u}\|_{L^p(0,T;H)} \leqslant R$ implies that $\boldsymbol{u} \in M_R^p$.

4 Continuity of J. We show that $v_n \to v$ in $L^p(0,T;H)$ implies that $J(v_n) \to J(v)$ in $W(0,T)$. $u_n = J(v_n)$ and $u = J(v)$ such that $\bar{u}_n = u_n - u$ solves

$$\bar{u}_n' + \mathcal{A}\,\bar{u}_n + \tilde{\mathcal{B}}_\varepsilon(v,\bar{u}_n) = \tilde{\mathcal{B}}_\varepsilon(\underbrace{v - v_n}_{\to 0}, \underbrace{u_n}_{\text{bd.}}).$$

It can be show that $\tilde{\mathcal{B}}_\varepsilon(v - v_n, u_n) \to 0$ in $L^2(0,T;V^*)$ such that the continuity follows from the continuity of the solution operator.

By Schauder's fixed-point theorem, there exists a solution $\boldsymbol{u}_\varepsilon$ for $\varepsilon > 0$ of (P_ε).

(Tim Alternative: Also, for $u := J\bar{u}$ and $v := J\bar{v}$ with $\bar{u},\bar{v} \in \mathcal{M}_R$ we find

$$\begin{aligned}&|\langle \mathcal{B}_\varepsilon(\bar{u},u) - \mathcal{B}_\varepsilon(\bar{v},v), u - v\rangle| = |\langle \mathcal{B}_\varepsilon(\bar{u}-\bar{v},u) + \mathcal{B}_\varepsilon(\bar{v},u-v), u-v\rangle| \\ &= |\langle \mathcal{B}_\varepsilon(\bar{u}-\bar{v},u), u-v\rangle| \leqslant c2R^2\|\rho_\varepsilon * (\bar{u}-\bar{v})\|_{L^\infty((0,T)\times\Omega)} \leqslant C(\varepsilon)2R^2\|\bar{u}-\bar{v}\|_{L^2(0,T;H)},\end{aligned}$$

where we used

$$\begin{aligned}|(\rho_\varepsilon * (\bar{u}-\bar{v}))(t,x)| &\leqslant \int_\Omega |\rho_\varepsilon^\Omega(x-y)| \int_0^T |\rho_\varepsilon^T(t-s)||\bar{u}(s,y) - \bar{v}(s,y)|\,\mathrm{d}s\,\mathrm{d}y \\ &\leqslant c\|\rho_\varepsilon^\Omega \rho_\varepsilon^T\|_{L^\infty((0,T)\times\Omega)}\|\bar{u}-\bar{v}\|_{L^2(0,T;H)}.\end{aligned}$$

Hence, J is continuous, as we obtain by testing with $u - v$ the relation

$$0 = |u(T) - v(T)|^2 + \nu\|u - v\|^2_{L^2(0,T;V)} + \langle \mathcal{B}_\varepsilon(\bar{u},u) - \mathcal{B}_\varepsilon(\bar{v},v), u - v\rangle,$$

where the last term on the right vanishes for $\bar{v} \to \bar{u}$ in $L^2(0,T;H)$. Thus, in the limit

$$\|u - v\|^2_{L^2(0,T;H)} \leqslant c\|u - v\|^2_{L^2(0,T;V)} = 0.$$

By Aubin-Lions

$$J(L^2(0,T;H)) \subset W^{1,2,2}(0,T;V,V^*) \overset{c}{\hookrightarrow} L^2(0,T;H)$$

and for a solution u to $\bar{u} \in \mathcal{M}_R$ we find

$$\begin{aligned}\|u'\|_{L^2(0,T;V^*)} &= \|f - \mathcal{A}\,u - \mathcal{B}_\varepsilon(\bar{u},u)\|_{L^2(0,T;V^*)} \\ &\leqslant \|f\|_{L^2(0,T;V^*)} + \big(\nu + C(\varepsilon)\|\bar{u}\|_{L^2(0,T;H)}\big)\|u\|_{L^2(0,T;V)},\end{aligned}$$

where each solution u is bounded in $L^2(0,T;V)$ by the same constant. We find that $J(L^2(0,T;H))$ is bounded in $W^{1,2,2}(0,T;V,V^*)$ and thus relatively compact in $L^2(0,T;H)$. Therefore, J is compact and Schauder's fixed-point theorem furnishes a solution $u_\varepsilon \in W^{1,2,2}(0,T;V,V^*)$ to

$$\begin{cases} u_\varepsilon' + \mathcal{A}\,u_\varepsilon + \mathcal{B}_\varepsilon(u_\varepsilon,u_\varepsilon) = f & \text{in} \quad L^2(0,T;V^*), \\ u_\varepsilon(0) = u_0 & \text{in} \quad H. \end{cases}$$

)

5 **Passing to the limit.** We obtain a-priori estimate by testing (P_ε) by $\boldsymbol{u}_\varepsilon$, which is allowed due to $u_\varepsilon \in W(0,T)$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_\varepsilon|^2 + \nu\|\boldsymbol{u}_\varepsilon\|^2 + \underbrace{\langle \mathcal{B}_\varepsilon(\boldsymbol{u}_\varepsilon), \boldsymbol{u}_\varepsilon \rangle}_{=0} = \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon \rangle.$$

Estimating the RHS by YOUNG's inequality we deduce that

$$|\boldsymbol{u}_\varepsilon(t)|^2 + \nu\int_0^t \|\boldsymbol{u}_\varepsilon(s)\|^2\,\mathrm{d}s \leqslant |\boldsymbol{u}_0|^2 + \frac{1}{\nu}\int_0^t \|\boldsymbol{f}(s)\|^2.$$

We infer that

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(0,T;H)} + \|\boldsymbol{u}_\varepsilon\|^2_{L^2(0,T;V)} \leqslant C$$

for all $\varepsilon > 0$. By the Theorem of BANACH-ANAOGLU there exists a subsequence such that $\boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $L^\infty(0,T;H)$ and $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;H)$.

We need some strong convergence to pass to the limit in the nonlinear term $\mathcal{B}_\varepsilon$, for which we need information about the time derivative.

6 Time derivative We observe that

$$\begin{aligned}
\|\boldsymbol{u}_\varepsilon'(t)\|_* &= \sup_{\substack{v\in V\\ \|v\|=1}} \langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{v} \rangle = \sup_{\substack{v\in V\\ \|v\|=1}} \langle \boldsymbol{f}(t) - \mathcal{A}\,\boldsymbol{u}_\varepsilon(t) - \mathcal{B}(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v} \rangle \\
&\leqslant \|\boldsymbol{f}(t)\|_* + \|\mathcal{A}\,\boldsymbol{u}_\varepsilon(t)\|_* + \|\mathcal{B}(\boldsymbol{u}_\varepsilon(t))\|_* \\
&\overset{(H)}{\leqslant} \|\boldsymbol{f}(t)\|_* + \nu\|\boldsymbol{u}_\varepsilon(t)\|_* + \|(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon)(t)\|_{L^4}\|\boldsymbol{u}_\varepsilon)(t)\|_{L^4} \\
&\leqslant \|\boldsymbol{f}(t)\|_* + \nu\|\boldsymbol{u}_\varepsilon(t)\|_* + \|\boldsymbol{u}_\varepsilon(t)\|^2_{L^4} \\
&\overset{2}{\leqslant} \|\boldsymbol{f}(t)\|_* + \nu\|\boldsymbol{u}_\varepsilon(t)\|_* + \|\boldsymbol{u}_\varepsilon(t)\|^{\frac{3}{2}}|\boldsymbol{u}_\varepsilon(t)|^{\frac{1}{2}}. \qquad (d=3)
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\boldsymbol{u}_\varepsilon(t)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,T;V*)} &\leqslant C\int_0^T \|\boldsymbol{f}(t)\|_*^{\frac{4}{3}} + \nu^{\frac{4}{3}}\|\boldsymbol{u}_\varepsilon(t)\|_*^{\frac{4}{3}} + \|\boldsymbol{u}_\varepsilon(t)\|^2|\boldsymbol{u}_\varepsilon(t)|^{\frac{2}{3}}\,\mathrm{d}t \\
&\leqslant C\Big(\|\boldsymbol{f}\|^{\frac{4}{3}}_{L^2(0,T;V*)} + \nu^{\frac{4}{3}}\|\boldsymbol{u}_\varepsilon\|^{\frac{4}{3}}_{L^2(0,T;V*)} \\
&\qquad + \|\boldsymbol{u}_\varepsilon\|^2_{L^2(0,T;V*)}|\boldsymbol{u}_\varepsilon(t)|^{\frac{2}{3}}_{L^\infty(0,T;H)}\Big)
\end{aligned}$$

by using JENSEN's inequality (???).

We extract another subsequence such that

$$\boldsymbol{u}_\varepsilon' \overset{*}{\rightharpoonup} \boldsymbol{u} \qquad \text{in } L^{\frac{4}{3}}(0,T;V^*).$$

By LIONS-AUBIN we infer strong convergence such that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \qquad \text{in } L^2(0,T;H).$$

Using this, we want to show that we can pass to the limit, identifying the nonlinear term:

$$\underbrace{\boldsymbol{u}_\varepsilon'}_{\overset{*}{\rightharpoonup}\boldsymbol{u}'} + \underbrace{\mathcal{A}\,\boldsymbol{u}_\varepsilon}_{\overset{*}{\rightharpoonup}\mathcal{A}\,\boldsymbol{u}} + \underbrace{\mathcal{B}_\varepsilon(\boldsymbol{u}_\varepsilon)}_{\rightharpoonup ???} = \boldsymbol{f}.$$

But first, we deduce strong pointwise convergence. By the inverse of LEBESGUE's theorem, we know that there exists another subsequence such that

$$\boldsymbol{u}_\varepsilon(x,t) \to \boldsymbol{u}(x,t) \qquad \text{almost everywhere in } \Omega\times(0,T)$$

and that there exists a dominating function in $L^2(\Omega \times (0,T))$. Due to that pointwise convergence, we have

$$(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon)(x,t) \to u(x,t) \qquad \text{almost everywhere in } \Omega \times (0,T).$$

Since the norm of $\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon$ is dominated by $\boldsymbol{u}_\varepsilon$ in the L^2-norm, there is a dominating function in L^2 for $\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon$.

For all $\psi \in \mathcal{C}_c^\infty(0,T;\mathcal{V})$ we have

$$\begin{aligned}
&\int_0^t |b(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \psi) - b(\boldsymbol{u}, \boldsymbol{u}, \psi)| \,\mathrm{d}t \\
\leqslant &\int_0^t |b(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}, \psi)| + |b(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon - \boldsymbol{u}, \boldsymbol{u}, \psi)| \,\mathrm{d}t \\
\leqslant &\int_0^t \underbrace{\|\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon\|_{L^2(\Omega)}}_{\substack{\text{converges}\\ \text{as it is bd.}}} \underbrace{\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{L^2(\Omega)}}_{\xrightarrow{\varepsilon \searrow 0} 0} + \underbrace{\|\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{L^2(\Omega)}}_{\xrightarrow{\varepsilon \searrow 0} 0} \|\boldsymbol{u}\|_{L^2(\Omega)} \|\psi\|_{W^{1,\infty}} \,\mathrm{d}t.
\end{aligned}$$

Hence

$$\langle \mathcal{B}_\varepsilon(\boldsymbol{u}_\varepsilon), \psi \rangle \xrightarrow{\varepsilon \searrow 0} \langle \mathcal{B}(\boldsymbol{u}), \psi \rangle \qquad \forall \psi \in \mathcal{C}_c^\infty(0,T;\mathcal{V}).$$

By density and by boundedness of $\mathcal{B}$ in $L^{\frac{4}{3}}(0,T;V^*)$ (for $d=3$), we get

$$\mathcal{B}_\varepsilon(\boldsymbol{u}_\varepsilon) \overset{*}{\rightharpoonup} \mathcal{B}(\boldsymbol{u}) \qquad \text{in } L^{\frac{4}{3}}(0,T;V^*).$$

Convergence of the initial values follows since they are fixed for all $\varepsilon > 0$.

(Tim's alternative: We have

$$|u_\varepsilon(t)|^2 + \nu \|u_\varepsilon\|^2_{L^2(0,t;V)} \leqslant |u_0|^2 + \frac{1}{\nu}\|f\|^2_{L^2(0,T;V)}$$

for a.e. $t \in (0,T)$. Thus, (u_ε) is bounded in $L^2(0,T;V)$ and $L^\infty(0,T;H)$ and we find

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(0,T;V) \quad \text{and} \quad u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;H).$$

We will now show boundedness of (u'_ε) in $L^{4/3}(0,T;V^*)$. To this end, first notice

$$\begin{aligned}
\|B_\varepsilon(u_\varepsilon, u_\varepsilon)\|_*^{4/3} &\leqslant \|\rho_\varepsilon * u_\varepsilon\|^{4/3}_{L^4(\Omega)} \|u_\varepsilon\|^{4/3}_{L^4(\Omega)} \\
&\leqslant \|\rho_\varepsilon^T * u_\varepsilon\|^{4/3}_{L^4(\Omega)} \|u_\varepsilon\|^{4/3}_{L^4(\Omega)} \leqslant c\|\rho_\varepsilon^T * u_\varepsilon\| \|u_\varepsilon\| |\rho_\varepsilon^T * u_\varepsilon|^{1/3} |u_\varepsilon|^{1/3},
\end{aligned}$$

hence

$$\begin{aligned}
&\|\mathcal{B}_\varepsilon(u_\varepsilon, u_\varepsilon)\|^{4/3}_{L^{4/3}(0,T;V)} \\
&\leqslant c\|\rho_\varepsilon^T * u_\varepsilon\|_{L^2(0,T;V)} \|u_\varepsilon\|_{L^2(0,T;V)} \|\rho_\varepsilon^T * u_\varepsilon\|^{1/3}_{L^\infty(0,T;H)} \|u_\varepsilon\|^{1/3}_{L^\infty(0,T;H)} \\
&\leqslant c\|u_\varepsilon\|^2_{L^2(0,T;V)} \|u_\varepsilon\|^{2/3}_{L^\infty(0,T;H)}.
\end{aligned} \tag{45}$$

We have

$$\begin{aligned}
\|u'_\varepsilon\|^{4/3}_{L^{4/3}(0,T;V^*)} &= \|f - \mathcal{A}\, u_\varepsilon - \mathcal{B}_\varepsilon(u_\varepsilon, u_\varepsilon)\|^{4/3}_{L^{4/3}(0,T;V^*)} \\
&\leqslant c\left(\|f\|^2_{L^2(0,T;V^*)} + \left(\nu^{4/3} + \|u_\varepsilon\|^{2/3}_{L^\infty(0,T;H)}\right)\|u_\varepsilon\|^2_{L^2(0,T;V)}\right).
\end{aligned}$$

Thus,

$$u'_\varepsilon \rightharpoonup \dot{u} \quad \text{in} \quad L^{4/3}(0,T;V^*).$$

Taking arbitrary $\varphi \in \mathcal{C}_c^\infty(0,T)$ and $v \in V$ we find

$$-\langle u, v\varphi' \rangle \longleftarrow -\langle u_\varepsilon, v\varphi' \rangle = \langle u'_\varepsilon, \varphi v \rangle \to \langle \dot{u}, \varphi v \rangle,$$

hence

$$-\left\langle \int_0^T u\varphi' \,\mathrm{d}t, v \right\rangle = -\langle u, v\varphi' \rangle = \langle \dot{u}, \varphi v \rangle = \left\langle \int_0^T \dot{u}\varphi \,\mathrm{d}t, v \right\rangle$$

and so

$$\int_0^T u\varphi' \,\mathrm{d}t = \int_0^T \dot{u}\varphi \,\mathrm{d}t, \qquad \text{hence} \qquad u' = \dot{u}.$$

By Aubin-Lions we find strong convergence

$$u_\varepsilon \to u \quad \text{in} \quad L^2(0,T;H).$$

Using it, we obtain

$$\begin{aligned}
|\langle \mathcal{B}_\varepsilon(u_\varepsilon,u_\varepsilon) - \mathcal{B}\,u, \varphi\rangle| &\leqslant \int_0^T |\langle B(\rho_\varepsilon * u_\varepsilon - u, u_\varepsilon), \varphi\rangle| + |\langle B(u, u_\varepsilon - u), \varphi\rangle| \,\mathrm{d}t \\
&\leqslant \left(\|\rho_\varepsilon * u_\varepsilon - u\|_{L^2(0,T;H)}\|u_\varepsilon\|_{L^2(0,T;H)} + \|u\|_{L^2(0,T;H)}\|u_\varepsilon - u\|_{L^2(0,T;H)}\right)\|\nabla\varphi\|_{L^\infty((0,T)\times\Omega)} \to 0
\end{aligned}$$

for $\varepsilon \to 0$, thus

$$\langle \mathcal{B}_\varepsilon(u_\varepsilon,u_\varepsilon) - \mathcal{B}\,u, \varphi\rangle \to 0 \quad \text{for all} \quad \varphi \in \mathcal{C}_c^\infty((0,T)\times\Omega).$$

By density and (45) we find

$$\langle \mathcal{B}_\varepsilon(u_\varepsilon,u_\varepsilon) - \mathcal{B}\,u, v\rangle \to 0 \quad \text{for all} \quad v \in L^4(0,T;V)$$

or just

$$\mathcal{B}_\varepsilon(u_\varepsilon,u_\varepsilon) \overset{*}{\rightharpoonup} \mathcal{B}\,u \quad \text{in} \quad L^{4/3}(0,T;V^*).$$

Clearly, also

$$\langle \mathcal{A}\,u_\varepsilon - \mathcal{A}\,u, v\rangle = \nu\langle u_\varepsilon - u, v\rangle_{L^2(0,T;V^*)\times L^2(0,T;V)} \to 0,$$

hence

$$\mathcal{A}\,u_\varepsilon \overset{*}{\rightharpoonup} \mathcal{A}\,u \quad \text{in} \quad L^2(0,T;V^*).$$

To identify the initial value we use that

$$u_m, u \in \mathrm{AC}([0,T];V^*)$$

holds. Let $v \in \mathcal{C}^\infty([0,T];V)$ with $v(0) = w \in V$ and $v(T) = 0$ be arbitrary. We find

$$\begin{aligned}
\langle u_0, w\rangle = \langle u_\varepsilon(0), w\rangle &= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle u_\varepsilon, v\rangle \,\mathrm{d}t = -\int_0^T \langle u_\varepsilon', v\rangle + \langle u_\varepsilon, v'\rangle \,\mathrm{d}t \\
&\to -\int_0^T \langle u', v\rangle + \langle u, v'\rangle \,\mathrm{d}t = -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle u, v\rangle \,\mathrm{d}t = \langle u(0), w\rangle.
\end{aligned}$$

As $w \in V$ can be chosen freely, we obtain $u(0) = u_0$ in V^*.

Overall, we obtain

$$f = u_\varepsilon' + \mathcal{A}\,u_\varepsilon + \mathcal{B}_\varepsilon(u_\varepsilon,u_\varepsilon) \overset{*}{\rightharpoonup} u' + \mathcal{A}\,u + \mathcal{B}\,u \quad \text{in} \quad L^{4/3}(0,T;V^*)$$

and

$$u(0) = u_0 \quad \text{in} \quad V^*,$$

where

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap W^{1,1}(0,T;V^*).$$

) □

5.5 | Fractional time derivative

The regularisation we used in the above proof left us in a good position since we were allowed to test with V. If we used a GALERKIN approximation we would have need V-stability of the H-projection onto the GALERKIN subspaces. We need additional regularity of the boundary in order to have such a stability, and this often does not hold for FEM.

We need some information on the time derivative of the approximate sequence in order to deduce strong convergence, which is crucial for passing to the limit in the nonlinear term (weak convergence does not suffice!). But the sequence $(\boldsymbol{u}_\varepsilon')_{\varepsilon>0}$ is not bounded in $L^2(0,T;V^*)$. We need something weaker: either $L^{\frac{4}{3}}(0,T;V^*)$ or a SOBOLEV-SLOBODECKIJ space. The space $W^{\sigma,p}(0,T;H)$ can be obtained by properly interpolating between $L^p(0,T;H)$ and $W^{1,p}(0,T;H)$.

Definition 5.5.1 (Sobolev-Slobodeckij space)
Let $p \in [1,\infty)$ and $\sigma \in (0,1)$. Then

$$W^{\sigma,p}(0,T;H) := \{\boldsymbol{u} \in L^p(0,T;H) : \|\boldsymbol{u}\|_{W^{\sigma,p}(0,T;H)} < \infty\},$$

where

$$|\boldsymbol{u}|_{W^{\sigma,p}(0,T;H)} := \int_0^T \int_0^T \frac{|\boldsymbol{u}(t)-\boldsymbol{u}(s)|^p}{|t-s|^{1+\sigma p}} \, \mathrm{d}t \, \mathrm{d}s$$

and

$$\|\boldsymbol{u}\|_{W^{\sigma,p}(0,T;H)} := \left(\|\boldsymbol{u}\|^p_{W^{\sigma,p}(0,T;H)} + |\boldsymbol{u}|^p_{W^{\sigma,p}(0,T;H)}\right)^{\frac{1}{p}}.$$

We abbreviate $H^\sigma(0,T;H) := W^{\sigma,2}(0,T;H)$.

We have

$$H^1(0,T;H) \hookrightarrow H^\sigma(0,T;H) \hookrightarrow L^2(0,T;H)$$

(Exercise!) and furthermore for $\sigma \in (0,\frac{1}{2})$

$$H^\sigma(0,T;H) = H_0^\sigma(0,T;H) := \operatorname{clos}_{\|\cdot\|_{\sigma,2}} \mathcal{C}_c^\infty(0,T;H),$$

which means that if we take derivative of order lower that $\frac{1}{2}$, we don't *see* the boundary conditions. For $\sigma > \frac{1}{2}$, it doesn't matter if we first take the closure and then interpolate or the other way around.

Lemma 5.5.2
Let $V \overset{c}{\hookrightarrow} H$ for Banach *spaces V and H. Then we have*

$$L^2(0,T;V) \cap H^\sigma(0,T;H) \overset{c}{\hookrightarrow} L^2(0,T;H) \qquad \forall \sigma > 0.$$

Lemma 5.5.3
Let $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ be the sequence of approximate solutions. Then it holds for $\sigma < \frac{1}{8}$ (even holds for $\sigma < \frac{1}{4}$, see Exercises) that $|\boldsymbol{u}_\varepsilon|_{\sigma,2} \leqslant C$, where C is independent of ε.

Proof. It suffices to prove that $|\boldsymbol{u}_\varepsilon|_{\sigma,2} \leqslant C$ for all $\varepsilon > 0$. We have

$$\begin{aligned}
|\boldsymbol{u}_\varepsilon(t)-\boldsymbol{u}_\varepsilon(s)|^2 &= \big(\boldsymbol{u}_\varepsilon(t)-\boldsymbol{u}_\varepsilon(s), \boldsymbol{u}_\varepsilon(t)-\boldsymbol{u}_\varepsilon(s)\big) \\
&= \left(\int_s^t \boldsymbol{u}_\varepsilon'(\tau)\,\mathrm{d}\tau, \boldsymbol{u}_\varepsilon(t)-\boldsymbol{u}_\varepsilon(s)\right) \\
&= \int_s^t \big(\boldsymbol{f}(\tau) - \mathcal{A}\,\boldsymbol{u}_\varepsilon(\tau) - \mathcal{B}\,\boldsymbol{u}_\varepsilon(\tau), \boldsymbol{u}_\varepsilon(t)-\boldsymbol{u}_\varepsilon(s)\big)\,\mathrm{d}\tau \\
&\leqslant \int_s^t (\|\boldsymbol{f}(\tau)\|_* + \nu\|\boldsymbol{u}_\varepsilon(\tau)\| + C\|(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon)(\tau)\|_{L^4}\|\boldsymbol{u}_\varepsilon(\tau)\|_{L^4}) \\
&\qquad \cdot (\|\boldsymbol{u}_\varepsilon(t)\| + \|\boldsymbol{u}_\varepsilon(s)\|)\,\mathrm{d}\tau.
\end{aligned}$$

By 2, as before,

$$\|(\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon)(\tau)\|_{L^4}\|\boldsymbol{u}_\varepsilon(\tau)\|_{L^4} \leqslant \|\boldsymbol{u}_\varepsilon\|^2_{L^4} \leqslant \tilde{C}\|\boldsymbol{u}_\varepsilon\|^{\frac{3}{2}}_{L^4}|\boldsymbol{u}_\varepsilon|^{\frac{1}{2}}.$$

We use the abbreviation

$$g(\tau) := \|\boldsymbol{f}(\tau)\|_* + \nu\|\boldsymbol{u}_\varepsilon(\tau)\| + \tilde{C}\|\boldsymbol{u}_\varepsilon\|^{\frac{3}{2}}_{L^4}|\boldsymbol{u}_\varepsilon|^{\frac{1}{2}}.$$

We can now find

$$
\begin{aligned}
|\boldsymbol{u}_\varepsilon|_{\sigma,2}^2 &= \int_0^T \int_0^T \frac{|\boldsymbol{u}_\varepsilon(t) - u_\varepsilon(s)|^2}{|t-s|^{1+2\sigma}} \,\mathrm{d}t \,\mathrm{d}s \\
&= \int_0^T \int_0^T |t-s|^{-1-2\sigma} \int_s^t g(\tau) \left(\|\boldsymbol{u}_\varepsilon(t)\| + \|\boldsymbol{u}_\varepsilon(s)\|\right) \mathrm{d}\tau \,\mathrm{d}t \,\mathrm{d}s \\
&= 2 \int_0^T \int_0^T \int_{\min(s,t)}^{\max(s,t)} |t-s|^{-1-2\sigma} g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}s \,\mathrm{d}t \\
&= 2 \int_0^T \int_0^T \int_s^t |t-s|^{-1-2\sigma} g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}s \,\mathrm{d}t \\
&\quad + 2 \int_0^T \int_0^T \int_t^s |t-s|^{-1-2\sigma} g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}s \,\mathrm{d}t \\
&\overset{\text{(F)}}{=} 2 \int_0^T \int_0^T \int_0^\tau |t-s|^{-1-2\sigma} \,\mathrm{d}s g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}t \\
&\quad + 2 \int_0^T \int_0^T \int_\tau^T |t-s|^{-1-2\sigma} \,\mathrm{d}s g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}t \\
&= \frac{1}{\sigma} \int_0^T \int_0^T \left((t-\tau)^{-2\sigma} - t^{-2\sigma}\right) g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}t \\
&\quad + \frac{1}{\sigma} \int_0^T \int_0^T \left((\tau-t)^{-2\sigma} - (T-t)^{-2\sigma}\right) g(\tau) \|\boldsymbol{u}_\varepsilon(t)\| \,\mathrm{d}\tau \,\mathrm{d}t \\
&= \text{TODO}
\end{aligned}
$$

□

We may use the compact embedding

$$L^2(0,T;V) \cap H^\sigma(0,T;H) \overset{c}{\hookrightarrow} L^2(0,T;H).$$

We deduce $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in $L^2(0,T;H)$. Now proceed as in the previous proof to prove the convergence

$$\mathcal{B}_\varepsilon(\boldsymbol{u}_\varepsilon) \overset{*}{\rightharpoonup} \mathcal{B}(\boldsymbol{u}) \qquad \text{in } L^{\frac{4}{3}}(0,T;H).$$

5.6 Weak-strong uniqueness

Recall that we have at least one solution $\boldsymbol{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ with

$$\boldsymbol{u}' \in \begin{cases} L^2(0,T;V^*) & \text{for } d = 2, \\ L^{\frac{4}{3}}(0,T;V^*) & \text{for } d = 3, \end{cases} \subset L^1(0,T;V^*)$$

(e.g. less time integrability for the time derivative in $d = 3$) and

$$\boldsymbol{u} \in \begin{cases} \mathcal{C}([0,T];H) & \text{for } d = 2, \\ \mathcal{C}_{\mathrm{w}}([0,T];H) & \text{for } d = 3. \end{cases}$$

Furthermore, $\boldsymbol{u}$ enjoys the fractional time regularity

$$\boldsymbol{u} \in H^\sigma(0,T;H) \quad \text{for } \sigma < \begin{cases} \frac{1}{2} & \text{for } d = 2, \\ \frac{1}{4} & \text{for } d = 3. \end{cases}$$

Weak-strong uniqueness means that if there is a solution admitting additional regularity, then this *strong solution* is unique in the class of weak solutions. So whenever there is a strong solution, this strong solution coincides with every weak solution to the same initial values and right hand side $\boldsymbol{f}$.

It is a major drawback of weak solutions, that one cannot show weak-strong uniqueness; one has to get an additional ingredient - the energy inequality, which comes from the physical insight (energy is conserved).

Definition 5.6.1 (Suitable weak solutions)
A weak solution to the Navier-Stokes equations is suitable if it fulfills the energy inequality

$$\frac{1}{2}|\boldsymbol{u}(t)|^2 + \nu \int_0^t \|\boldsymbol{u}(s)\|^2 \,\mathrm{d}s \leqslant \frac{1}{2}|\boldsymbol{u}_0|^2 + \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s) \rangle \,\mathrm{d}s \qquad \forall t \in (0,T). \tag{46}$$

For $d = 2$, the energy inequality is an equality.

Theorem 5.6.1: Existence of suitable weak solutions

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain for $d \in \{2,3\}$. Then there exists a suitable weak solution to the Navier-Stokes equations.

Proof. In dimension 2. Since $\boldsymbol{u} \in W(0,T)$, we test the abstract equation by $\boldsymbol{u}$ in order to infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}(t)|^2 + \nu\|\boldsymbol{u}(t)\|^2 = \langle \boldsymbol{u}'(t), \boldsymbol{u}(t) \rangle + \nu\big((\boldsymbol{u}(t), \boldsymbol{u}(t))\big) + \underbrace{b\big(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)\big)}_{=0} = \langle \boldsymbol{f}(t), \boldsymbol{u}(t) \rangle.$$

Integrating in time from 0 to $t \in (0,T)$ yields

$$\frac{1}{2}|\boldsymbol{u}(t)|^2 + \nu \int_0^t \|\boldsymbol{u}(s)\|^2 \,\mathrm{d}s = \frac{1}{2}|\boldsymbol{u}_0|^2 + \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s) \rangle \,\mathrm{d}s,$$

so the energy inequality is an equality (as mentioned above).

In dimension 3 the problem is that we are not allowed to test with the solution $\boldsymbol{u}$, so we have to go back to the approximation scheme (P_ε) and do this calculation there. On the approximate level, we are allowed to test with $\boldsymbol{u}_\varepsilon \in W(0,T)$, yielding

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_\varepsilon(t)|^2 + \nu\|\boldsymbol{u}_\varepsilon(t)\|^2 &= \langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{u}_\varepsilon(t) \rangle + \nu\big((\boldsymbol{u}_\varepsilon(t), \boldsymbol{u}_\varepsilon(t))\big) + \underbrace{b\big((\boldsymbol{\rho}_\varepsilon * \boldsymbol{u}_\varepsilon)(t), \boldsymbol{u}_\varepsilon(t), \boldsymbol{u}_\varepsilon(t)\big)}_{=0} \\ &= \langle \boldsymbol{f}(t), \boldsymbol{u}_\varepsilon(t) \rangle.\end{aligned}$$

Integrating in time from 0 to $t \in (0,T)$ yields

$$\frac{1}{2}|\boldsymbol{u}_\varepsilon(t)|^2 + \nu \int_0^t \|\boldsymbol{u}_\varepsilon(s)\|^2 \,\mathrm{d}s = \frac{1}{2}|\boldsymbol{u}_0|^2 + \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}_\varepsilon(s) \rangle \,\mathrm{d}s, \tag{47}$$

We now want to pass to limit with $\varepsilon \searrow 0$. In the right hand side the $\boldsymbol{u}_\varepsilon$ appears linearly, so we may pass to the limit using the weak convergence. For the integral left hand side we have to use that the norm is weakly lower semi-continuous and that we have weak convergence in $L^2(0,T;V^*)$. We need a way to pass to the limit pointwise in the very first term to get an equality that holds everywhere and not only almost everywhere, so we need something more than L^∞.

We already established that

$$\|\boldsymbol{u}_\varepsilon'\|_{L^{\frac{4}{3}}(0,T;V^*)} \leqslant c$$

independent of $\varepsilon > 0$. Therefore, $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^\infty(0,T;H) \subset L^1(0,T;V^*)$ and $(\boldsymbol{u}_\varepsilon')_{\varepsilon>0} \subset L^{\frac{4}{3}}(0,T;V^*) \subset L^1(0,T;V^*)$. Hence

$$(\boldsymbol{u}_\varepsilon')_{\varepsilon>0} \subset W^{1,1}(0,T;V^*) \hookrightarrow \mathrm{AC}([0,T];V^*) \subset \mathcal{C}_w([0,T];V^*).$$

We know $\mathcal{C}_w([0,T];V^*) \cap L^\infty(0,T;H) \subset \mathcal{C}_w([0,T];H)$ by lemma 5.3.5. So a weakly continuous function in a weaker space which is bounded in a stronger space, is also weakly continuous in the stronger space.

We observe that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathcal{C}_w([0,T];H)$. We need some information on the time derivative and thus we can get this weak convergence pointwise - L^∞ is not enough to get weak convergence pointwise.

There exists a subsequence such that

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \qquad \text{in } \mathcal{C}_w([0,T];H),$$

which means that for every $t \in [0,T]$ holds that

$$\boldsymbol{u}_\varepsilon'(t) \rightharpoonup \boldsymbol{u}(t) \qquad \text{in } H.$$

For all $t \in [0,T]$, we observe that by the lower semi-continuity of the H-norm we have

$$|\boldsymbol{u}(t)| \leqslant \liminf_{\varepsilon \searrow 0} |\boldsymbol{u}_\varepsilon(t)|.$$

The same holds for the weak solution. Similarly, since $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;H)$, we have

$$\int_0^t \|\boldsymbol{u}(s)\|^2 \,\mathrm{d}s \leqslant \liminf_{\varepsilon \searrow 0} \int_0^t \|\boldsymbol{u}_\varepsilon(s)\|^2 \,\mathrm{d}s \qquad \forall t \in (0,T).$$

From $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;V)$, we deduce

$$\int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}_\varepsilon(s) \rangle \,\mathrm{d}s \to \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s) \rangle \,\mathrm{d}s \qquad \forall t \in [0,T].$$

We have now derived for convergence for every term and end up with an inequality. Passing to the limit $\varepsilon \searrow 0$ in the energy equality (47), we deduce the energy inequality (46). $\square$

We will now consider the uniqueness of solutions in 2D and the weak-strong uniqueness of solutions in 3D.

Lemma 5.6.2 (Intermediate regularity result)
Let $\boldsymbol{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Then it holds that

$$\boldsymbol{u} \in \begin{cases} L^4(0,T;L^4(\Omega)^d) \cong L^4(\Omega \times (0,T))^d, & \textit{for } d = 2, \\ L^{\frac{8}{3}}(0,T;L^4(\Omega)^d), & \textit{for } d = 3. \end{cases}$$

Proof. For $d = 2$ we have by [2]

$$\int_0^T \|\boldsymbol{u}(t)\|_{L^4(\Omega)}^4 \,\mathrm{d}t \leqslant C \int_0^T |\boldsymbol{u}(t)|^2 \|\boldsymbol{u}(t)\|^2 \,\mathrm{d}t \leqslant C \|\boldsymbol{u}\|_{L^\infty(0,T;H)}^2 \|\boldsymbol{u}\|_{L^2(0,T;V)}^2.$$

For $d = 3$ we have by [2]

$$\int_0^T \|\boldsymbol{u}(t)\|_{L^4(\Omega)}^{\frac{8}{3}} \,\mathrm{d}t \leqslant C \int_0^T \|\boldsymbol{u}(t)\|^{\frac{3}{4}\cdot\frac{8}{3}} |\boldsymbol{u}(t)|^{\frac{1}{4}\frac{8}{3}} \,\mathrm{d}t \leqslant C \|\boldsymbol{u}\|_{L^\infty(0,T;H)}^{\frac{2}{3}} \|\boldsymbol{u}\|_{L^2(0,T;H)}^2. \qquad \square$$

Theorem 5.6.2: Uniqueness for $d = 2$

For $d = 2$ there exists a unique solution $\boldsymbol{u} \in W(0, T)$ to the Navier-Stokes equations.

Proof. Let $\boldsymbol{u}, \tilde{\boldsymbol{u}}$ be two solutions such that

$$\begin{cases} (\boldsymbol{u} - \tilde{\boldsymbol{u}})' + \mathcal{A}(\boldsymbol{u} - \tilde{\boldsymbol{u}}) + \mathcal{B}(\boldsymbol{u}) - \mathcal{B}(\tilde{\boldsymbol{u}}) = 0, & \text{in } L^2(0, T; V^*), \\ (\boldsymbol{u} - \tilde{\boldsymbol{u}})(0) = \boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0, & \text{in } H. \end{cases}$$

Since $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in W(0, T)$, we may test the equations by $\boldsymbol{u} - \tilde{\boldsymbol{u}}$, inferring

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{u} - \tilde{\boldsymbol{u}}|^2 + \nu \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2 + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \tilde{\boldsymbol{u}}) - b(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}}) = 0.$$

We observe that

$$\begin{aligned} |b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \tilde{\boldsymbol{u}}) - b(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}})| &= \Big| \underbrace{b(\boldsymbol{u}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}})}_{=0} - b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}}) \Big| \\ &= b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) \\ &\overset{\text{(H)}}{\leqslant} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^4} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\| \|\tilde{\boldsymbol{u}}\|_{L^4} \\ &\overset{2}{\leqslant} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^{\frac{3}{2}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}|_{L^4}^{\frac{1}{2}} \|\tilde{\boldsymbol{u}}\|_{L^4} \\ &\overset{\text{(Y)}}{\leqslant} \frac{\nu}{2} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2 + C_\nu |\boldsymbol{u} - \tilde{\boldsymbol{u}}|^2 \|\tilde{\boldsymbol{u}}\|_{L^4}, \end{aligned}$$

where (H) is the generalised Hölder inequality and (Y) is Young's inequality. By Gronwall's inequality, we have for all $t \in (0, T)$

$$\begin{aligned} |\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)|^2 + \underbrace{\int_0^t \nu \|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|^2 \exp\left(\int_s^t C_\nu \|\tilde{\boldsymbol{u}}\|_{L^4}^4 \,\mathrm{d}\tau \right) \mathrm{d}s}_{\geqslant 0} \\ \leqslant |\boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0|^2 \exp\left(\int_s^t C_\nu \|\tilde{\boldsymbol{u}}\|_{L^4}^4 \,\mathrm{d}\tau \right) \mathrm{d}s. \end{aligned}$$

□

Theorem 5.6.3: Weak-strong uniqueness property (Serrin 1962, Prodi 1959)

Assume that a suitable weak solution $\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$ to the Navier-Stokes equations fulfills additionally

$$\boldsymbol{u} \in L^s(0, T; L^r(\Omega)^d)$$

with $s \in [2, \infty]$, $r \geqslant d$ such that

$$\frac{2}{s} + \frac{d}{r} \leqslant 1.$$

Then the solution is unique in the class of suitable weak solutions.

Example. 5.6.3
Standard value combinations for s and r include

- In 2D, we can choose $(s, r) = (4, 4)$ or $(s, r) = (\infty, 2)$. We get the function space $L^4(0, T; L^4(\Omega)^d)$ and $L^\infty(0, T; L^2(\Omega)^d)$.

- In 3D, there is at most one solution in $L^8(0,T;L^4(\Omega)^d)$, that is, $(s,r)=(8,4)$. As $8>\frac{8}{3}$, this function is class is smaller than the class considered beforehand. $\diamond$

Remark. 5.6.4 In three dimensions, only a weak-strong uniqueness results holds: if there exists a solution fulfilling the additional regularity $L^8(0,T;L^4(\Omega)^d)$, all weak solutions emanating from the same initial value coincide with the regular solutions (it is unique).

Problem for well-posedness in 3D. Existence in $L^{\frac{8}{3}}(0,T;L^4(\Omega)^d)$ and uniqueness in $L^8(0,T;L^4(\Omega)^d)$. There is a regularity gap, uniqueness only in a smaller space.

Proof. (of Theorem 5.6.3) Let $\boldsymbol{u}$ be a suitable weak solution to the NAVIER-STOKES equations. Then $\boldsymbol{u}\in L^\infty(0,T;H)\cap L^2(0,T;V)\cap W^{1,\frac{4}{3}}(0,T;V^*)$.

Let $\tilde{\boldsymbol{u}}$ be a more regular solution with $\tilde{\boldsymbol{u}}\in L^8(0,T;L^4(\Omega)^d)$. Now we can get better estimates for the nonlinear operator $\mathcal{B}$ (we will see that it maps to $L^2(0,T;V^*)$).

Then

$$\|\mathcal{B}\tilde{\boldsymbol{u}}\|_{L^2(0,T;V^*)}=\int_0^T\|\mathcal{B}\tilde{\boldsymbol{u}}(t)\|_*^2\,\mathrm{d}t\leqslant c\int_0^T(\|\tilde{\boldsymbol{u}}(t)\|_{L^4(\Omega)}^2)^2\,\mathrm{d}t\leqslant c\|\tilde{\boldsymbol{u}}\|_{L^4(0,T;L^4(\Omega)^d)}^4<\infty.$$

This implies $\tilde{\boldsymbol{u}}\in W(0,T)$ by $\tilde{\boldsymbol{u}}'=\boldsymbol{f}-\mathcal{A}\tilde{\boldsymbol{u}}-\mathcal{B}(\tilde{\boldsymbol{u}})$ in $L^2(0,T;V^*)$. Hence testing with this function is allowed such that we infer the energy equality

$$\frac{1}{2}|\tilde{\boldsymbol{u}}|^2+\int_0^t\nu\|\tilde{\boldsymbol{u}}(s)\|^2\,\mathrm{d}s=\frac{1}{2}|\tilde{\boldsymbol{u}}_0|^2+\int_0^t\langle\boldsymbol{f}(s),\tilde{\boldsymbol{u}}(s)\rangle\mathrm{d}s\qquad\forall t\in(0,T).$$

We now show an integration-by-parts rule:

$$\big(\boldsymbol{u}(t),\boldsymbol{v}(t)\big)-\int_0^t\big(\boldsymbol{u}(s),\partial_t\boldsymbol{v}(s)\big)-\big(\boldsymbol{u}_0,\boldsymbol{v}_0\big)=\int_0^t\big(\partial_t\boldsymbol{u}(s),\boldsymbol{v}(s)\big)\,\mathrm{d}s$$

for all $\boldsymbol{u}\in\mathcal{C}_w(0,T;H)\cap W^{1,\frac{4}{3}}(0,T;V^*)\cap L^2(0,T;V)$ as well as $\boldsymbol{v}\in W^{(,0},T)\cap L^8(0,T;L^4(\Omega)^d)$. This holds for continuously differentiable function, then generalised by density arguments.

For the weak solution $\boldsymbol{u}$, it holds

$$\begin{aligned}\big(\boldsymbol{u}(t),\boldsymbol{v}(t)\big)-\int_0^t\langle\boldsymbol{v}'(t),\boldsymbol{u}(s)\rangle\mathrm{d}s-\big(\boldsymbol{u}_0,\boldsymbol{v}(0)\big)+\int_0^t\nu((\boldsymbol{u}(s),\boldsymbol{v}(s)))+\langle\mathcal{B}(\boldsymbol{u}(s)),\boldsymbol{v}(s)\rangle\\ =\int_0^t\langle\boldsymbol{f}(s),\boldsymbol{v}(s)\rangle\mathrm{d}s,\end{aligned}$$

which can be shown by integration by parts formula above or one multiplies by a function v which is only supported on $(0,t)$ and then integrates by parts.

This formulation is well defined, observe for the nonlinear operator $\mathcal{B}$:

$$\begin{aligned}\int_0^t b(\boldsymbol{u}(s),\boldsymbol{u}(s),\boldsymbol{v}(s))\,\mathrm{d}s&\leqslant\int_0^t\|\boldsymbol{u}(s)\|_{L^4}\|\boldsymbol{u}(s)\|\|\boldsymbol{v}(s)\|_{L^4}\,\mathrm{d}s\\&\overset{2}{\leqslant}\int_0^t\|\boldsymbol{u}(s)\|^{\frac{3}{2}}\|\boldsymbol{v}(s)\|_{L^4}\,\mathrm{d}s\\&\overset{(\mathrm{Y})}{\leqslant}\int_0^t\frac{\nu}{2}\|\boldsymbol{u}(s)\|^2+C_\nu|\boldsymbol{u}(s)|^2\|\boldsymbol{v}(s)\|_{L^4}^4\,\mathrm{d}s,\end{aligned}$$

so all terms are well-defined.

We now use the relative energy approach. We start by defining the relative energy

$$\boxed{\mathcal{R}(\boldsymbol{u}\mid\tilde{\boldsymbol{u}}):=\frac{1}{2}|\boldsymbol{u}-\tilde{\boldsymbol{u}}|^2,}$$

the relative dissipation

$$\boxed{\mathcal{W}(\boldsymbol{u} \mid \tilde{\boldsymbol{u}}) := \nu \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2.}$$

We have by using the energy inequality and the energy inequality

$$\begin{aligned}
\mathcal{R}(\boldsymbol{u} \mid \tilde{\boldsymbol{u}})(t) + \int_0^t \mathcal{W}(\boldsymbol{u} \mid \tilde{\boldsymbol{u}})(s)\,\mathrm{d}s &= \frac{1}{2}|\boldsymbol{u}|^2 + \nu \int_0^t \|\boldsymbol{u}(s)\|^2\,\mathrm{d}s \\
&\quad + \frac{1}{2}|\tilde{\boldsymbol{u}}|^2 + \nu \int_0^t \|\tilde{\boldsymbol{u}}(s)\|^2\,\mathrm{d}s \\
&\quad - \big(\boldsymbol{u}(t), \widetilde{\boldsymbol{u}(t)}\big) - \int_0^t 2\nu((\boldsymbol{u}(s), \tilde{\boldsymbol{u}}(s)))\,\mathrm{d}s \\
&\leqslant \frac{1}{2}|\boldsymbol{u}_0|^2 + \int_0^t \langle\, \boldsymbol{f}(s), \boldsymbol{u}(s) \,\rangle\,\mathrm{d}s + \frac{1}{2}|\tilde{\boldsymbol{u}}_0|^2 \\
&\quad + \cancel{\int_0^t \langle\, \boldsymbol{f}(s), \tilde{\boldsymbol{u}}(s) \,\rangle}\,\mathrm{d}s - \big(\boldsymbol{u}_0, \tilde{\boldsymbol{u}}(0)\big) \\
&\quad - \int_0^t \nu((\boldsymbol{u}(s), \tilde{\boldsymbol{u}}(s)))\,\mathrm{d}s - \int_0^t \langle\, \tilde{\boldsymbol{u}'}(s), \boldsymbol{u}(s) \,\rangle\,\mathrm{d}s \\
&\quad + \int_0^t \langle\, \tilde{\mathcal{B}}(\boldsymbol{u}(s)), \tilde{\boldsymbol{u}}(s) \,\rangle\,\mathrm{d}s - \cancel{\int_0^t \langle\, \boldsymbol{f}(s), \tilde{\boldsymbol{u}}(s) \,\rangle\,\mathrm{d}s} \\
&= \mathcal{R}(\boldsymbol{u}_0 \mid \tilde{\boldsymbol{u}}(0)) + \cancel{\int_0^t \langle\, \boldsymbol{f}(s), \boldsymbol{u}(s) \,\rangle}\,\mathrm{d}s + \int_0^t \langle\, \mathcal{B}(\boldsymbol{u}), \tilde{\boldsymbol{u}} \,\rangle\,\mathrm{d}s \\
&\quad + \int_0^t \langle\, \mathcal{B}(\tilde{\boldsymbol{u}}), \boldsymbol{u} \,\rangle\,\mathrm{d}s - \cancel{\int_0^t \langle\, \boldsymbol{f}(s), \boldsymbol{u}(s) \,\rangle}\,\mathrm{d}s \\
&= \mathcal{R}(\boldsymbol{u}_0 \mid \tilde{\boldsymbol{u}}(0)) + \int_0^t b(\boldsymbol{u}, \boldsymbol{u}, \tilde{\boldsymbol{u}})\,\mathrm{d}s + \int_0^t b(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{u})\,\mathrm{d}s \\
&= \mathcal{R}(\boldsymbol{u}_0 \mid \tilde{\boldsymbol{u}}(0)) + \int_0^t b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u}, \tilde{\boldsymbol{u}})\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u}, \tilde{\boldsymbol{u}})\,\mathrm{d}s &= \int_0^t b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}})\,\mathrm{d}s \\
&\leqslant \int_0^t \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\| \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^4} \|\tilde{\boldsymbol{u}}\|_{L^4}\,\mathrm{d}s \\
&\overset{2}{\leqslant} \int_0^t \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^{\frac{7}{4}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}|^{\frac{1}{4}} \|\tilde{\boldsymbol{u}}\|_{L^4}\,\mathrm{d}s \\
&\overset{(\mathrm{Y})}{\leqslant} \int_0^t \frac{\nu}{2} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2 + C_\nu |\boldsymbol{u} - \tilde{\boldsymbol{u}}|^2 \|\tilde{\boldsymbol{u}}\|_{L^4}^8\,\mathrm{d}s.
\end{aligned}$$

We conclude again by GRONWALL's Lemma

$$\begin{aligned}
&\mathcal{R}\big(\boldsymbol{u}(t) \mid \tilde{\boldsymbol{u}}(t)\big) + \int_0^t \mathcal{W}\big(\boldsymbol{u}(s) \mid \tilde{\boldsymbol{u}}(s)\big) \exp\left(\int_s^t C_\nu \|\tilde{\boldsymbol{u}}(\tau)\|_{L^4}^8\,\mathrm{d}\tau\right) \mathrm{d}s \\
&\leqslant \mathcal{R}\big(\boldsymbol{u}_0 \mid \tilde{\boldsymbol{u}}(0)\big) \exp\left(\int_0^t C_\nu \|\tilde{\boldsymbol{u}}(\tau)\|_{L^4}^8\,\mathrm{d}\tau\right).
\end{aligned}$$

This proves weak-strong uniqueness but also continuous dependence, if such a ?? solution exists. □

5.7 Local existence and uniqueness in three dimensions

In the previous section we assumed a more regular solution exists. Now we want to prove that this is actually the case, but we will only be able to show this on a small time interval or with small data.

We want to use the property of the STOKES-operator A to construct the smooth solution we talked about. if we want to prove something like this, one would test the equation not with u, but with Au or Δu such that we are again in the solenoidal functions. Even if we have the resulting estimates, we need some discretisation argument to make the proof rigorous. We do this here by constructing an appropriate GALERKIN scheme.

Idea: find an estimate for $\|\boldsymbol{u}(t)\|_V$. We want to ask for which $\boldsymbol{v}$ we have

$$\|\boldsymbol{v}\|^2 = \|\boldsymbol{v}\|^2_{H_0^1(\Omega)} = \langle A\boldsymbol{v}, \boldsymbol{v} \rangle = (\boldsymbol{v}, A\boldsymbol{v})_{L^2(\Omega)^3},$$

that is, for which $v \in V$ we have $A\boldsymbol{v} \in H$.

Consider the elliptic problem

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \end{cases}$$

whose solution operator is the STOKES-operator.

Preliminaries

Let $A\colon V \to V^*$ be a linear, bounded, strongly positive and symmetric operator:

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle = ((\boldsymbol{u}, \boldsymbol{v})) = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_{L^2(\Omega)^d}$$

with a GELFAND triple $V \overset{c}{\hookrightarrow} H \overset{d}{\hookrightarrow} V^*$. By LAX-MILGRAM, there exists a solution operator $A^{-1}\colon V^* \to V$, which again is linear, bounded, strongly positive and symmetric.

We consider the restriction $A_F^{-1} := A^{-1}\big|_H \colon V^* \supset H \to \mathcal{D}(A) \subset V \subset H$, where $\mathcal{D}(A) := \operatorname{ran}(A^{-1}|_H)$. Then is A_F^{-1} is bounded in H: for $\boldsymbol{v} \in H$ we have (as $V \hookrightarrow H \hookrightarrow V^*$ implies $|\cdot| \leqslant c\|\cdot\|$ and $\|\cdot\|_* \leqslant c|\cdot|$)

$$|A_F^{-1}\boldsymbol{v}| \leqslant c\|A_F^{-1}\boldsymbol{v}\| = c\|A^{-1}v\| \leqslant c\frac{1}{\mu}\|\boldsymbol{v}\|_* \leqslant c^2\frac{1}{\mu}|\boldsymbol{v}|.$$

As A_F^{-1} is linear and symmetric:

$$(A_F^{-1}\boldsymbol{u}, \boldsymbol{v}) = \langle A^{-1}\boldsymbol{u}, \boldsymbol{v} \rangle = \langle A^{-1}\boldsymbol{v}, \boldsymbol{u} \rangle = (A_F^{-1}\boldsymbol{v}, \boldsymbol{u})$$

Thus it is self-adjoint.

Furthermore, A_F^{-1} is compact: let $(g_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then $(A_F^{-1}g_n)_{n\in\mathbb{N}} \subset V$ is bounded in V. From $V \overset{c}{\hookrightarrow} H$, we infer that $(A_F^{-1}g_n)_{n\in\mathbb{N}} \subset H$ is relatively compact.

By the Spectral Theorem (from Functional Analysis II) there exists an ONB consisting of eigenfunctions of A_F^{-1}, that is, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ of eigenvalues that is bounded and converges to 0 such that the eigenspaces $\ker(A - \lambda_n I)$ are finite dimensional. We find an an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}} \subset H$ consisting of eigenfunctions with $A_F^{-1}\varphi_n = \lambda_n\varphi_n$ (the eigenvalue have to be counted according to their geometric multiplicity). Furthermore, $H = \ker(A_F^{-1}) \oplus \overline{\operatorname{span}}(\{\varphi_n\}_{n\in\mathbb{N}})$ and A^{-1} is strongly positive such that

$$\lambda_n|\varphi_n|^2 = \lambda_n(\varphi_n, \varphi_n) = (\varphi_n, A_F^{-1}\varphi_n) = \nu\|A^{-1}\varphi_n\|^2 \geqslant \frac{\nu}{\beta}\|\varphi_n\|_*^2 \geqslant 0,$$

so $\lambda_n \geqslant 0$ for all $n \in \mathbb{N}$.

We observe that A_F^{-1} is invertible, since $\ker(A_F^{-1}) = \{0\}$, so A_F^{-1} is injective. It is additionally surjective by definition, since it is restricted to $\mathcal{D}(A)$:

$$A_F \colon H \supset \mathcal{D}(A) := \{\boldsymbol{u} \in V : A\boldsymbol{u} \in H\} \to H.$$

The operator A_F is called the Friedrich's extension. In this case the domain can be identified via $\mathcal{D}(A) = V \cap H^2(\Omega)^d$ for the Friedrich's extension (this identification only works when $\partial\Omega$ is of class $\mathcal{C}^2$). As φ_n are eigenfunctions to the eigenvalue λ_n of A^{-1}, they are also eigenfunctions of A_F to the eigenvalue λ_n^{-1}:

$$\lambda_n A_F\varphi_n = A_F(\lambda_n\varphi_n) = A_F(A_F^{-1}\varphi_n) = \varphi_n.$$

We consider the orthogonal projections on $V_m := \operatorname{span}\{\varphi_1, \dots, \varphi_m\}$ defined by $P_m \colon H \to V_m \subset H$ defined via $P_m\boldsymbol{v} := \sum_{j=1}^m (\boldsymbol{v}, \varphi_i)\varphi_i$. Therewith, we infer $P_m\boldsymbol{v} \to \boldsymbol{v}$ for all $\boldsymbol{v} \in H$ as $m \to \infty$ (limit closedness). From the definition, we find $\varphi_j \in \mathcal{D}(A)$ (since $\varphi_j = \lambda_j^{-1}A_F^{-1}\varphi_j \in \mathcal{D}(A)$). Thus $V_m \subset \mathcal{D}(A)$.

Lemma 5.7.1

The eigenfunctions $(\varphi_i)_{i\in\mathbb{N}}$ are a Galerkin *basis in V.*

Proof. TODO □

Theorem 5.7.1: Cattabriga

Let $\Omega \subset \mathbb{R}^d$ be a convex (not needed) bounded domain with $\partial\Omega \in \mathcal{C}^2$. There exists a $c > 0$ such that

$$|A\boldsymbol{v}| \leqslant \|\boldsymbol{v}\|_{H^2(\Omega)^d} \leqslant c|A\boldsymbol{v}| \qquad \forall \boldsymbol{v} \in \mathcal{D}(A) = V \cap H^2(\Omega)^d.$$

Proof. TODO □

Remark. 5.7.2 The first inequality is obvious. From the previous theorem, it follows that the norm $|A\,\cdot\,|$ is equivalent to the full norm on $H^2(\Omega) \cap V$. The assumptions on Ω can be generalised.

Theorem 5.7.2: Weak solution on small time interval

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^2$. To $\boldsymbol{u}_0 \in C$ and $f \in L^\infty(0, T; H)$ there exists a $T_0 := \min(T, T_*)$ with

$$T_* \leqslant \frac{C\nu}{(1 + \|\boldsymbol{u}_0\|^2)^2} \min\left(\nu^2, \frac{1}{\|f\|_{L^\infty(0,T;H)}}\right), \tag{48}$$

such that the Navier-Stokes problem admits a unique weak solution on $[0, T_0)$ with $\boldsymbol{u} \in L^\infty(0, T_0, V) \cap L^2(0, T; \mathcal{D}(A))$.

Remark. 5.7.3 If the data $\|\boldsymbol{u}_0\|$, $\|f\|_{L^\infty(0,T;H)}$ and $\mathrm{Re} = \frac{1}{\nu}$ are small enough, there exists a global (on $[0,T]$) solution.

Proof. TODO $\square$

In the last theorem we proved that for (48), $(u_m)_{m\in\mathbb{N}} \subset L^\infty(0,T_*;V) \cap L^2(0,T_*;\mathcal{D}(A))$ is bounded. As usual the existence of a solution

$$\boldsymbol{u} \in L^\infty(0,T;V) \cap L^2(0,T_*;\mathcal{D}(A))$$

follows. Moreover, $\boldsymbol{u} \in \mathcal{C}([0,T_*);V) \cap W^{1,2}(0,T_*;\mathcal{D}(A))$, which implies uniqueness, i.e. we may test the equation by $\boldsymbol{u}$ in order to find the energy inequality and the uniqueness follows from the previous weak-strong uniqueness theorem as $L^\infty(0,T;V) \hookrightarrow L^8(0,T;L^4(\Omega)^d)$.

Corollary 5.7.4 (Uniqueness in 2D)
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^2$. To $\boldsymbol{u}_0 \in V$ and $\boldsymbol{f} \in L^\infty(0,T;H)$ there exists a unique solution

$$\boldsymbol{u} \in L^\infty(0,T;V) \cap L^2(0,T;\mathcal{D}(A))$$

in dimension 2.

5.8 Existence of the pressure

The Navier-Stokes equations have four unknowns ($\boldsymbol{u} \in \mathbb{R}^3$ and $p \in \mathbb{R}$) but in the weak formulation, the pressure vanishes because we test with solenoidal functions. How do we get the pressure back?

Formally, we may write

$$\nabla p = \boldsymbol{f} - \boldsymbol{u}' + \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} =: \boldsymbol{g}.$$

For all $\boldsymbol{v} \in V$ it holds that

$$\langle \boldsymbol{g}, \boldsymbol{v} \rangle = \langle \boldsymbol{f} - \boldsymbol{u}' + \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v} \rangle = 0.$$

We may use the result of the exercise.

Theorem 5.8.1: deRham

Let $\boldsymbol{g} \in H^{-1}(\Omega)^d$ with

$$\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{H^{-1}(\Omega)^d, H_0^1(\Omega)^d} = 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}.$$

Then there exists a $p \in L^2(\Omega)$ with $\int_\Omega p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$ such that $\nabla p = \boldsymbol{g}$.

In contrast to our previous result, we now assume that $\boldsymbol{f} \in L^2(0,T;H^{-1}(\Omega)^d)$ (instead of $L^2(0,T;V^*)$). This is a more restrictive assumption. Since $V \subset H_0^1(\Omega)^d$ implies $H^{-1}(\Omega)^d \subset V^*$.

Theorem 5.8.2: Simon (1988)

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, $\boldsymbol{u}_0$, $\boldsymbol{f} \in L^2(0,T;H^{-1}(\Omega)^d)$. Then there exists a pair

$$(\boldsymbol{u}, p) \in \big(L^\infty(0,T;H) \cap L^2(0,T;V) \cap \mathcal{C}_w([0,T];H)\big) \times W^{-1,\infty}(0,T;L^2(\Omega)/_{\mathbb{R}})$$

satisfying the Navier-Stokes equation in the weak sense.

Remark. 5.8.1 The main problem is that $V^* \subsetneq \mathcal{D}'(\Omega)$. The elements of the dual space V^* are no distributions. [Sim99] even showed that there exists no Hausdorff space (minimal assumptions to distinguish two elements) such that V^* and $H^{-1}(\Omega)^d$ can be embedded into this Hausdorff space.

Remark. 5.8.2 Formally, we may get from $\nabla p = \boldsymbol{f} - \boldsymbol{u}' + \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ by applying the divergence

$$\Delta p = \nabla \cdot \big(\boldsymbol{f} - \boldsymbol{u}' + \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) = \nabla \cdot f - \operatorname{tr}((\nabla\boldsymbol{u})^2)$$

in $\Omega \times (0,T)$ and

$$\boldsymbol{n}\cdot\nabla p = \big(\boldsymbol{f} - \boldsymbol{u}' + \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big)\cdot\boldsymbol{n} = \big(\boldsymbol{f} + \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big)\cdot\boldsymbol{n}$$

on $\partial\Omega \times (0,T)$. From $\boldsymbol{u}$, one may deduce p by solving formally the above Neumann problem.

Remark. 5.8.3 (Difficulties for proving additional regularity) When trying to prove additional regularity of solutions, we need additional compatibility assumptions for these solutions. This leads to problems! Let $\boldsymbol{u}_0 \in V$ and let the compatibility condition $\boldsymbol{u}'(0) = \nu\Delta\boldsymbol{u}_0 - (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 - \boldsymbol{f}(0) \in V$ be fulfilled. Due to the original equation, we find

$$\boldsymbol{u}'(0) + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 - \nu\Delta\boldsymbol{u}_0 + \nabla p(0) = \boldsymbol{f}(0).$$

From $\nabla\boldsymbol{u}_0 = 0$ ad $\nabla\cdot\boldsymbol{u}'(0) = 0$, we find that

$$\Delta p_0 = \nabla\cdot\big(\boldsymbol{f}(0) + \nu\Delta\boldsymbol{u}_0 - \boldsymbol{u}'(0) - (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0\big) = \nabla\cdot\big(\boldsymbol{f}(0) - \operatorname{tr}((\nabla\boldsymbol{u}_0)^2)\big)$$

and

$$\nabla p_0 = \boldsymbol{f}(0) - \boldsymbol{u}'(0) + \nu\Delta\boldsymbol{u}_0 - (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 = \boldsymbol{f}(0) + \nu\Delta\boldsymbol{u}_0$$

on $\partial\Omega$. The boundary terms for $\boldsymbol{u}_0$ vanish since $\boldsymbol{u}_0 \equiv 0$ on $\partial\Omega$. This is an overdetermined system and does not possess a solution in general.

Singular limits and long-time behaviour

We want to consider the singular limit $\nu \to 0$ and the long behaviour for solutions of the Navier-Stokes equations. As a tool, we will use the relative-energy inequality. We already implicitly used it to prove the weak strong uniqueness result.

Definition 5.8.4 (Relative energy, solution operator)
The relative energy is

relative energy

$$\mathcal{R}\colon H \times H \to \mathbb{R}_+, \qquad (\boldsymbol{v} \mid \tilde{\boldsymbol{v}}) \mapsto \frac{1}{2}\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}$$

and the solution operator $\mathcal{A}_\nu$ is defined by

solution operator

$$\langle \mathcal{A}_\nu(\tilde{\boldsymbol{v}}), \cdot\rangle := \langle \partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} - \nu\Delta\tilde{\boldsymbol{v}} - \boldsymbol{f}, \cdot\rangle,$$

which has to be understood in a weak sense, at least with respect to space.

If $\nabla \cdot \boldsymbol{v} = 0$, then

$$\nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = (\nabla \boldsymbol{v})\boldsymbol{v},$$

so the NAVIER-STOKES equations can be rewritten as

$$\begin{cases} \partial_t \boldsymbol{v} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{v} = 0, & \text{in } \Omega \times (0, T), \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 & \text{in } \Omega, \\ \nu(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{v} = 0 \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{v} = 0, & \text{on } \partial\Omega \times (0, T). \end{cases}$$

By writing the boundary conditions in this way, the system incorporates the NAVIER-STOKES system with no-slip conditions for $\nu > 0$ and the EULER equations for $\nu = 0$. Indeed, for $\nu > 0$, the tangential and normal part of the velocity field vanish such that this is equivalent to $\boldsymbol{v} = 0$ on $\partial\Omega \times (0, T)$. For the friction-less case of $\nu = 0$, only the normal component vanishes on the boundary.

THEOREM 5.8.3: LOCAL EXISTENCE OF SMOOTH SOLUTIONS

Let $\Omega = \mathbb{R}^3$ be a bounded domain with smooth enough boundary and $\boldsymbol{u}_0 \in H_0^s(\Omega) \cap H$ with $s \geqslant 3$. Then there exists a $T_0 > 0$ such that there exists a solution

$$\boldsymbol{u} \in L^\infty(0, T_0; H^s(\Omega) \cap H)$$

solving the EULER equations.

Proof. Exercises. □

THEOREM 5.8.4: SINGULAR LIMIT FOR $\nu \searrow 0$

For $\nu > 0$, let $\boldsymbol{u}_\nu \in L^2(0, T; H) \cap L^2(0, T; V)$ be a weak solution to the NAVIER-STOKES equations for $\boldsymbol{u}_0^\nu \in H$ for $f \equiv 0$. Additionally, let $\boldsymbol{u} \in L^1(0, T; W^{1,\infty}(\Omega)) \cap L^\infty(0, T; H)$ solve the EULER equations in a weak sense. Then for almost all $t \in (0, T)$ it holds that

$$\frac{1}{2}|\boldsymbol{u}_\nu(t) - \boldsymbol{u}(t)|^2 \leqslant c\left(|\boldsymbol{u}_0^\nu - \boldsymbol{u}_0|^2 + \frac{\sqrt{\nu}}{2}\|\boldsymbol{u}\|_{L^2(0,T;V)}\right)^2 \exp\left(\int_0^t \|(\nabla \boldsymbol{u})_{\text{sym}}\|_\infty \,\mathrm{d}s\right).$$

THEOREM 5.8.5: LONG-TERM BEHAVIOUR

We consider the case $T = \infty$. For $\boldsymbol{f} \in L^1(0, T; L^2(\Omega))$ and $\boldsymbol{v}_0 \in H$, assume there exists a weak solution $\boldsymbol{u}$ to the NAVIER-STOKES equations. Then, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ such that $\boldsymbol{u}(t_n) \to 0$ in V for $n \to \infty$.

5.9 Energy-variational solutions

Motivation

Weak solutions have some drawbacks

- they yield existence only, no uniqueness,

- natural energy estimates alone do not suffice to pass to the limit, thus the time-derivative has to be estimated,
- weak solutions without energy conservation are known to display non-physical behaviour,
- weak-strong uniqueness only for suitable weak solutions,
- the motivation for the weak solution is questionable.

In the process of modelling, one assumes that the functions involved are regular enough to write down pointwise relations to come up with the PDE. Then in existence theory, the equations are generalised for scenarios of less regularity. But maybe, the pointwise equations do not describe the physical behaviour away from the right regularity?

This motivates a different approach – the energy-variational solution concept. The variation of the energy inequality, since this relation should hold for every reasonable solution, is taken with respect to more regular functions, for which the pointwise relation given by the PDE makes sense.

Preliminaries

Lemma 5.9.1
Let $A \subset \mathbb{R}^{d+1}$ be a bounded open set and

$$f\colon A \times \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty)$$

with $d, n, m \geqslant 1$ a measurable non-negative function such that

- *$f(x, \cdot, \cdot)$ is lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^m$ for a.e. $x \in A$,*
- *$f(x, y, \cdot)$ is convex for fixed $x \in A$ and $y \in \mathbb{R}^n$.*

For sequences $(u_k)_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$ and $(v_k)_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$ as well as functions $u \in L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$ and $v \in L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$ with

$$u_k \to u \qquad \text{a.e. in } A \quad \text{and} \quad v_k \rightharpoonup v \qquad \text{in } L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$$

it holds

$$\liminf_{k\to\infty} \int_A f(x, u_k(x), v_k(x))\,\mathrm{d}x \geqslant \int_A f(x, u(x), v(x))\,\mathrm{d}x.$$

Lemma 5.9.2
Let $f \in L^1(0,T)$ and $g \in L^\infty(0,T)$ with $g \geqslant 0$ a.e. in $(0,T)$. Then

$$-\int_0^T \varphi'(\tau) g(\tau)\,\mathrm{d}\tau - g(0) + \int_0^T \varphi(\tau) f(\tau)\,\mathrm{d}\tau \leqslant 0$$

holds for all $\varphi \in \tilde{\mathcal{C}}[0,T]$ if and only if

$$g(t) - g(s) \leqslant \int_s^t f(\tau)\,\mathrm{d}\tau$$

for almost all $s, t \in (0,T)$, where

$$\boxed{\tilde{\mathcal{C}}[0,T] := \{\psi \in \mathcal{C}^1[0,T] \mid \psi \geqslant 0,\ \psi' \leqslant 0,\ \psi(0) = 1,\ \psi(T) = 0\}.}$$

Definitions

We define the spaces

- $X := L^\infty(0,T;H) \cap L^2(0,T;V)$ for solutions,
- $Y := H^2(\Omega) \cap V \cap L^d(\Omega)$ for test functions,
- $Z := L^2(0,T;H^{-1}(\Omega)) \oplus L^1(0,T;L^2(\Omega))$ for the right-hand side,

where H denotes the solenoidal $L^2(\Omega)$ and V the solenoidal $H^1_0(\Omega)$ functions.

Definition 5.9.3 (Energy-variational solution)
A function u is called an energy-variational solution, if $u \in X$ and the relative energy inequality

energy-variational solution

$$\mathcal{R}(u(t) \mid v(t)) + \int_0^t \left(\mathcal{W}_\nu(u \mid v) + \langle \mathcal{A}_\nu(v), u - v\rangle\right) e^{\int_s^t \mathcal{K}_\nu(v)\,\mathrm{d}\tau}\,\mathrm{d}s \leqslant \mathcal{R}(u_0 \mid v(0)) e^{\int_0^t \mathcal{K}_\nu(v)\,\mathrm{d}s}$$

holds for a.e. $t \in (0,T)$, where

$$\begin{aligned} \mathcal{R}(u(t) \mid v(t)) &= \frac{1}{2}\|u(t) - v(t)\|^2_{L^2(\Omega)} \\ \text{and} \quad \langle \mathcal{A}_\nu(v(s)), \cdot\rangle &= \langle \partial_t v(s) + (v(s)\cdot\nabla)v(s) - \nu\Delta v(s) - f(s), \cdot\rangle, \end{aligned}$$

and for all $v \in \mathcal{C}^1([0,T];Y)$ and all convex non-negative potentials $\mathcal{K}_\nu \colon Y \to [0,\infty)$ such that

$$\begin{aligned} &\mathcal{W}_\nu \colon V \times Y \to [0,\infty), \\ &\mathcal{W}_\nu(u \mid v) = \nu\|\nabla u - \nabla v\|^2_{L^2(\Omega)} - \int_\Omega ((u-v)\cdot\nabla)(u-v)\cdot v\,\mathrm{d}x + \mathcal{K}_\nu(v)\frac{1}{2}\|u-v\|^2_{L^2(\Omega)} \end{aligned}$$

is convex in u and continuous in v.

Remark. 5.9.4 The solution concept fulfills the standard requirements for a generalized solution concept

- existence of generalized solution,
- weak-strong uniqueness of solutions,
- additional regularity implies uniqueness,
- convergence to stationaly states.

Remark. 5.9.5 Advantages over weak solutions are

- existence in every space dimension,
- only relies on classical energy estimates,
- the convex solution set allows to define a descent selection criterion to find the physically relevant solution dissipating most energy.

Remark. 5.9.6 (Well-definedness of the solution concept) The convection term is well-defined, since we have $V \overset{c}{\hookrightarrow} L^{2d/(d-2)}(\Omega)$, thus

$$\begin{aligned} &\int_0^T \int_\Omega ((u-v)\cdot\nabla)(u-v)\cdot v\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \leqslant \|u-v\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))}\|u-v\|_{L^2(0,T;V)}\|v\|_{L^\infty(0,T;L^d(\Omega))} < \infty. \end{aligned}$$

Moreover, the set of possible K_ν's is non-empty. To see this consider

$$\mathcal{K}_\nu \colon Y \to [0,\infty), \qquad \mathcal{K}_\nu(v) = c\|v\|^s_{L^r(\Omega)} \quad \text{for} \quad \frac{2}{s} + \frac{d}{r} \leqslant 1.$$

Then by the Gagliardo-Nirenberg inequality

$$\|v\|_{L^p(\Omega)} \leqslant \|\nabla v\|^{\alpha}_{L^2(\Omega)}\|v\|^{1-\alpha}_{L^2(\Omega)}$$

for $\alpha = \frac{d(p-2)}{2p}$ and $d \leqslant \frac{2d}{p-2}$ (TODO: this simplifies to $p \leqslant 4$...) with

$$\frac{1}{p} = \left(\frac{1}{2} - \frac{1}{d}\right)\alpha + \frac{1}{2}(1-\alpha)$$

we obtain

$$\begin{aligned}\left|\int_\Omega ((u-v)\cdot\nabla)(u-v)\cdot v \,\mathrm{d}x\right| &\leqslant \|u-v\|_{L^p(\Omega)}\|\nabla u - \nabla v\|_{L^2(\Omega)}\|v\|_{L^{2p/(p-2)}(\Omega)} \\ &\leqslant C\|u-v\|^{1-\alpha}_{L^2(\Omega)}\|\nabla u - \nabla v\|^{1+\alpha}_{L^2(\Omega)}\|v\|_{L^{2p/(p-2)}(\Omega)} \\ &\leqslant \frac{\nu}{2}\|\nabla u - \nabla v\|^2_{L^2(\Omega)} + c\|v\|^{2/(1-\alpha)}_{L^{2p/(p-2)}(\Omega)}\frac{1}{2}\|u-v\|^2_{L^2(\Omega)}.\end{aligned}$$

Existence

Let us give a proof of existence, which does not require any time-derivative estimate.

Theorem 5.9.1: Navier-Stokes Existence

Let $\Omega \subset \mathbb{R}^d$ for $d \geqslant 2$ be a bounded Lipschitz domain, $\nu \geqslant 0$ and $\mathcal{R}, \mathcal{W}_\nu, \mathcal{K}_\nu$ and $\mathcal{A}_\nu$ as above. Then there exists at least one energy-variational solution $u \in X$ to every $u_0 \in H$ and $f \in Z$. Moreover, the set of solutions is convex and weakly sequentially closed.

Proof. We attack by Galerkin discretization. As $\mathcal{V}$ is dense in V there exists a Galerkin-Scheme $(V_m)_{m\in\mathbb{N}} \subset \mathcal{V}$ of V. The m'th approximate problem reads

$$\begin{cases} (\partial_t u_m, v) + ((u_m\cdot\nabla)u_m, v) + \nabla((u_m, v)) = \langle f, v\rangle, & \text{for all } v \in V_m \text{ and } t \in [0,T], \\ u_m^0 = P_m u_0, \end{cases} \tag{49}$$

where $P_m\colon H \to V_m$ is the orthogonal projection from H onto the subspace V_m.

Via the Theorem of Caratheodory we find solutions u_m to the approximate problems given by 49 on time intervals $[0, T_m)$ with $T_m > 0$. In fact, when we show the apriori estimates in the next step, we excluded the possibility of finite-time blowups and thus $T_m = T$ will follow for all $m \in \mathbb{N}$.

Testing by u_m itself leads to the apriori estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m|^2 + \nu\|u_m\|^2 = \langle f, u_m\rangle,$$

where

$$f = f_1 + f_2 \quad \text{with} \quad f_1 \in L^1(0,T;L^2(\Omega)) \quad \text{and} \quad f_2 \in L^2(0,T;H^{-1}(\Omega)),$$

hence

$$\begin{aligned}\langle f, u_m\rangle = \langle f_1, u_m\rangle + \langle f_2, u_m\rangle &\leqslant \|f_1\|_{L^2(\Omega)}\|u_m\| + \|f_2\|_{H^{-1}(\Omega)}\|u_m\| \\ &\leqslant \|f_1\|_{L^2(\Omega)}(1+|u_m|^2) + \frac{\nu}{2}\|u_m\|^2 + \frac{1}{2\nu}\|f_2\|^2_{H^{-1}(\Omega)}.\end{aligned}$$

Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}|u_m|^2 + \nu\|u_m\|^2 \leqslant 2\|f_1\|_{L^2(\Omega)}(1+|u_m|^2) + \frac{1}{\nu}\|f_2\|^2_{H^{-1}(\Omega)} \tag{50}$$

and via GRONWALL

$$\begin{aligned}|u_m(t)|^2 &+ \int_0^t \nu\|u_m\|^2\,\mathrm{d}s \\ &\leqslant \left(|u_0|^2 + 2\|f_1\|_{L^1(0,T;L^2(\Omega))} + \frac{1}{\nu}\|f_2\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)e^{2\|f_1\|_{L^1(0,T;L^2(\Omega))}}\end{aligned}$$

a.e. in $(0,T)$. The GRONWALL argument can be seen as follows: integrating (50) and using $\int_0^t \|\ldots\|\,\mathrm{d}s \leqslant \int_0^T \|\ldots\|\,\mathrm{d}s$ yields

$$|u_m(t)|^2 \leqslant |u_m(0)|^2 - \nu\int_0^t \|u_m(s)\|^2\,\mathrm{d}s + 2\|f_1\|_{L^1(0,T;L^2(\Omega))} + \frac{1}{\nu}\|f_2\|^2_{L^2(0,T;H^{-1}(\Omega))} + \int_0^t 2\|f_1(s)\|_{L^2(\Omega)}|u_m(s)|^2\,\mathrm{d}s.$$

Now, GRONWALL's Lemma (again using $\int_0^t \|\ldots\|\,\mathrm{d}s \leqslant \int_0^T \|\ldots\|\,\mathrm{d}s$) implies

$$|u_m(t)|^2 \leqslant \left(|u_m(0)|^2 - \nu\int_0^t \|u_m(s)\|^2\,\mathrm{d}s + 2\|f_1\|_{L^1(0,T;L^2(\Omega))} + \frac{1}{\nu}\|f_2\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)e^{\|f_1\|_{L^1(0,T;L^2(\Omega))}}.$$

Since the exponential term is greater than one, the $-\nu$-term can be put on the other side of the inequality and estimate such that the above equation holds.

This boundedness implies

$$u_m \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;H) \quad \text{and} \quad u_m \rightharpoonup u \quad \text{in} \quad L^2(0,T;V),$$

hence

$$u_m \overset{*}{\rightharpoonup} u \quad \text{in} \quad X = L^\infty(0,T;H) \cap L^2(0,T;V).$$

We will proceed by establishing an approximate relative-energy inequality. We already deduced the relative energy inequality for each u_m, i.e.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|u_m|^2 + \nu\|u_m\|^2 = \langle f, u_m\rangle. \tag{51}$$

We test the Erstaz-problem by $P_m v$, where $v \in \mathcal{C}^1([0,T];Y)$, to obtain

$$(\partial_t u_m, P_m v) + ((u_m\cdot\nabla)u_m, P_m v) + \nu((u_m, P_m v)) = \langle f, P_m v\rangle. \tag{52}$$

Moreover, we find for the solution operator

$$\langle \mathcal{A}_\nu(P_m v), u_m\rangle = (\partial_t P_m v, u_m) + ((P_m v\cdot\nabla)P_m v, u_m) + \nu((P_m v, u_m)) - \langle f, u_m\rangle \tag{53}$$

and

$$\langle \mathcal{A}_\nu(P_m v), P_m v\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|P_m v|^2 + \nu\|P_m v\|^2 - \langle f, P_m v\rangle. \tag{54}$$

Combining (51) - (52) - (53) + (54) gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}&|u_m - P_m v|^2 + \nu\|u_m - P_m v\|^2 \\ &+ \big(((u_m - P_m v)\cdot\nabla)P_m v, u_m\big) + \langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle = 0.\end{aligned}$$

Multiplying by $e^{-\int_0^t \mathcal{K}_\nu(P_m v)\,\mathrm{d}s}$ and simplifying yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(|u_m - P_m v|^2 e^{-\int_0^t \mathcal{K}_\nu(P_m v)\,\mathrm{d}s}\right) + \frac{1}{2}\mathcal{K}_\nu(P_m v)|u_m - P_m v|^2 e^{-\int_0^t \mathcal{K}_\nu(P_m v)\,\mathrm{d}s} \\ &+ \Bigg(\nu\|u_m - P_m v\|^2 + \big(((u_m - P_m v)\cdot\nabla)P_m v, u_m\big) \\ &+ \langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle\Bigg)e^{-\int_0^t \mathcal{K}_\nu(P_m v)\,\mathrm{d}s} = 0.\end{aligned}$$

Integrating from 0 to t gives

$$\begin{aligned}
&\mathcal{R}(u_m(t) \mid P_m v(t)) e^{-\int_0^t \mathcal{K}_\nu(P_m v)\,\mathrm{d}s} \\
&\qquad + \int_0^t \big(\mathcal{W}_\nu(u_m \mid P_m v) + \langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle\big) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\qquad\qquad \leqslant \mathcal{R}(u_m(0) \mid P_m v(0)).
\end{aligned} \tag{55}$$

Take any $\varphi \in \tilde{\mathcal{C}}[0,T]$, multiply (55) by $-\varphi'$, integrate from 0 to T and use integration by parts to arrive at

$$\begin{aligned}
&-\int_0^T \varphi' \mathcal{R}(u_m \mid P_m v) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\qquad + \int_0^T \varphi\big(\mathcal{W}_\nu(u_m \mid P_m v) + \langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle\big) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\qquad\qquad \leqslant \mathcal{R}(u_m(0) \mid P_m v(0)).
\end{aligned} \tag{56}$$

We proceed by passage to the limit for $m \to \infty$. Since $\mathcal{V}$ is dense in Y, we have

$$P_m v(t) \to v(t) \qquad \text{in } Y \text{ for each } t \in [0,T]. \tag{57}$$

Appealing to Lemma 5.9.1 and an approximation argument we obtain

$$\begin{aligned}
&-\int_0^T \varphi' \mathcal{R}(u \mid v) e^{-\int_0^s \mathcal{K}_\nu(v)\,\mathrm{d}\tau}\,\mathrm{d}s + \int_0^T \varphi \mathcal{W}_\nu(u \mid v) e^{-\int_0^s \mathcal{K}_\nu(v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\leqslant \liminf_{m\to\infty} \left(-\int_0^T \varphi' \mathcal{R}(u_m \mid P_m v) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s + \int_0^T \varphi \mathcal{W}_\nu(u_m \mid P_m v) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s \right),
\end{aligned}$$

where the weak convergence of u_m to u and (57) were used. Since $P_n \to \mathrm{id}_H$, we find for the initial values

$$\mathcal{R}(u_m(0) \mid P_m v(0)) = \frac{1}{2}|P_m(u_0 - v(0))|^2 \to \frac{1}{2}|u_0 - v(0)|^2 = \mathcal{R}(u_0 \mid v(0)).$$

And for the solution operator part we have

$$(\partial_t P_m v, u_m - P_m v) = (P_m \partial_t v, u_m - P_m v) = (\partial_t v, u_m - P_m v),$$

as P_m is an orthogonal projection, thus

$$\begin{aligned}
&\langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle - \langle \mathcal{A}_\nu(v), u_m - P_m v\rangle \\
&= \big((P_m v \cdot \nabla)(P_m v - v) + ((P_m v - v)\cdot\nabla)v, u_m - P_m v\big) \\
&+ \nu(\nabla P_m v - \nabla v, \nabla u_m - \nabla P_m v) \\
&\leqslant \|P_m v\|_{L^d(\Omega)} \|\nabla P_m v - \nabla v\|_{L^2(\Omega)} \|u_m - P_m v\|_{L^{2d/(d-2)}(\Omega)} \\
&+ \|P_m v - v\|_{L^{d/2}(\Omega)} \|\nabla v\|_{L^{2d/(d-2)}(\Omega)} \|u_m - P_m v\|_{L^{2d/(d-2)}(\Omega)} \\
&+ \nu \|\nabla P_m v - \nabla v\|_{L^2(\Omega)} \|\nabla u_m - \nabla P_m v\|_{L^2(\Omega)},
\end{aligned}$$

and so

$$
\begin{aligned}
&\int_0^T \varphi\big(\langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle - \langle \mathcal{A}_\nu(v), u_m - P_m v\rangle\big) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\leqslant \|P_m v\|_{L^\infty(0,T;L^d(\Omega))} \|\nabla P_m v - \nabla v\|_{L^2(0,T;L^2(\Omega))} \|u_m - P_m v\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} \\
&\qquad + \|P_m v - v\|_{L^2(0,T;L^{d/2}(\Omega))} \|\nabla v\|_{L^\infty(0,T;L^{2d/(d-2)}(\Omega))} \|u_m - P_m v\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} \\
&\qquad\qquad + \nu \|\nabla P_m v - \nabla v\|_{L^2(0,T;L^2(\Omega))} \|\nabla u_m - \nabla P_m v\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}
\tag{58}
$$

where $0 \leqslant \varphi \leqslant 1$ was used. Using $u_m \rightharpoonup u$ in $L^2(0,T;V)$ and (57) we find

$$
\begin{aligned}
\|u_m - P_m v\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} &\leqslant \|u_m - P_m v\|_{L^2(0,T;V)} < \infty, \\
\|\nabla u_m - \nabla P_m v\|_{L^2(0,T;L^2(\Omega))} &= \|u_m - P_m v\|_{L^2(0,T;V)} < \infty,
\end{aligned}
$$

and that the right-hand side of (58) vanishes. This implies

$$
\int_0^T \varphi\big(\langle \mathcal{A}_\nu(P_m v), u_m - P_m v\rangle - \langle \mathcal{A}_\nu(v), u - v\rangle\big) e^{-\int_0^s \mathcal{K}_\nu(P_m v)\,\mathrm{d}\tau}\,\mathrm{d}s \to 0.
$$

Therefore,

$$
\begin{aligned}
&-\int_0^T \varphi' \mathcal{R}(u \mid v) e^{-\int_0^s \mathcal{K}_\nu(v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\qquad + \int_0^T \varphi\big(\mathcal{W}_\nu(u \mid v) + \langle \mathcal{A}_\nu(v), u - v\rangle\big) e^{-\int_0^s \mathcal{K}_\nu(v)\,\mathrm{d}\tau}\,\mathrm{d}s \\
&\qquad\qquad\qquad \leqslant \mathcal{R}(u(0) \mid v(0)).
\end{aligned}
$$

holds for all $\varphi \in \tilde{\mathcal{C}}[0,T]$. Lemma 5.9.2 then implies the relative-energy inequality. □

Homework 12.1: Assume that in the definition of energy-variational solution for the Navier-Stokes equations, equality holds for a function u. Show that u is a weak solution to the Navier-Stokes equations.

6 Ericksen-Leslie equations for the description of nematic liquid crystals

6.1 Motivation and Applications

This equation is even more complicated than the Navier-Stokes equation due to the presence of an additional variable modelling the anisotropy (dt.: *Richtungsabhängigkeit*) in the fluid, giving rise to different applications:

TODO (w/ images)

6.2 Modelling

This is more or less a static problem: if you only want to consider how the molecules are distributed without any movement in them (identifying the static states), then one ends up with the Oseen-Frank energy, modelling the behaviour of the liquid crystals.

A solution to the stationary problem solves the minimisation problem for the Oseen-Frank energy

$$F_{OF}(\boldsymbol{d}, \nabla \boldsymbol{d}) := k_1(\nabla \cdot \boldsymbol{d})^2 + k_2|\boldsymbol{d} \times (\nabla \times \boldsymbol{d})|^2 + k_3(\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d}))^2 \tag{59}$$

for all $d \in H^1(\Omega) \cap L^\infty(\Omega)$. The parameters k_1, k_2 and k_3 depend on the material. (Generally, k_1 and k_3 are of similar order, while k_2 is different.) This also allows for singularities.

(One could also write $F_{OF}(a, B) = k_1(\mathrm{tr}(B))^2 + k_2|(B - B^T)a|^2 + \frac{k_3}{4}([a]_\times : (B - B^T))^2$.)

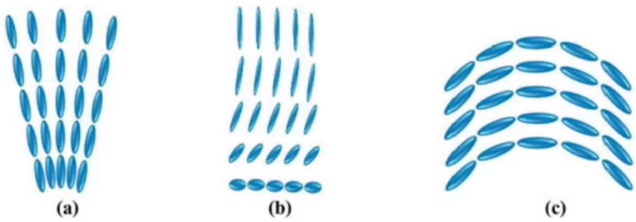

Fig. 7: The k_1-term in (59) corresponds to (a), the splay of the material. The k_2-term (shown in (b)) is the twist term and is modelled by the curl of the director, where the direction is orthogonal to the molecules, that is, $(\nabla \times \boldsymbol{d}) \perp \boldsymbol{d}$. In (c) we can see the bending term (the k_3-term), where $(\nabla \times \boldsymbol{d}) \parallel \boldsymbol{d}$.

The associated (static) minimisation problem is

$$\min_{\boldsymbol{d} \in H^1(\Omega)} \int_\Omega F_{OF}(\boldsymbol{d}, \nabla \boldsymbol{d})\, \mathrm{d}\boldsymbol{x} \qquad \text{with } |\boldsymbol{d}(\boldsymbol{x})| = 1 \text{ a.e. in } \Omega \text{ and } \boldsymbol{d} = \boldsymbol{d}_1 \text{ on } \partial\Omega.$$

The norm restriction $|\boldsymbol{d}(\boldsymbol{x})| = 1$ encodes that we are only interested in the direction and not the magnitude of $\boldsymbol{d}$. The velocity field is, as beforehand, modelling the velocity of the fluid in some container and $\boldsymbol{d}$ models the direction of the dispersed molecules.

We have the following symmetry of the Oseen-Frank enery: if we change the signs of both inputs, the value remains the same.

For $k = k_1 = k_2 = k_3$, the Oseen-Frank energy simplifies to

$$F_{OF}(\boldsymbol{d}, \nabla \boldsymbol{d}) = k|\nabla d|^2$$

(see Exercises). This is the simplification we are going to consider from now on. One can still, with some more work, show the existence of energy-variational solutions when $k_1 \neq k_2 \neq k_3$.

How is this system going to evolve? We don't only have to minimise the energy, but also the dissipation. Given a dissipation potential $D(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})$ (which describes how energy is dissipated in this mechanism) and a free energy $\mathcal{F}(\boldsymbol{d}, \nabla \boldsymbol{d}, \boldsymbol{v})$ the system of equations can be formally derived by

$$\underbrace{\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}}_{\text{material time derivative}} \underbrace{-\nabla \cdot \left(\frac{\partial \mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})}{\partial(\nabla \boldsymbol{v})} - p\boldsymbol{I} \right) + \frac{\partial \mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})}{\partial \boldsymbol{v}}}_{\text{dissipative forces}} = f.$$

$$\frac{\partial \mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})}{\partial(\partial_t \boldsymbol{d})} + \underbrace{\frac{\partial \mathcal{F}(\boldsymbol{d}, \nabla \boldsymbol{d}, \boldsymbol{v})}{\partial \boldsymbol{d}} - \nabla \cdot \left(\frac{\partial \mathcal{F}(\boldsymbol{d}, \nabla \boldsymbol{d}, \boldsymbol{v})}{\partial(\nabla \boldsymbol{d})} \right)}_{\text{total derivative of the energy}} + \lambda_2 \boldsymbol{d} = 0. \tag{60}$$

The LAGRANGIAN multipliers p and λ_2 are due to the algebraic restrictions $\nabla \cdot \boldsymbol{v} = 0$ and $|\boldsymbol{d}| = 1$, respectively. By choosing (the simplest energy we can think about)

$$\mathcal{F}(\boldsymbol{d}, \nabla \boldsymbol{d}, \boldsymbol{v}) := \frac{1}{2}|\boldsymbol{v}|^2 + \frac{1}{2}|\nabla \boldsymbol{d}|^2 \tag{61}$$

and

$$\mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d}) := \nu \left|(\nabla \boldsymbol{v})_{\text{sym}}\right|^2 + \frac{1}{2}\left|\partial_t \boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d}\right|^2 \tag{62}$$

we find formally by the proposed scheme the simplified version of the ERICKSEN-LESLIE equations

$$\boxed{\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p - \nu\Delta \boldsymbol{v} + \nabla\cdot(\nabla \boldsymbol{d}^{\mathsf{T}}\nabla \boldsymbol{d}) &= \boldsymbol{f} \qquad \text{in } \Omega\times(0,T) \\ \partial_t \boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - (\boldsymbol{I} - \boldsymbol{d}\otimes\boldsymbol{d})\Delta \boldsymbol{d} &= 0 \qquad \text{in } \Omega\times(0,T) \\ \nabla\cdot\boldsymbol{v} = 0, \quad |\boldsymbol{d}| &= 1 \qquad \text{in } \Omega\times(0,T). \end{aligned}}$$

Proof. We have

$$\begin{aligned} \nabla \cdot \frac{\partial \mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})}{\partial \nabla \boldsymbol{v}} &\overset{(62)}{=} \nabla \cdot 2\nu(\nabla \boldsymbol{v})_{\text{sym}} = \nu\Delta \boldsymbol{v}, \\ \frac{\partial \mathcal{F}(\boldsymbol{d}, \nabla \boldsymbol{d}, \boldsymbol{v})}{\partial(\nabla \boldsymbol{d})} &\overset{(61)}{=} -\nabla\cdot\nabla \boldsymbol{d} = -\Delta \boldsymbol{d}, \\ \frac{\partial \mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})}{\partial(\partial_t \boldsymbol{d})} &\overset{(62)}{=} \partial_t \boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d}. \end{aligned}$$

Hence (as by (61), $\frac{\partial \mathcal{F}}{\partial \boldsymbol{d}} = 0$) the second equation (60) becomes

$$\partial_t \boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - \Delta \boldsymbol{d} + \lambda_2 \boldsymbol{d} = 0.$$

Multiplying by $\boldsymbol{d}$ yields

$$\frac{1}{2}(\partial_t + (\boldsymbol{v}\cdot\nabla))|\boldsymbol{d}|^2 - \Delta \boldsymbol{d}\cdot\boldsymbol{d} + \lambda_2|\boldsymbol{d}|^2 = 0.$$

By choosing $\lambda_2 := \Delta \boldsymbol{d}\cdot\boldsymbol{d}$, the norm restriction holds. We end up with

$$\partial_t \boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - (I - \boldsymbol{d}\otimes\boldsymbol{d})\Delta \boldsymbol{d} = 0. \tag{63}$$

As $\Delta \boldsymbol{d}$ is the derivative of the energy, $(I - \boldsymbol{d}\otimes\boldsymbol{d})$ acts as a projection onto the sphere.

As $|\boldsymbol{d}| = 1$, $\nabla|\boldsymbol{d}| = 0$ and thus

$$\begin{aligned}
\frac{\partial \mathcal{D}(\boldsymbol{v}, \nabla \boldsymbol{v}, \partial_t \boldsymbol{d}, \boldsymbol{d})}{\partial \boldsymbol{v}} &\overset{(62)}{=} \nabla \boldsymbol{d}^\mathsf{T} (\partial_t \boldsymbol{d} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{d}) \overset{(63)}{=} \nabla \boldsymbol{d}^\mathsf{T} (I - \boldsymbol{d} \otimes \boldsymbol{d}) \Delta \boldsymbol{d} \\
&= \nabla \boldsymbol{d}^\mathsf{T} \Delta \boldsymbol{d} - \frac{1}{2} \underbrace{\nabla |\boldsymbol{d}|^2}_{=0} (\boldsymbol{d} \cdot \Delta \boldsymbol{d}) \\
&= \nabla \cdot (\nabla \boldsymbol{d}^\mathsf{T} \nabla \boldsymbol{d}) - \frac{1}{2} \nabla |\nabla \boldsymbol{d}|^2 \\
&= \sum_{i,j,k=1}^{d} \partial_{x_i} \boldsymbol{d}_j \partial_{x_k}^2 \boldsymbol{d}_j \\
&= \sum_{i,j,k=1}^{d} \left(\partial_{x_k} \partial_{x_j} \boldsymbol{d}_j \partial_{x_k} \boldsymbol{d}_j\right) - \partial_{x_j x_k}^2 \boldsymbol{d}_j \partial \boldsymbol{d}_j \\
&= \nabla \cdot (\nabla \boldsymbol{d}^\mathsf{T} \nabla \boldsymbol{d}) - \frac{1}{2} \nabla |\nabla \boldsymbol{d}|^2.
\end{aligned}$$

We redefine $\tilde{p} := p - \frac{1}{2}|\nabla \boldsymbol{d}|^2$. □

6.3 Preliminaries

In the Navier-Stokes equation we had the condition that v vanished on the boundary. As we want to have inhomogeneous boundary conditions, we add a constant (in time) function which fulfills the boundary condition. We didn't do this for the Navier-Stokes equation, but here we cannot easily obtain homogeneous Dirichlet boundary conditions by, say, a linear transformation.

Lemma 6.3.1 (Extension operator)
There exists a linear continuous operator $E\colon H^{3/2}(\partial\Omega) \to H^2(\Omega)$, where Ω is of class $\mathcal{C}^{1,1}$. This operator is the right-inverse of the trace operator, that is, for all right hand sides $\boldsymbol{g} \in H^{3/2}(\partial\Omega)$, we have $E\boldsymbol{g} = \boldsymbol{g}$ on $\partial\Omega$ in the sense of the trace operator. There exists a constant $c > 0$ such that

$$\|E\boldsymbol{g}\|_{H^2(\Omega)} \leqslant c\|\boldsymbol{g}\|_{H^{3/2}(\partial\Omega)} \qquad \forall \boldsymbol{g} \in H^{3/2}(\partial\Omega).$$

We can write down this operator E via the Fourier transform on the boundary as well as on Ω. Another equivalent description is the solution h of

$$\begin{cases} -\Delta h = 0, & \text{in } \Omega, \\ h = g, & \text{on } \partial\Omega. \end{cases}$$

Assume that $\boldsymbol{d} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^3)$. Then $(\nabla \boldsymbol{d})_{i,j} := \partial_{x_j} \boldsymbol{d}_i$ for $i, j \in \{1, \dots, 3\}$ and thus $\nabla \boldsymbol{d} \in \mathbb{R}^{3\times 3}$.

Definition 6.3.2 (Skew-symmetric part, curl)
The skew-symmetric part of a vector field is skew-symmetric part

$$(\nabla d)_{\text{skew}} := \frac{1}{2}\left(\nabla \boldsymbol{d} - (\nabla \boldsymbol{d})^\mathsf{T}\right) = \frac{1}{2} \begin{pmatrix} 0 & \partial_{x_2} d_1 - \partial_{x_1} d_2 & \partial_{x_3} d_1 - \partial_{x_1} d_2 \\ \partial_{x_1} d_2 - \partial_{x_2} d_1 & 0 & \partial x_3 d_2 - \partial_{x_2} d_2 \\ \partial_{x_1} d_2 - \partial_{x_3} d_1 & \partial_{x_2} d_2 - \partial_{x_3} d_2 & 0 \end{pmatrix}$$

and its curl is curl

$$\nabla \times \boldsymbol{d} := \begin{pmatrix} \partial_{x_2} d_3 - \partial_{x_3} d_2 \\ \partial_{x_3} d_1 - \partial_{x_1} d_3 \\ \partial_{x_1} d_2 - \partial_{x_2} d_1 \end{pmatrix},$$

where the i-th component of $\boldsymbol{d}$ is denoted by d_i.

We have

$$2|(\nabla \boldsymbol{d})_{\text{skew}}|^2 = |\nabla \times \boldsymbol{d}|^2,$$

where $|A|^2 = A : A$ and $A : B := \sum_{i,j=1}^{n} A_{i,j} B_{i,j}$ is the Frobenius product and

$$|\nabla \boldsymbol{d}|^2 = |\nabla \times \boldsymbol{d}|^2 + \operatorname{tr}(\nabla \boldsymbol{d}^2)$$

or, differently put, (for lemma 6.3.4)

$$|\nabla \boldsymbol{d}|^2 = (\nabla \cdot \boldsymbol{d})^2 + |\nabla \times \boldsymbol{d}|^2 + \operatorname{tr}(\nabla \boldsymbol{d}^2) - (\nabla \cdot \boldsymbol{d})^2.$$

Definition 6.3.3 (Cross-product-matrix)
We define the mapping

$$[\cdot]_\times : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \qquad h \mapsto \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}.$$

Let $\boldsymbol{I}$ be the identity matrix. For all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ we have

$$[\boldsymbol{a}]_\times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \qquad \text{and} \qquad [\boldsymbol{a}]_\times^\top [\boldsymbol{b}]_\times = (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{I} - \boldsymbol{b} \otimes \boldsymbol{a}. \tag{64}$$

Additionally, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^3)$ we have

$$\nabla \cdot [\boldsymbol{a}]_\times = -\nabla \times \boldsymbol{a}, \quad [\boldsymbol{a}]_\times : \nabla \boldsymbol{b} = [\boldsymbol{a}]_\times : (\nabla \boldsymbol{b})_{\text{skew}} = \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b}), \quad [\nabla \times \boldsymbol{a}]_\times = 2(\nabla \boldsymbol{a})_{\text{skew}}.$$

Lemma 6.3.4 (Exercise 12.1)
We have

$$\arg\min \int_\Omega |\nabla \boldsymbol{d}|^2 \, \mathrm{d}x = \arg\min \int_\Omega (\nabla \cdot \boldsymbol{d})^2 + |\boldsymbol{d} \times \nabla \times \boldsymbol{d}|^2 + (\boldsymbol{d} \cdot \nabla \times \boldsymbol{d})^2 \, \mathrm{d}x$$

for all $\boldsymbol{d} \in H^1(\Omega)$ *with* $|\boldsymbol{d}| = 1$ *almost everywhere in* Ω *and* $\boldsymbol{d} = \boldsymbol{d}_1 \in \mathcal{C}^1(\partial\Omega)$ *with* $|\boldsymbol{d}_1| = 1$ *on* $\partial\Omega$.

Proof. We have

$$(\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d}))^2 = \left(\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_2 d_3 - \partial_3 d_2 \\ \partial_3 d_1 - \partial_1 d_3 \\ \partial_1 d_2 - \partial_2 d_1 \end{pmatrix} \right)^2 =$$

and

$$|\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d})|^2 = \ldots$$

Adding both quantities and simplifying, we obtain

$$(\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d}))^2 + |\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d})|^2 = |\boldsymbol{d}|^2 |\nabla \cdot \boldsymbol{d}|^2.$$

We note further, that we can write the term

$$\operatorname{tr}(\nabla \boldsymbol{d}^2) - (\nabla \cdot \boldsymbol{d})^2 = \nabla \cdot (\nabla \boldsymbol{d}\boldsymbol{d} - \nabla \cdot \boldsymbol{d}\boldsymbol{d})$$

can be written as the divergence of a vector field. By the divergence theorem, we obtain

$$\int_\Omega \nabla\cdot(\nabla \boldsymbol{d}\boldsymbol{d} - \nabla\cdot\boldsymbol{d}\boldsymbol{d})\,\mathrm{d}\boldsymbol{d} = \int_{\partial\Omega}(\nabla \boldsymbol{d}\boldsymbol{d} - \nabla\cdot\boldsymbol{d}\boldsymbol{d})\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\partial\Omega}(\nabla \boldsymbol{d}_1\boldsymbol{d}_1 - \nabla\cdot\boldsymbol{d}_1\boldsymbol{d}_1)\cdot\boldsymbol{n}\,\mathrm{d}s,$$

which is therefore independent of the particular choice for $\boldsymbol{d}$. Therefore,

$$\begin{aligned}\arg\min\int_\Omega|\nabla\boldsymbol{d}|^2\,\mathrm{d}\boldsymbol{x} &= \arg\min\int_\Omega(\nabla\cdot\boldsymbol{d})^2 + |\nabla\times\boldsymbol{d}|^2 + \operatorname{tr}(\nabla\boldsymbol{d}^2) - (\nabla\cdot\boldsymbol{d})^2\,\mathrm{d}\boldsymbol{x}\\ &= \arg\min\int_\Omega(\nabla\cdot\boldsymbol{d})^2 + (\boldsymbol{d}\cdot(\nabla\times\boldsymbol{d}))^2 + |\boldsymbol{d}\times\nabla\times\boldsymbol{d}|^2 + \operatorname{tr}(\nabla\boldsymbol{d}^2) - (\nabla\cdot\boldsymbol{d})^2\,\mathrm{d}\boldsymbol{x}\\ &= \arg\min\int_\Omega(\nabla\cdot\boldsymbol{d})^2 + (\boldsymbol{d}\cdot(\nabla\times\boldsymbol{d}))^2\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega}(\nabla\boldsymbol{d}_1\boldsymbol{d}_1 - \nabla\cdot\boldsymbol{d}_1\boldsymbol{d}_1)\cdot\boldsymbol{n}\,\mathrm{d}s\\ &= \arg\min\int_\Omega(\nabla\cdot\boldsymbol{d})^2 + (\boldsymbol{d}\cdot(\nabla\times\boldsymbol{d}))^2\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

for all $\boldsymbol{d}\in H^1(\Omega)$ with $|\boldsymbol{d}|=1$ almost everywhere in Ω and $\boldsymbol{d}=\boldsymbol{d}_1\in\mathcal{C}^1(\partial\Omega)$ with $|\boldsymbol{d}_1|=1$ almost everywhere on $\partial\Omega$. □

Lemma 6.3.5 (Exercise 12.2)
Prove that

$$\boldsymbol{d}\times\Delta\boldsymbol{d} = \nabla\cdot([\boldsymbol{d}]_\times\nabla\boldsymbol{d})$$

for all $\boldsymbol{d}\in\mathcal{C}^2(\overline{\Omega})$.

Proof. For all $\boldsymbol{d}\in\mathcal{C}^2(\overline{\Omega})$ we have

$$\begin{aligned}\nabla\cdot([\boldsymbol{d}_\times]\nabla\boldsymbol{d}) &= \nabla\cdot\begin{pmatrix}\boldsymbol{d}\times\partial_1\boldsymbol{d} & \boldsymbol{d}\times\partial_2\boldsymbol{d} & \boldsymbol{d}\times\partial_3\boldsymbol{d}\end{pmatrix}\\ &= \partial_1\big(\boldsymbol{d}\times(\partial_1\boldsymbol{d})\big) + \partial_2\big(\boldsymbol{d}\times(\partial_2\boldsymbol{d})\big) + \partial_3\big(\boldsymbol{d}\times(\partial_3\boldsymbol{d})\big)\\ &= (\partial_1\boldsymbol{d})\times(\partial_1\boldsymbol{d}) + \boldsymbol{d}\times(\partial_1^2\boldsymbol{d}) + (\partial_2\boldsymbol{d}\times\partial_2\boldsymbol{d})\\ &\qquad + (\boldsymbol{d}\times\partial_2^2\boldsymbol{d}) + (\partial_3\boldsymbol{d})\times(\partial_3\boldsymbol{d}_3) + \boldsymbol{d}\times(\partial_3^2\boldsymbol{d})\\ &= \boldsymbol{d}\times(\partial_1^2\boldsymbol{d} + \partial_2^2\boldsymbol{d} + \partial_3^2\boldsymbol{d}) = \boldsymbol{d}\times(\Delta\boldsymbol{d}),\end{aligned}$$

where we have used that $\boldsymbol{a}\times\boldsymbol{a}=0$, the product rule $\partial_i(\boldsymbol{a}\times\boldsymbol{b}) = (\partial_i\boldsymbol{a})\times\boldsymbol{b} + \boldsymbol{a}\times(\partial_i\boldsymbol{b})$ and that the cross product has the distributive property. □

We consider the following simplified Ericksen-Leslie system

$$\begin{aligned}\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p - \nu\Delta\boldsymbol{v} + \nabla\cdot(\nabla\boldsymbol{d}^\mathsf{T}\nabla\boldsymbol{d}) &= \boldsymbol{f} &&\text{in } \Omega\times(0,T) &(65)\\ \partial_t\boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - (I - \boldsymbol{d}\otimes\boldsymbol{d})\Delta\boldsymbol{d} &= 0 &&\text{in } \Omega\times(0,T) &(66)\\ \nabla\cdot\boldsymbol{v} = 0,\quad |\boldsymbol{d}| &= 1 &&\text{in } \Omega\times(0,T) &(67)\\ \boldsymbol{v} = 0,\quad \boldsymbol{d} &= \boldsymbol{d}_1 &&\text{on } \partial\Omega\times(0,T) &(68)\\ \boldsymbol{v}(0) = \boldsymbol{v_0},\quad \boldsymbol{d}(0) &= \boldsymbol{d_0} &&\text{in } \Omega &(69)\end{aligned}$$

for which we want to prove some existence result.

Energy principle of the model.

Formally, testing (multiplying, integrating over Ω and applying integration by parts to the last term) (65) by $\boldsymbol{v}$, we observe (as v is velocity-free, the ∇p-term vanishes)

$$\frac{1}{2}\frac{\partial}{\partial t}|\boldsymbol{v}|^2 + \frac{1}{2}\underbrace{\int_\Omega(\boldsymbol{v}-\nabla|\boldsymbol{v}|^2)\,\mathrm{d}\boldsymbol{x}}_{=0} + \int_\Omega\nu|\nabla\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x} - \int_\Omega\nabla\boldsymbol{d}^\mathsf{T}\nabla\boldsymbol{d} : \nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \langle f, \boldsymbol{v}\rangle.$$

Formally, testing (66) by $-\Delta \boldsymbol{d}$, we obtain (using integration by parts on the first term)

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla \boldsymbol{d}\|_{L^2}^2 - \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{d}\Delta\boldsymbol{d} - \underbrace{(I-\boldsymbol{d}\otimes\boldsymbol{d})}_{=[\boldsymbol{d}]_x^{\mathsf{T}}[\boldsymbol{d}]_x}\Delta\boldsymbol{d}\cdot\Delta\boldsymbol{d}\,\mathrm{d}\boldsymbol{x} = 0.$$

An integration-by-parts provides ($\boldsymbol{v}$ vanishes on the boundary, so there is no boundary term)

$$\begin{aligned}-\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{d}\Delta\boldsymbol{d}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \nabla((\boldsymbol{v}\cdot\nabla)\boldsymbol{d}) : \nabla\boldsymbol{d}\,\mathrm{d}\boldsymbol{x}\\ &\overset{(\star)}{=} \underbrace{\int_\Omega (\boldsymbol{v}\cdot\nabla)\nabla\boldsymbol{d} : \nabla\boldsymbol{d}}_{=\int_\Omega \frac{1}{2}(\boldsymbol{v}\cdot\nabla)|\nabla\boldsymbol{d}|^2\,\mathrm{d}\boldsymbol{x}=0} + \nabla\boldsymbol{d}^{\mathsf{T}}\nabla\boldsymbol{d} : \nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\boldsymbol{d}^{\mathsf{T}}\nabla\boldsymbol{d} : \nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where $(\star)$ is the product rule. Cancelling everything and integrating in time provides the formal energy estimate

$$\begin{aligned}\frac{1}{2}\int_\Omega |\boldsymbol{v}(t)|^2 + |\nabla\boldsymbol{d}(t)|^2\,\mathrm{d}\boldsymbol{x} + \underbrace{\int_0^t\int_\Omega \nabla|\nabla\boldsymbol{v}|^2 + |\boldsymbol{d}\times\Delta\boldsymbol{d}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s}_{\text{dissipative parts}}\\ = \frac{1}{2}\int_\Omega |\boldsymbol{v}_0|^2 + |\nabla\boldsymbol{d}_0|^2\,\mathrm{d}x + \int_0^t \langle \boldsymbol{f},\boldsymbol{v}\rangle\mathrm{d}s.\end{aligned}$$

6.4 Definitions and energy-variational formulation

We define

$$\boxed{X := \big(L^\infty(0,T;H)\cap L^2(0,T;V), L^\infty(0,T;H^1(\Omega))\big).}$$

Definition 6.4.1 (Relative energy, dissipation, Hamiltonian, form) The relative energy is

$$\mathcal{R}(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) := \frac{1}{2}\left(\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\|_{L^2(\Omega)}^2\right).$$

The relative dissipation is

$$\begin{aligned}\mathcal{W}_D(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) := {}&\nu\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + \|\boldsymbol{d}\times\Delta\boldsymbol{d}-\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\|_{L^2(\Omega)}^2\\ &+ \big(\boldsymbol{d}\times\Delta\boldsymbol{d}-\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}, (\tilde{\boldsymbol{d}}-\boldsymbol{d})\times\Delta\tilde{\boldsymbol{d}}\big)\\ &- \big(\nabla(\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}); (\tilde{\boldsymbol{d}}-\boldsymbol{d})\times(\nabla\tilde{\boldsymbol{d}}-\nabla\boldsymbol{d})\big),\end{aligned}$$

the relative Hamiltonian is

$$\begin{aligned}\mathcal{W}_H(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) := {}&(\boldsymbol{v}-\tilde{\boldsymbol{v}}, (\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}(\boldsymbol{v}-\tilde{\boldsymbol{v}})) + \left(\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right)^{\mathsf{T}}\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right); (\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\right)\\ &- \left(\nabla\tilde{\boldsymbol{d}}^{\mathsf{T}}\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right), \nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\right) + \left(((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)\nabla\tilde{\boldsymbol{d}}; \nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right),\end{aligned}$$

where $(\cdot,\cdot)$ denotes the L^2 inner product and the relative form is

$$\mathcal{W}(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) := \mathcal{W}_D(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) + \mathcal{W}_H(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) + \mathcal{K}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\mathcal{R}(\boldsymbol{v},\boldsymbol{d}\,|\,\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}), \tag{70}$$

where the regularity measure can be chosen as

$$\begin{aligned}\mathcal{K}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) = {}&C\|\Delta\tilde{\boldsymbol{d}}\|_{L^3(\Omega)}^2 + C\|\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\|_{W^{1,3}(\Omega)} + 2\|(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\|_{L^\infty(\Omega)}\\ &+ \frac{1}{\nu}\|\nabla\tilde{\boldsymbol{d}}\|_{L^\infty(\Omega)}^2 + \frac{C}{\nu}\|\nabla^2\tilde{\boldsymbol{d}}\|_{L^\infty(\Omega)}^2.\end{aligned}$$

Definition 6.4.2 (Solution operator)
The solution operator is

$$\mathcal{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\colon\ \mathcal{C}^1([0,T];Y)\to X,\quad \mathcal{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}):=\begin{pmatrix}\partial_t\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}-\nu\Delta\tilde{\boldsymbol{v}}+\nabla\cdot(\nabla\tilde{\boldsymbol{d}}^{\mathsf{T}}\nabla\tilde{\boldsymbol{d}})-\boldsymbol{f}\\ \partial_t\tilde{\boldsymbol{d}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}+\tilde{\boldsymbol{d}}\times(\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}).\end{pmatrix}.$$

The relative energy inequality can be expressed via

$$\begin{aligned}&\mathcal{R}(\boldsymbol{v},\boldsymbol{d}\mid\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\\ &\quad+\int_0^t\left(W(\boldsymbol{v},\boldsymbol{d}\mid\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})+\left\langle\mathcal{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\boldsymbol{v}-\tilde{\boldsymbol{v}}\\ \Delta\tilde{\boldsymbol{d}}-\Delta\boldsymbol{d}\end{pmatrix}\right\rangle\right)\exp\left(\int_s^t K(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}\tau\right)\mathrm{d}s\\ &\leqslant R(\boldsymbol{v_0},\boldsymbol{d_0}\mid\tilde{\boldsymbol{v}}(0),\tilde{\boldsymbol{d}(0)})\exp\left(\int_0^t K(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s\right).\end{aligned}\tag{71}$$

Definition 6.4.3 (Energy-variational solution)
A tuplet of functions $(\boldsymbol{v},\boldsymbol{d})\in X$ is an energy-variational solution to the Ericksen-Leslie system, if the additional regularity assumptions

energy-variational solution

$$(\boldsymbol{d}\times\Delta\boldsymbol{d})\in L^2(0,T;L^2(\Omega))\qquad\text{and}\qquad\partial_t\tilde{\boldsymbol{d}}\in L^2(0,T;L^{\frac{3}{2}}(\Omega))$$

hold as well as $|\boldsymbol{d}|=1$ almost everywhere in $\Omega\times(0,T)$ and it fulfills the relative energy inequality (71) for all $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\in\mathcal{C}^1([0,T];Y)$ for almost all $t\in(0,T)$ and for all $K\colon Y\to\mathbb{R}_+$ such that the form W from (70) is convex and lower semicontinuous in $(\boldsymbol{v},\boldsymbol{d},\boldsymbol{d}\times\Delta\boldsymbol{d})$ and continuous in $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$. Additionally, the director equation holds:

director equation

$$\boldsymbol{d}\times\partial_t\boldsymbol{d}+\boldsymbol{d}\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{d}+\boldsymbol{d}\times\Delta\boldsymbol{d}=0\qquad\text{almost everywhere in }\Omega\times(0,T)\tag{72}$$

with $\boldsymbol{d}=\boldsymbol{d_1}$ on $\partial\Omega$ and $\boldsymbol{d}(0)=\boldsymbol{d}_0$ in Ω both fulfilled in the sense of the respective trace operator.

Lemma 6.4.4 (Norm restriction)
For a function $\boldsymbol{d}\in L^\infty(0,T;H^1(\Omega))$ and $v\in X$ such that

$$\partial_t\boldsymbol{d}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{d}-\boldsymbol{d}\times\boldsymbol{a}=0\tag{73}$$

for some $\boldsymbol{a}\in L^2(\Omega)$ with $\boldsymbol{d}=\boldsymbol{d_1}$ on $\partial\Omega$ and $\boldsymbol{d}(0)=\boldsymbol{d}_0$ in Ω and $|\boldsymbol{d}_0|=1$, we have

Note that $(\boldsymbol{d}\times\boldsymbol{a})\cdot\boldsymbol{d}=\boldsymbol{a}^{\mathsf{T}}\underbrace{[\boldsymbol{d}]_\times^{\mathsf{T}}\boldsymbol{d}}_{=0}=0$.

$$|\boldsymbol{d}(\boldsymbol{x},t)|=1\qquad\textit{almost everywhere in }\Omega\times(0,T).$$

Proof. Multiplying (73) by $\operatorname{sgn}(|\boldsymbol{d}|^2-1)\boldsymbol{d}$ and integrating over Ω, we observe

$$\partial_t\int_\Omega\big||\boldsymbol{d}|^2-1\big|\,\mathrm{d}\boldsymbol{x}+\int_\Omega(\boldsymbol{v}\cdot\nabla)\big||\boldsymbol{d}|^2-1\big|\,\mathrm{d}\boldsymbol{x}=0,\tag{74}$$

where we used that $\boldsymbol{d}\times\boldsymbol{d}=0$ in the last term of (73). Since $\boldsymbol{v}$ is a solenoidal vector field ($\nabla\cdot v=0$), the second term on the left-hand side of the previous equality (74) vanishes. Integrating in (74) in time, we observe

$$\int_\Omega\big||\boldsymbol{d}(t)|^2-1\big|\,\mathrm{d}\boldsymbol{x}=\int_\Omega\big||\boldsymbol{d}_0|^2-1\big|\,\mathrm{d}\boldsymbol{x},$$

which proves the assertion. □

Theorem 6.4.1: Existence of energy-variational solution to the Ericksen-Leslie equations

Let $\Omega \subset \mathbb{R}^3$ be bounded domain of class $\mathcal{C}^{1,1}$ and let $\boldsymbol{v}_0 \in H$ and $\boldsymbol{d}_0 \in H^1(\Omega;\mathbb{S}^2)$ with $\operatorname{tr}(\boldsymbol{d_0}) = \boldsymbol{d_1}$[a] on $\partial\Omega$ for $\boldsymbol{d_1} \in H^{\frac{3}{2}}(\partial\Omega)$. Then there exists an energy-variational solution.

[a] $\operatorname{tr}(\boldsymbol{d_0}) = \boldsymbol{d_1}$ is a compatibility condition in order to fulfil the boundary conditions: this would be the same for linear PDE (third chapter) in order to deduce H^2-regularity. We will need this additional regularity on the approximate level.

Remark. 6.4.5 The energy-variational solution for the Ericksen-Leslie model fulfills the same standard assumptions on a solvability concept as in the Navier-Stokes case:

- existence of generalised solutions,
- weak-strong uniqueness of solutions,
- additional regularity implies uniqueness,
- convergence to stationary states.

Assume that $(\boldsymbol{v}, \boldsymbol{d})$ solves the Ericksen-Leslie system of equations. Formally, we may derive the relative energy inequality. We may add (65) tested by $\boldsymbol{v} - \tilde{\boldsymbol{v}}$ and (66) by $-(\Delta \boldsymbol{d} - \Delta\tilde{\boldsymbol{d}})$ and perform integration by parts in the $\Delta\boldsymbol{v}$-term, the $\nabla\cdot(\nabla\boldsymbol{d}^{\mathsf{T}}\nabla\boldsymbol{d})$-term and the $\partial_t\boldsymbol{d}$-term (the $((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{v})$-term vanishes because $\nabla\cdot v = 0$??):

$$
\begin{aligned}
(\partial_t \boldsymbol{v}, \boldsymbol{v} - \tilde{\boldsymbol{v}}) - ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \tilde{\boldsymbol{v}}) + \nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}) - (\nabla\boldsymbol{d}^{\mathsf{T}}\nabla\boldsymbol{d}, \nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \boldsymbol{v} - \tilde{\boldsymbol{v}}\rangle \\
+ (\nabla\partial_t\boldsymbol{d}, \nabla\boldsymbol{d} - \nabla\tilde{\boldsymbol{d}}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{d}, -\Delta\boldsymbol{d} + \Delta\tilde{\boldsymbol{d}}) + ((I - d\otimes d)\Delta d, \Delta d - \Delta\tilde{d}) = 0,
\end{aligned}
\tag{75}
$$

where $(\cdot,\cdot)$ denotes the L^2 scalar product and the red terms cancel by integration by parts (as beforehand).

$$
\begin{aligned}
\left\langle \mathcal{A}(\tilde{v},\tilde{d}), \begin{pmatrix} \boldsymbol{v} - \tilde{\boldsymbol{v}} \\ -\Delta\boldsymbol{d} + \Delta\tilde{\boldsymbol{d}} \end{pmatrix} \right\rangle - \Big(& (\partial_t\tilde{\boldsymbol{v}}, \boldsymbol{v} - \tilde{\boldsymbol{v}}) + ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}, \boldsymbol{v}) \\
& + \nu(\nabla\tilde{\boldsymbol{v}}, -\nabla\boldsymbol{v} + \nabla\tilde{\boldsymbol{v}}) - (\nabla\tilde{\boldsymbol{d}}^{\mathsf{T}}\nabla\tilde{\boldsymbol{d}}, \nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}) \\
& - \langle \boldsymbol{f}, \boldsymbol{v} - \tilde{\boldsymbol{v}}\rangle \\
& + (\nabla\partial_t\tilde{\boldsymbol{d}}, \nabla\boldsymbol{d} - \nabla\tilde{\boldsymbol{d}}) - ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}, \Delta\boldsymbol{d} - \Delta\tilde{\boldsymbol{d}}) \\
& + ((I - \tilde{d}\otimes\tilde{d})\Delta\tilde{d}, \Delta d - \Delta\tilde{d}) \Big) = 0.
\end{aligned}
\tag{76}
$$

(We have $\boldsymbol{d}\times(\boldsymbol{d}\times\Delta\boldsymbol{d}) = -(\boldsymbol{I} - \boldsymbol{d}\otimes\boldsymbol{d})\Delta\boldsymbol{d}$ if $|\boldsymbol{d}| = 1$.) Again, the red terms vanish, as above, by integration by parts. Adding (75) and (76), the f-terms cancel and we find

$$
\begin{aligned}
0 = & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\boldsymbol{v} - \tilde{\boldsymbol{v}}|_H^2 + \|\nabla\boldsymbol{d} - \nabla\tilde{\boldsymbol{d}}\|_{L^2}^2\right) + \nu\|\nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}\|_{L^2}^2 \\
& + \|\boldsymbol{d}\times\Delta\boldsymbol{d} - \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\|_{L^2}^2 + 2(\boldsymbol{d}\times\Delta\boldsymbol{d}, \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}) - (\boldsymbol{d}\times\Delta\boldsymbol{d}, \boldsymbol{d}\times\Delta\tilde{\boldsymbol{d}}) - (\tilde{\boldsymbol{d}}\times\Delta\boldsymbol{d}, \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}) \\
& - \underbrace{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \tilde{\boldsymbol{v}}) - ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}, \boldsymbol{v}) + (\nabla\tilde{\boldsymbol{d}}^{\mathsf{T}}\nabla\tilde{\boldsymbol{d}}, \nabla\boldsymbol{v}) + ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}, \Delta\boldsymbol{d})(\nabla\boldsymbol{d}^{\mathsf{T}}\nabla\boldsymbol{d}, \nabla\tilde{\boldsymbol{v}}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{d}, \Delta\tilde{\boldsymbol{d}})}_{\text{convective terms}}.
\end{aligned}
$$

We now only consider

$$
\begin{aligned}
& 2(\boldsymbol{d}\times\Delta\boldsymbol{d}, \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}) - (\boldsymbol{d}\times\Delta\boldsymbol{d}, \boldsymbol{d}\times\Delta\tilde{\boldsymbol{d}}) - (\tilde{\boldsymbol{d}}\times\Delta\boldsymbol{d}, \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}) - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \tilde{\boldsymbol{v}}) \\
& -((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}, \boldsymbol{v}) + (\nabla\tilde{\boldsymbol{d}}^{\mathsf{T}}\nabla\tilde{\boldsymbol{d}}, \nabla\boldsymbol{v}) - ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}, \Delta\boldsymbol{d}) + (\nabla\boldsymbol{d}^{\mathsf{T}}\nabla\boldsymbol{d}, \nabla\tilde{\boldsymbol{v}}) - ((\boldsymbol{v}\cdot\nabla)\boldsymbol{d}, \Delta\tilde{\boldsymbol{d}}).
\end{aligned}
$$

We have

$$\begin{aligned}
&2(\boldsymbol{d}\times\Delta\boldsymbol{d},\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}})-(\boldsymbol{d}\times\Delta\boldsymbol{d},\boldsymbol{d}\times\Delta\tilde{\boldsymbol{d}})-(\tilde{\boldsymbol{d}}\times\Delta\boldsymbol{d},\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}})\\
&=\big(\boldsymbol{d}\times\Delta\boldsymbol{d},(\tilde{\boldsymbol{d}}-\boldsymbol{d})\times\Delta\tilde{\boldsymbol{d}}\big)+\big((\boldsymbol{d}-\tilde{\boldsymbol{d}})\times\Delta\boldsymbol{d},\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\big)\\
&=\big(\boldsymbol{d}\times\Delta\boldsymbol{d}-\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}},(\tilde{\boldsymbol{d}}-\boldsymbol{d})\times\Delta\tilde{\boldsymbol{d}}\big)+\big(\underbrace{(\boldsymbol{d}-\tilde{\boldsymbol{d}})\times\Delta(\boldsymbol{d}-\tilde{\boldsymbol{d}})}_{\nabla\cdot([\boldsymbol{d}-\tilde{\boldsymbol{d}}]_\times\nabla(\boldsymbol{d}-\tilde{\boldsymbol{d}}))},\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\big)\\
&\overset{(\star)}{=}\big(\boldsymbol{d}\times\Delta\boldsymbol{d}-\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}},(\tilde{\boldsymbol{d}}-\boldsymbol{d})\times\Delta\tilde{\boldsymbol{d}}\big)-\big([\boldsymbol{d}-\tilde{\boldsymbol{d}}]_\times\nabla(\boldsymbol{d}-\tilde{\boldsymbol{d}}),\nabla(\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}})\big),
\end{aligned}$$

where in $(\star)$ we use integration by parts and the boundary term vanishes, because both $\boldsymbol{d}$ and $\tilde{\boldsymbol{d}}$ have to fulfil the same DIRICHLET boundary conditions. On the other hand,

$$\begin{aligned}
-\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\tilde{\boldsymbol{v}}\big)-\big((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}},\boldsymbol{v}\big)&=\big((\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{v}},\boldsymbol{v}-\tilde{\boldsymbol{v}}\big)-\big((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}},\boldsymbol{v}-\tilde{\boldsymbol{v}}\big)\\
&=\Big(\big((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla\big)\tilde{\boldsymbol{v}},\boldsymbol{v}-\tilde{\boldsymbol{v}}\Big)\\
&=\big((\boldsymbol{v}-\tilde{\boldsymbol{v}})\otimes(\boldsymbol{v}-\tilde{\boldsymbol{v}});(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\big).
\end{aligned}$$

Lastly, by integration in the last terms

$$\begin{aligned}
&\big(\nabla\boldsymbol{d}^\mathsf{T}\nabla\boldsymbol{d},\nabla\tilde{v}\big)+\big(\nabla\tilde{\boldsymbol{d}}^\mathsf{T}\nabla\tilde{\boldsymbol{d}},\nabla\boldsymbol{v}\big)+\underbrace{\big((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}},\Delta\boldsymbol{d}\big)+\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{d},\Delta\tilde{\boldsymbol{d}}\big)}_{=-\big((\tilde{\boldsymbol{v}}\cdot\nabla)\nabla\tilde{\boldsymbol{d}},\nabla\boldsymbol{d}\big)-\big(\nabla\tilde{\boldsymbol{v}},\nabla\tilde{\boldsymbol{d}}^\mathsf{T}\nabla\boldsymbol{d}\big)}\\
&=\big((\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}})^\mathsf{T}\nabla\boldsymbol{d},\nabla\tilde{v}\big)\big((\nabla\tilde{\boldsymbol{d}}^\mathsf{T}-\nabla\boldsymbol{d}^\mathsf{T})\nabla\tilde{\boldsymbol{d}},\nabla\boldsymbol{v}\big)-\big((\tilde{\boldsymbol{v}}\cdot\nabla)\nabla\tilde{\boldsymbol{d}},\nabla\boldsymbol{d}\big)-\big((\boldsymbol{v}\cdot\nabla)\nabla\boldsymbol{d},\nabla\tilde{\boldsymbol{d}}\big)\\
&=\big((\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}})^\mathsf{T}(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}),\nabla\tilde{\boldsymbol{v}}\big)+\big((\nabla\tilde{\boldsymbol{d}}^\mathsf{T}-\nabla\boldsymbol{d}^\mathsf{T})\nabla\tilde{\boldsymbol{d}}),\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\big)\\
&\quad+\Big(\big(\boldsymbol{v}-\tilde{\boldsymbol{v}}\big)\cdot\nabla\big)\nabla\tilde{\boldsymbol{d}},\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\Big).
\end{aligned}$$

This should formally be the terms that arise in the definition of $\mathcal{W}_H$ and $\mathcal{W}_D$.

Proof. (of Theorem 6.4.1) 1 We use a semi-discretisation in space as an approximate system. Let $(W_n)\subset V$ be a sequence of space that form an GALERKIN approximation of the space V. We denote the $L^2(\Omega)$-orthogonal projection onto W_n by P_n. For the approximation of the direction equation (72) we use an $L^2(\Omega)$ orthonormal GALERKIN basis consisting of eigenfunctions $\boldsymbol{y_1},\boldsymbol{y_2},\ldots$ of the differential operator corresponding to the boundary value problem

$$\begin{cases}-\Delta\boldsymbol{z}=\boldsymbol{h} & \text{in }\Omega,\\ \boldsymbol{z}=0 & \text{on }\partial\Omega.\end{cases}\tag{77}$$

The above problem is a symmetric strongly elliptic system that possesses a unique weak solution $z\in H_0^1(\Omega)$ for any $\boldsymbol{h}\in H^{-1}(\Omega)$. Its solution operator is thus a compact selfadjoint operator in L^2. Hence there exists an orthogonal basis of eigenfunctions $\boldsymbol{y_1},\boldsymbol{y_2},\ldots$ in $L^2(\Omega)$, which are orthonormal in $L^2(\Omega)$. A regularity result provides regularity of the eigenfunctions such that $Y_n:=\operatorname{span}\{\boldsymbol{y_1},\ldots,\boldsymbol{y_n}\}\subset H^2(\Omega)\cap H^1(\Omega)$. The associated orthogonal $L^2(\Omega)$-projection is denoted by $R_n\colon L^2(\Omega)\to Y_n$, $\boldsymbol{d}\mapsto\sum_{i=1}^n(\boldsymbol{d},\boldsymbol{y}_i)_{L^2}\boldsymbol{y}_i$. So for each eigenvector $\boldsymbol{y_i}$ there is an eigenvalue λ_i. Then by partial integration (as the eigenfunctions fulfil the homogeneous boundary conditions) in the first step and third step and inserting the definition of R_n in the second step we have

$$\begin{aligned}
(\nabla R_n\boldsymbol{d},\nabla h)=-(\Delta R_n\boldsymbol{d},h)&=\sum_{i=1}^n(\boldsymbol{d},\boldsymbol{y}_i)(\lambda_i\boldsymbol{y}_i,h)\\
&=\sum_{i=1}^n(\boldsymbol{d},\lambda_i\boldsymbol{y}_i)(\boldsymbol{y}_i,h)=\sum_{i=1}^n(\nabla\boldsymbol{d},\nabla\boldsymbol{y}_i)(\boldsymbol{y}_i,h)=(\nabla\boldsymbol{d},\nabla R_nh)
\end{aligned}$$

for $h \in H^{-1}(\Omega)$. This shows that R_n is also an orthogonal projection in the space H^1.

Note that the projection R_n is $H^1(\Omega)$-stable, that is, there exists a constant $c > 0$ (which can explicitly calculated by inserting $h = R_n \boldsymbol{d}$ in the above term and using PLANCHAREL's inequality) such that $\|R_n \boldsymbol{y}\|_{H^1(\Omega)} \leqslant c\|\boldsymbol{y}\|_{H^1(\Omega)}$ for all $\boldsymbol{y} \in H^1(\Omega)$.

(2) Let $n \in \mathbb{N}$. We consider the ansatz (recall that $-\Delta \mathbb{E}\, d_1 = 0$ in Ω and $\mathbb{E}\, d_2 = d_1$ on $\partial\Omega$)

$$\boldsymbol{v}_n(t) = \sum_{i=1}^{n} v_n^i(t)\boldsymbol{w}_i, \qquad \boldsymbol{d}_n = \mathbb{E}\,\boldsymbol{d}_1 + \sum_{i=1}^{n} d_n^i(t)\boldsymbol{z}_i,$$

where (v_n^i, d_n^i) are absolutely continuous functions on $(0,T)$ for all $i \in \{1,\dots,n\}$. The approximate system is

approximate system

$$\big(\partial_t \boldsymbol{v}_n + (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{w}\big) + \big(\nu\nabla\boldsymbol{v_n} - \nabla\boldsymbol{d}_n^{\mathsf{T}}\nabla\boldsymbol{d_n}, \nabla\boldsymbol{w}\big) = \langle\, \boldsymbol{f}, \boldsymbol{w}\,\rangle \qquad \forall \boldsymbol{w} \in W_n, \tag{78}$$

$$\big(\partial_t \boldsymbol{d}_n + (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n - (|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)\Delta\boldsymbol{d}_n, \boldsymbol{y}\big) = 0 \qquad \forall \boldsymbol{y} \in Y_n \tag{79}$$

such that $\boldsymbol{v}_n = P_n \boldsymbol{v}_0$ and $\boldsymbol{d}_n(0) = \mathbb{E}\,\boldsymbol{d}_1 + R_n(\boldsymbol{d}_0 - \mathbb{E}\,\boldsymbol{d}_1)$.

A classical existence theorem provides, for every $n \in \mathbb{N}$, the existence of a maximally extended solution to the above approximate problem on an interval $[0, T_n)$ in the sense of CARATHEÓDORY.

(3) **A priori estimates.** Choosing $w := v_n$ in (78) and $z := -\Delta d_n$ (79), which are suitable test functions, by adding (78) and (79) and integrating by parts in the ∂_t-terms we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{d}_n\|_{L^2(\Omega)}^2\Big) + \nu\|\nabla\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\boldsymbol{d}_n \times \Delta\boldsymbol{d}_n\|_{L^2(\Omega)}^2$$
$$+ \underbrace{\big((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{v}_n\big)}_{=-\frac{1}{2}\int_\Omega \nabla\cdot\boldsymbol{v}_n|\boldsymbol{v}_n|^2\,\mathrm{d}\boldsymbol{x}=0} \underbrace{- \big(\nabla\boldsymbol{d}_n^{\mathsf{T}}\nabla\boldsymbol{d}_n, \nabla\boldsymbol{v}_n\big) \underbrace{-\big((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n, \Delta\boldsymbol{d}_n\big)}_{=(\nabla\boldsymbol{v}_n;\nabla\boldsymbol{d}_n^{\mathsf{T}}\nabla\boldsymbol{d})+\frac{1}{2}\underbrace{((\boldsymbol{v}_n\cdot\nabla)\nabla|\boldsymbol{d}_n|^2)}_{=0}}}_{=0}$$
$$= \langle\, \boldsymbol{f}, \boldsymbol{v}_n\,\rangle.$$

Hence the above equation reduces to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{d}_n\|_{L^2(\Omega)}^2\Big) + \nu\|\nabla\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\boldsymbol{d}_n \times \Delta\boldsymbol{d}_n\|_{L^2(\Omega)}^2 = \langle\, \boldsymbol{f}, \boldsymbol{v}_n\,\rangle.$$

Integrating provides

$$\frac{1}{2}\Big(\|\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{d}_n\|_{L^2(\Omega)}^2\Big)\Big|_0^t + \int_0^t \nu\|\nabla\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\boldsymbol{d}_n \times \Delta\boldsymbol{d}_n\|_{L^2(\Omega)}^2\,\mathrm{d}s = \langle\, \boldsymbol{f}, \boldsymbol{v}_n\,\rangle.$$

If we assume that $f \in L^2(0,T;H^{-1}(\Omega)) \oplus L^1(0,T;L^2(\Omega))$, then

$$\begin{aligned}\langle\, \boldsymbol{f}, \boldsymbol{v}_n\,\rangle &= \langle\, \boldsymbol{f}_1, \boldsymbol{v}_n\,\rangle + \langle\, \boldsymbol{f}_2, \boldsymbol{v}_n\,\rangle \\ &\leqslant \|\boldsymbol{f}_1\|_{L^2(\Omega)}^2(1 + \|\boldsymbol{v}_n\|^2) + \frac{C}{2\nu}\|\boldsymbol{f}_2\|_{H^{-1}(\Omega)}^2 + \frac{\nu}{2}\|\nabla\boldsymbol{v}_n\|_{L^2(\Omega)}^2,\end{aligned}$$

where we used YOUNG's inequality and the constant $C > 0$ stems from the POINCARÉ embedding.

We find that by GRONWALL

$$\begin{aligned}&\Big(\|\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{d}_n\|_{L^2(\Omega)}^2\Big)(t) + \int_0^t \nu\|\nabla\boldsymbol{v}_n\|_{L^2(\Omega)}^2 + 2\|\boldsymbol{d}_n \times \Delta\boldsymbol{d}_n\|_{L^2(\Omega)}^2\,\mathrm{d}s \\ &\leqslant \Big(\|\boldsymbol{v}_0^n\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{d}_0^n\|_{L^2(\Omega)}^2\Big)\exp\Big(\int_0^t 2\|\boldsymbol{f}_1\|_{L^2(\Omega)}\,\mathrm{d}s\Big) \\ &\quad + \frac{C}{\nu}\|\boldsymbol{f}_1\|_{L^2(0,T;H^{-1}(\Omega))}\exp\Big(\int_0^t 2\|\boldsymbol{f}_1\|_{L^2(\Omega)\,\mathrm{d}s}\Big).\end{aligned}$$

Hence

$$\|\boldsymbol{v}_n\|_{L^\infty(0,T;H)} + \|\boldsymbol{v}_n\|_{L^2(0,T;V)} + \|\boldsymbol{d}_n\|_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n\|_{L^2(\Omega\times(0,T))} \leqslant C.$$

4 **Estimate for the time derivative.** Let $y \in H_0^1(\Omega)$, then test (79) by $R_n\boldsymbol{y} \in Y_n$ and integrate in time to obtain

$$\int_0^t |(\partial_t \boldsymbol{d}_n, R_n\boldsymbol{y})|\,\mathrm{d}s \leqslant \int_0^t y|(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n, R_n y)| + \Big|(\underbrace{[\boldsymbol{d}_n]_\times^\mathsf{T}[\boldsymbol{d}_n]_\times \Delta\boldsymbol{d}_n}_{=-\boldsymbol{d}_n\times(\boldsymbol{d}_n\times\Delta\boldsymbol{d}_n)}, R_n\boldsymbol{y})\Big|\,\mathrm{d}s$$

$$\overset{(\mathrm{H})}{\leqslant} \|\boldsymbol{v}_n\|_{L^2(0,T;L^6(\Omega))}\|\boldsymbol{d}_n\|_{L^\infty(0,T;H^{-1}(\Omega)}\|R_n\boldsymbol{y}\|_{L^2(0,T;L^3(\Omega)}$$
$$+ \|\boldsymbol{d}_n\|_{L^\infty(0,T;L^6(\Omega))}\|\boldsymbol{d}_n \times \Delta\boldsymbol{d}_n\|_{L^2(\Omega\times(0,T))}\|R_n\boldsymbol{y}\|_{L^2(0,T;L^3(\Omega))},$$

where in (H) we use HÖLDER's inequality with "$\frac{1}{2}+\frac{1}{\infty}+\frac{1}{2} = 1$" in time and $\frac{1}{6}+\frac{1}{2}+\frac{1}{3} = 1$ in space and we use that in the three-dimensional space Ω, we have $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

We have

$$\|R_n\boldsymbol{y}\|_{L^2(0,T;L^3(\Omega))} \leqslant C\|R_n\boldsymbol{y}\|_{L^2(0,T;H^1(\Omega))} \leqslant C\|\boldsymbol{y}\|_{L^2(0,T;H^1(\Omega))},$$

where the last inequality is due to the $H^1(\Omega)$-stability of R_n. Hence by the definition of the dual norm we have

$$\|\partial_t\boldsymbol{d}_n\|_{L^2(0,T;H^{-1}(\Omega))} = \sup_{\substack{\boldsymbol{y}\in L^2(0,T;H_0^1(\Omega)):\\ \|\boldsymbol{y}\|_{L^2(0,T;H_0^1(\Omega))}=1}} |\langle\, \partial_t\boldsymbol{d}_n, \boldsymbol{y}\,\rangle|$$
$$\leqslant C\Big(\|\boldsymbol{v}_n\|_{L^2(0,T;L^6(\Omega))}\|\boldsymbol{d}_n\|_{L^\infty(0,T;H^{-1}(\Omega))}$$
$$+ \|\boldsymbol{d}_n\|_{L^\infty(0,T;L^6(\Omega))}\|\boldsymbol{d}_n \times \Delta\boldsymbol{d}_n\|_{L^2(\Omega\times(0,T))}\Big) \leqslant \tilde{C}.$$

5 **Converging subsequences.** We have

$$\boxed{\begin{aligned} \boldsymbol{v}_n &\overset{*}{\rightharpoonup} \boldsymbol{v} && \text{in } L^\infty(0,T;H)\cap L^2(0,T;V),\\ \boldsymbol{d}_n &\overset{*}{\rightharpoonup} \boldsymbol{d} && \text{in } L^\infty(0,T;H^1(\Omega)),\\ \boldsymbol{d}_n\times\Delta\boldsymbol{d}_n &\to \xi && \text{in } L^2(\Omega\times(0,T)),\\ \partial_t\boldsymbol{d}_n &\to \partial_t\boldsymbol{d} && \text{in } L^2(0,T;H^{-1}(\Omega)).\end{aligned}}$$

By the Lemma of AUBIN-LIONS, the compact embedding

$$L^\infty(0,T;H^1(\Omega)) \cap W^{1,2}(0,T;H^{-1}(\Omega)) \overset{c}{\hookrightarrow} L^2(0,T;L^2(\Omega))$$

holds. By some interpolation inequality, we find

$$L^\infty(0,T;H^1(\Omega)) \cap W^{1,2}(0,T;H^{-1}(\Omega)) \overset{c}{\hookrightarrow} L^p(0,T;L^q(\Omega))$$

with $p < \infty$ and $q < 6$ in $d = 3$.

6 **Identification of the limit of $(\boldsymbol{d}_n\times\Delta\boldsymbol{d}_n)_{n\in\mathbb{N}}$.** Recall $\boldsymbol{d}_n\times\Delta\boldsymbol{d}_n = \nabla\cdot([\boldsymbol{d}_n]_\times\nabla\boldsymbol{d}_n)$. Now observe that by HÖLDER's inequality we have

$$\|[\boldsymbol{d}_n]_\times\nabla\boldsymbol{d}_n\|_{L^\infty(0,T;L^3(\Omega))} \leqslant \|\boldsymbol{d}_n\|_{L^\infty(0,T;L^6(\Omega))}\|\nabla\boldsymbol{d}_n\|_{L^2(0,T;L^2(\Omega))}$$

which implies that

$$[\boldsymbol{d}_n]_\times\nabla\boldsymbol{d}_n \rightharpoonup [\boldsymbol{d}]_\times\nabla\boldsymbol{d} \qquad \text{in } L^p(0,T;L^r(\Omega))$$

for $p < \infty$ and $r < \frac{3}{2}$. Thus (as ∇ is linear)

$$\nabla \cdot ([\boldsymbol{d}_n]_\times \nabla \boldsymbol{d}_n) \rightharpoonup \nabla \cdot ([\boldsymbol{d}]_\times \nabla \boldsymbol{d}) \qquad \text{in } L^p(0,T;W^{1,r}(\Omega))$$

for $p < \infty$ and $r < \frac{3}{2}$. Since a weak limit is unique, we have

$$\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n = \nabla \cdot ([\boldsymbol{d}_n]_\times \nabla \boldsymbol{d}_n) \rightharpoonup \nabla \cdot ([\boldsymbol{d}]_\times \nabla \boldsymbol{d}) \qquad \text{in } L^2(\Omega \times (0,T)).$$

Hence

$$\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n \rightharpoonup \boldsymbol{d} \times \Delta \boldsymbol{d} \qquad \text{in } L^2(\Omega \times (0,T)).$$

7 **Approximate relative energy inequality** Define

$$Y := (W^{2,p}(\Omega) \cap V, H^{2,p}(\Omega))$$

for $p > d = 3$.

Testing (78) by $(\boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}})$ and (79) by $y = -\Delta(\boldsymbol{d}_n - R_n\tilde{\boldsymbol{d}})$ yields

$$\begin{aligned}
&(\partial_t \boldsymbol{v}_n, \boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}}) - \big((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, P_n\tilde{v}\big) + \nu(\nabla \boldsymbol{v}_n, \nabla \boldsymbol{v}_n - \nabla P_n\tilde{\boldsymbol{v}}) \\
&- (\nabla \boldsymbol{d}_n^\mathsf{T} \nabla \boldsymbol{d}_n, \cancel{\nabla \boldsymbol{v}_n} - \nabla P_n\tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}} \rangle + (\partial_t \nabla \boldsymbol{d}_n, \nabla \boldsymbol{d}_n - \nabla R_n\tilde{\boldsymbol{d}}) \\
&- \big(\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n, \cancel{\Delta \boldsymbol{d}_n} - \Delta R_n\tilde{\boldsymbol{d}}\big) + \big(\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n, \boldsymbol{d}_n \times (\Delta \boldsymbol{d}_n - \Delta R_n\tilde{\boldsymbol{d}})\big) = 0,
\end{aligned} \tag{80}$$

where the two terms cancel as beforehand by integration by parts.

Additionally, testing $\mathcal{A}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}})$ by $(P_n\tilde{\boldsymbol{v}} - v_n, \Delta \boldsymbol{d}_n - \Delta R_n\tilde{\boldsymbol{d}})$ yields

$$\begin{aligned}
&(\partial_t P_n\tilde{\boldsymbol{v}}, P_n\tilde{\boldsymbol{v}} - \boldsymbol{v}_n) + \big((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{v}}, \cancel{P_n\tilde{\boldsymbol{v}}} - \boldsymbol{v}_n\big) + \nu(\nabla P_n\tilde{\boldsymbol{v}}, \nabla P_n\tilde{\boldsymbol{v}} - \nabla \boldsymbol{v}_n) \\
&- (\nabla R_n\tilde{\boldsymbol{d}}^\mathsf{T} \nabla R\tilde{\boldsymbol{d}}, \cancel{\nabla P_n\tilde{\boldsymbol{v}}} - \nabla \boldsymbol{v}_n) - \langle \boldsymbol{f}, \nabla\tilde{\boldsymbol{v}} - \boldsymbol{v}_n \rangle + (\partial_t \nabla R_n\tilde{\boldsymbol{d}}, \nabla R_n\tilde{\boldsymbol{d}} - \nabla \boldsymbol{d}_n) \\
&- \big((P_n\tilde{\boldsymbol{v}} \cdot \nabla)R_n\tilde{\boldsymbol{d}}, \cancel{\Delta R_n\tilde{\boldsymbol{d}}} - \Delta \boldsymbol{d}_n\big) + \big(R_n\tilde{\boldsymbol{d}} \times \Delta R_n\tilde{\boldsymbol{d}}, R_n\tilde{\boldsymbol{d}} \times \Delta(R_n\tilde{\boldsymbol{d}} - \boldsymbol{d}_n)\big) = 0.
\end{aligned} \tag{81}$$

Adding (80) and (81) and integrating in time yields (the f-terms cancel each other out)

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}}\|_{L^2}^2 + \|\nabla \boldsymbol{d}_n - \nabla R_n\tilde{\boldsymbol{d}}\|_{L^2}^2\Big) \\
&+ \int_0^t \nu\|\nabla \boldsymbol{v}_n - \nabla P_n\tilde{v}\|_{L^2}^2 + \|\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n - R_n\tilde{\boldsymbol{d}} \times \Delta R_n\tilde{\boldsymbol{d}}\|_{L^2}^2 \,\mathrm{d}s \\
&+ \int_0^t 2(\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n, R_n\tilde{\boldsymbol{d}} \times \Delta R_n\tilde{\boldsymbol{d}}) - (\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n, \boldsymbol{d}_n \times R_n\tilde{\boldsymbol{d}}) - (R_n\tilde{\boldsymbol{d}} \times \Delta \boldsymbol{d}_n, R_n\tilde{\boldsymbol{d}} \times \Delta R_n\tilde{\boldsymbol{d}}) \,\mathrm{d}s \\
&- \int_0^t \big((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, P_n\tilde{\boldsymbol{v}}\big) + \big((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}_n\big) - (\nabla \boldsymbol{d}_n^\mathsf{T} \nabla \boldsymbol{d}_n, \nabla P_n\tilde{\boldsymbol{v}}) - (\nabla R_n\tilde{\boldsymbol{d}}^\mathsf{T} \nabla R_n\tilde{\boldsymbol{d}}, \nabla \boldsymbol{v}_n) \,\mathrm{d}s \\
&+ \int_0^t \big((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n, \Delta P_n\tilde{\boldsymbol{d}}\big) + \big((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{d}}, \Delta \boldsymbol{d}_n\big) \,\mathrm{d}s = 0.
\end{aligned}$$

On the discrete level it holds that (as the last three lines can be rewritten similarly to the calculation on the continuous level)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(\boldsymbol{v}_n\boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) + \mathcal{W}(\boldsymbol{v}_n\boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) + \left\langle \mathcal{A}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}), \begin{pmatrix} \boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}} \\ \Delta(R_n\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \end{pmatrix} \right\rangle \\
&+ \mathcal{K}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}})\mathcal{R}(\boldsymbol{v}_n\boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) = 0.
\end{aligned} \tag{82}$$

Since the GALERKIN spaces $((W_n, Y_n))_{n\in\mathbb{N}}$ are also limit-dense in Y, we infer

$$R_n\tilde{\boldsymbol{d}} \to \tilde{\boldsymbol{d}} \qquad \text{in } Y \text{ and almost everywhere in } (0,T).$$

This implies

$$\boxed{\begin{array}{cl} \Delta R_n\tilde{\boldsymbol{d}} \to \Delta\tilde{\boldsymbol{d}} & \text{in } L^2(0,T;L^3(\Omega)),\\ [R_n\tilde{\boldsymbol{d}}]_\times \nabla R_n\tilde{\boldsymbol{d}} \to [\tilde{\boldsymbol{d}}]_\times\nabla\tilde{\boldsymbol{d}} & \text{in } L^2(0,T;W^{2,3}(\Omega)),\\ (\nabla P_n\tilde{\boldsymbol{v}})_{\text{sym}} \to (\nabla\tilde{\boldsymbol{v}})_{\text{sym}} & \text{in } L^1(0,T;L^\infty(\Omega)),\\ \nabla R_n\tilde{\boldsymbol{d}} \to \nabla\tilde{\boldsymbol{d}} & \text{in } L^2(0,T;L^\infty(\Omega)),\\ R_n\tilde{\boldsymbol{d}} \to \tilde{\boldsymbol{d}} & \text{in } L^2(0,T;W^{2,3}(\Omega)). \end{array}}$$

due to the L^∞-bound (??).

We see that the projection R_n is constructed in such a way that we have more regular convergences.

Applying GRONWALL's Lemma to (71), we find by the Lemma for weak inequalities that

$$\begin{aligned} &-\int_0^T \Phi'\mathcal{R}(\boldsymbol{v}_n,\boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}_n, R_n\tilde{\boldsymbol{d}}_n)\exp\left(-\int_0^t \mathcal{K}(P_n\tilde{\boldsymbol{v}},R_n\tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t\\ &+\int_0^T \Phi\Bigg(\mathcal{W}(\boldsymbol{v}_n,\boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}_n, R_n\tilde{\boldsymbol{d}}_n)\\ &+\left\langle \mathcal{A}(P_n\tilde{\boldsymbol{v}},R_n\tilde{\boldsymbol{v}}), \begin{pmatrix}\boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}}\\ \Delta(R_n\tilde{\boldsymbol{d}} - d_n)\end{pmatrix}\right\rangle\Bigg)\exp\left(-\int_0^t \mathcal{K}(R_n\tilde{\boldsymbol{v}},P_n\tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t\\ &\quad\leqslant \Phi(0)\mathcal{R}(P_n\boldsymbol{v}_0, R_n\boldsymbol{d}_0 \mid P_n\tilde{\boldsymbol{v}}(0), R_n\tilde{\boldsymbol{d}}(0)). \end{aligned}$$

for all $\Phi\in\tilde{\mathcal{C}}([0,T])$.

8 Indeed,

$$\begin{aligned} \mathcal{W}_D(\boldsymbol{v},\boldsymbol{d}\mid\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) + \mathcal{W}_H(\boldsymbol{v},\boldsymbol{d}\mid\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) = {}& \nu\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\times\Delta\boldsymbol{d} - \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\|^2_{L^2(\Omega)}\\ &+\left(\boldsymbol{d}\times\Delta\boldsymbol{d} - \tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}, (\tilde{\boldsymbol{d}}-\boldsymbol{d})\times\Delta\tilde{\boldsymbol{d}}\right)\\ &-\left(\nabla(\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}); (\tilde{\boldsymbol{d}}-\boldsymbol{d})\times(\nabla\tilde{\boldsymbol{d}}-\nabla\boldsymbol{d})\right)\\ &\left(\boldsymbol{v}-\tilde{\boldsymbol{v}}, (\nabla\tilde{\boldsymbol{v}})_{\text{sym}}(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right)\\ &+\left(\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right)^{\mathsf{T}}\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right); (\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\right)\\ &\quad-\left(\nabla\tilde{\boldsymbol{d}}^{\mathsf{T}}\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right), \nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\right)\\ &+\left(((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)\nabla\tilde{\boldsymbol{d}}; \nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\right) =: (\star) \end{aligned}$$

and thus

$(\cdot,\cdot)$ denotes the L^2 scalar product for scalar and vector-valued functions while $(\cdot;\cdot)$ is for matrix-valued functions

$$\begin{aligned} (\star) \geqslant{}& \nu\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\times\Delta\boldsymbol{d}-\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{d}\times\Delta\boldsymbol{d}-\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}}\|^2_{L^2(\Omega)}\\ &-\frac{1}{2}\|\Delta\tilde{\boldsymbol{d}}\|^2_{L^3(\Omega)}\|\boldsymbol{d}-\tilde{\boldsymbol{d}}\|^2_{L^6(\Omega)} - \|\nabla(\tilde{\boldsymbol{d}}\times\Delta\tilde{\boldsymbol{d}})\|_{L^3(\Omega)}\|\boldsymbol{d}-\tilde{\boldsymbol{d}}\|_{L^6(\Omega)}\|\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\|_{L^2(\Omega)}\\ &-\|(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\|_{L^\infty(\Omega)}\left(\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \|\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\|^2_{L^2(\Omega)}\right) - \frac{\nu}{2}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\\ &-\frac{1}{2\nu}\|\nabla\tilde{\boldsymbol{d}}\|^2_{L^\infty(\Omega)}\|\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\|^2_{L^2(\Omega)} - \frac{\nu}{2C_\Omega}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L^6(\Omega)}\\ &+\frac{C_\Omega}{2\nu}\|\nabla^2\tilde{\boldsymbol{d}}\|^2_{L^3(\Omega)}\|\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\|^2_{L^2(\Omega)}\\ \geqslant{}& \mathcal{K}(\tilde{v},\tilde{d})\mathcal{R}(\boldsymbol{v},\boldsymbol{d}\mid\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}). \end{aligned}$$

We observe that $\mathcal{W}$ is quadratic in the terms

$$\boldsymbol{u} := \begin{pmatrix} \nabla \boldsymbol{v}_n - \nabla \tilde{\boldsymbol{d}} \\ \boldsymbol{d}_n \times \Delta \boldsymbol{d}_n - \boldsymbol{d} \times \Delta \boldsymbol{d} \\ \nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \\ \boldsymbol{d}_n - \tilde{\boldsymbol{d}} \\ \boldsymbol{v}_n - \tilde{\boldsymbol{v}}. \end{pmatrix}$$

Together, we deduce weakly lower semi-continuity

$$\begin{aligned} &- \liminf_{n\to\infty} \int_0^T \Phi' \mathcal{R}(\boldsymbol{v}_n, \boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) \exp\left(-\int_0^t \mathcal{K}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t \\ &\qquad + \liminf_{n\to\infty} \int_0^T \Phi \mathcal{W}(\boldsymbol{v}_n, \boldsymbol{d}_n \mid P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) \exp\left(\int_0^t \mathcal{K}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t \\ \geqslant &- \int_0^T \Phi' \mathcal{R}(\boldsymbol{v}, \boldsymbol{d} \mid \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}) \exp\left(-\int_0^t \mathcal{K}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t \\ &\qquad + \int_0^T \Phi \mathcal{W}(\boldsymbol{v}, \boldsymbol{d} \mid \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}) \exp\left(\int_0^t \mathcal{K}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t. \end{aligned}$$

From $P_n\boldsymbol{v}_0 \to \boldsymbol{v}_0$ in H and $R_n\boldsymbol{d}_0 \to \boldsymbol{d}_0$ in $H^1(\Omega)$ we find

$$\mathcal{R}\big(P_n\boldsymbol{v}_0, R_n\boldsymbol{d}_0 \mid P_n\tilde{\boldsymbol{v}}(0), R_n\tilde{\boldsymbol{d}}(0)\big) \to \mathcal{R}\big(\boldsymbol{v}_0, \boldsymbol{d}_0 \mid \tilde{\boldsymbol{v}}(0), \tilde{\boldsymbol{d}}(0)\big).$$

Finally, we have to prove that the consistency error vanishes, i.e.

$$\| \mathcal{A}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) - \mathcal{A}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}) \|_{L^2(0,T;V*)\otimes L^1(0,T;H^1(\Omega)} \xrightarrow{n\to\infty} 0.$$

For the time derivatives, we find

$$(\partial_t P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}}) = (P_n\partial_t\tilde{\boldsymbol{v}}, \boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}}) = (\partial_t\tilde{\boldsymbol{v}}, \tilde{v_n} - P_n\tilde{\boldsymbol{v}})$$

and

$$(\partial_t R_n\tilde{\boldsymbol{d}}, \Delta R_n\tilde{\boldsymbol{d}} - \Delta\boldsymbol{d}_n) = (R_n\partial_t\tilde{\boldsymbol{d}}, \Delta R_n\tilde{\boldsymbol{d}} - \Delta\boldsymbol{d}_n) = (\partial_t\tilde{\boldsymbol{d}}, \Delta R_n\tilde{\boldsymbol{d}} - \Delta\boldsymbol{d}_n)$$

since P_n and R_n are orthogonal projections and $\boldsymbol{v_n} - P_n\tilde{\boldsymbol{v}} \in W_n$ as well as $\Delta R_n\tilde{\boldsymbol{d}} - \Delta\boldsymbol{d}_n \in Y_n$ are elements of the respective approximate Galerkin space. Thus, we may estimate the consistency error as follows

$$\begin{aligned} &\| \mathcal{A}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}) - \mathcal{A}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}) \|_{L^2(0,T;V*)\otimes L^1(0,T;H^1(\Omega)} \\ &\leqslant \|(P_n\tilde{\boldsymbol{v}} \otimes P_n\tilde{\boldsymbol{v}}) - (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}})\|_{L^2(\Omega\times(0,T))} + \nu\|\nabla P_n\tilde{\boldsymbol{v}} - \nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega\times(0,T))} \\ &\quad + \|\nabla R_n\tilde{\boldsymbol{d}}^\mathsf{T}\nabla R_n\tilde{\boldsymbol{d}} - \nabla\tilde{\boldsymbol{d}}^\mathsf{T} - \nabla\tilde{\boldsymbol{d}}^\mathsf{T}\nabla R_n\tilde{\boldsymbol{d}}\|_{L^2(\Omega\times(0,T))} \\ &\quad + \|(P_n\tilde{\boldsymbol{v}}\cdot\nabla)R_n\tilde{\boldsymbol{d}} - (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}\|_{L^1(0,T;H^1(\Omega))} \\ &\quad + \|R_n\tilde{\boldsymbol{d}} \times (R_n\tilde{\boldsymbol{d}} \times \Delta\tilde{\boldsymbol{d}}) - \tilde{\boldsymbol{d}} \times (\tilde{\boldsymbol{d}} \times \Delta\tilde{\boldsymbol{d}})\|_{L^1(0,T;H^1(\Omega))} \to 0. \end{aligned}$$

The strong convergences in (23) allow to pass to the limit on the right hand side, which vanishes. This together with the weak converges (21a) and (21b) and (24), we infer

$$\begin{aligned} &\int_0^T \Phi \left\langle \mathcal{A}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}}), \begin{pmatrix} \boldsymbol{v}_n - P_n\tilde{\boldsymbol{v}} \\ \Delta R_n\tilde{\boldsymbol{d}} - \Delta\boldsymbol{d}_n \end{pmatrix} \right\rangle \exp\left(\int_0^t \mathcal{K}(P_n\tilde{\boldsymbol{v}}, R_n\tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t \\ &\quad \to \int_0^T \Phi \left\langle \mathcal{A}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}), \begin{pmatrix} \boldsymbol{v} - \tilde{\boldsymbol{v}} \\ \Delta\tilde{\boldsymbol{d}} - \Delta\boldsymbol{d} \end{pmatrix} \right\rangle \exp\left(\int_0^t \mathcal{K}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}})\,\mathrm{d}s\right)\mathrm{d}t. \end{aligned}$$

Via the lemma we check the ?? in time ??? It remains to consider the director equation

$$\big(\partial_t\boldsymbol{d}_n + (\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (|\boldsymbol{d}_n|^2 - \boldsymbol{d}_n\otimes\boldsymbol{d}_n)\Delta\boldsymbol{d}_n, y\big) = 0 \qquad \forall y \in Y_n,$$

which is equivalent to

$$\langle\, \partial_t \boldsymbol{d}_n, y \,\rangle - (\boldsymbol{d}_n \otimes \boldsymbol{v}_n, \nabla y) + (\boldsymbol{d}_n \times \Delta \boldsymbol{d}_n, \boldsymbol{d}_n \times y) = 0 \qquad \forall y \in Y_n$$

$$\dots$$

But $n \to \infty$ such that it holds for all $m \in \mathbb{N}$.

$$(\partial_t \boldsymbol{d}, \boldsymbol{y}) - (\boldsymbol{d} \otimes \boldsymbol{d}, \nabla \boldsymbol{y}) + (\boldsymbol{d} \times \Delta \boldsymbol{d}, \boldsymbol{d} \times \boldsymbol{y}) = 0 \qquad \forall \boldsymbol{y} \in L^2(0,T; L^3(\Omega)).$$

Therefore the equation holds even pointwise

$$\partial_t \boldsymbol{d} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{d} + \boldsymbol{d} \times (\boldsymbol{d} \times \Delta \boldsymbol{d}) = 0 \qquad \text{almost everywhere in } \Omega \times (0,T)$$

for $\boldsymbol{d}_0$ with $|\boldsymbol{d}_0| = 1$ almost everywhere in Ω. It follows that $|\boldsymbol{d}(\boldsymbol{x},t)| = 1$ almost everywhere in $\Omega \times (0,T)$. $\square$

Exercise 13.1: Let v be an energy-variational solution to the Ericksen-Leslie equations. Show that $v_\infty := \lim_{t\to\infty} v(t) \in \mathbb{R}$. To do so, show that there exists a sequence $t_n \to \infty$ such that

$$\|\nabla v(t_n)\|_2 + \|d(t_n) \times \Delta d(t_n)\|_2 \to 0 \qquad \text{as } t_n \to \infty.$$

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