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A function $u\colon [0,T] \to X$ is **Bochner measurable** (or strongly measurable) if there exists a sequence of simple functions $(u_n\colon [0,T] \to X)_{n\in\mathbb{N}}$ such that for almost all $t \in (0,T)$

$$\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0.$$

A function $u\colon [0,T] \to X$ is a **simple function** if there exist *finitely many pairwise disjoint* LEBESGUE *measurable* sets $(E_i \subset [0,T])_{i=1}^m$ such that u takes *constant values* $u_i \in X$ on each of these sets, that is $u = \sum_{i=1}^m u_i \mathbb{1}_{E_i}$. The *(*BOCHNER*) integral* of u is

$$\int_0^T u(t)\,\mathrm{d}t := \sum_{i=1}^m u_i |E_i| \in X.$$

A function $u\colon [0,T] \to X$ is *weakly* BOCHNER *measurable* if for all $f \in X^*$ the map $t \mapsto \langle f, u(t) \rangle$ is LEBESGUE *measurable.* Since strong convergence implies weak convergence, BOCHNER measurability implies weak BOCHNER measurability.

Let $u\colon [0,T] \to X$ be BOCHNER measurable. Then $\|u\|$ is LEBESGUE *measurable* on $[0,T]$.

As u is BOCHNER measurable, there exists a sequence of simple functions $(u_n : [0,T] \to X)_{n\in\mathbb{N}}$ such that $u_n(t) \to u(t)$ holds for almost all $t \in [0,T]$. For those $t \in [0,T]$ we thus have $|\|u_n(t)\| - \|u(t)\|| \leqslant \|u_n(t) - u(t)\| \xrightarrow{n\to\infty} 0$. The functions $(\|u_n\|: [0,T] \to \mathbb{R})_{n\in\mathbb{N}}$ are simple functions (and hence measurable), because the functions $u_n = \sum_{i=1}^{m_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}}$ are simple:

$$\|u_n(t)\| = \left\| \sum_{i=1}^{m_n} u_i^{(n)} \mathbb{1}_{E_i^{(n)}}(t) \right\| \overset{(\star)}{=} \sum_{i=1}^{m_n} \|u_i^{(n)}\| \mathbb{1}_{E_i^{(n)}}(t),$$

where in $(\star)$ we use that the $(E_i^{(n)})_{i=1}^{m_n}$ are *disjoint.* Hence $\|u\|$ is measurable as the *limit of the measurable functions* $\|u_n\|$.

Let $u\colon [0,T] \to X$ be BOCHNER measurable and $(u_n)_{n\in\mathbb{N}}$ a sequence of simple functions with $u_n(t) \to u(t)$ in X for almost all $t \in [0,T]$. Then u is **Bochner integrable** if $\int_0^T \|u_n(t) - u(t)\|\,\mathrm{d}t \to 0$. We set

$$\int_0^T u(t)\,\mathrm{d}t := \lim_{n\to\infty} \int_0^T u_n(t)\,\mathrm{d}t \in X.$$

For a measurable subset $B \subset [0,T]$, we set

$$\int_B u(t)\,\mathrm{d}t := \int_0^T u(t)\,\mathbb{1}_B(t)\,\mathrm{d}t.$$

A function $u\colon [0,T] \to X$ is *(essentially) separable valued* if it (up to a null set $N \subset [0,T]$) only takes values in a separable subset of X.

A function $u\colon [0,T] \to X$ is BOCHNER measurable if and only if u is weakly BOCHNER measurable and *essentially separable valued.*

Corollary: If X is *separable*, weak and strong BOCHNER measurability coincide.

Subsets of separable spaces are separable.

Let $u\colon [0,T] \to X$ be B-measurable. Then u is B-integrable if and only if $t \mapsto \|u(t)\|$ is *L-integrable.*

Let $u\colon [0,T] \to X$ be B-integrable. For all measurable subsets $B \subset [0,T]$ and for all $f \in X^*$ we have $\left\|\int_B u(t)\,\mathrm{d}t\right\| \leqslant \int_B \|u(t)\|\,\mathrm{d}t$ and $\left\langle f, \int_B u(t)\,\mathrm{d}t \right\rangle = \int_B \langle f, u(t)\rangle\,\mathrm{d}t$.

Let $(Y, \|\cdot\|_Y)$ be a BANACH space, $A \in L(X,Y)$ a linear bounded operator and $u\colon [0,T] \to X$ *B-integrable.* Then $Au\colon [0,T] \to Y$ is *B-integrable* with $\int_0^T (Au)(t)\,\mathrm{d}t = A\left(\int_0^T u(t)\,\mathrm{d}t\right)$.

The limit is well defined as each u_n and u are BOCHNER measurable and hence the function $\|u_n - u\|$ is LEBESGUE measurable by a Lemma.
For $n, m \geqslant M_\varepsilon$ we have, as $u_n - u_m$ is again a simple function and the triangle equality is an equality for simple functions, $\left\|\int_0^T u_n(t)\,\mathrm{d}t - \int_0^T u_m(t)\,\mathrm{d}t\right\|_X = \int_0^T \|u_n(t) - u_m(t)\|\,\mathrm{d}t \overset{\triangle\neq}{\leqslant} \int_0^T \|u_n(t) - u(t)\| + \|u(t) - u_m(t)\|\,\mathrm{d}t \leqslant 2\varepsilon$. Hence $\left(\int_0^T u_n(t)\,\mathrm{d}t\right)_{n\in\mathbb{N}}$ is a CAUCHY *sequence* in X and thus converges for $n \to \infty$ as X is a BANACH space.
The integral is independent of the approximating sequence of simple functions, as $\int_0^T \|u_n(t) - u(t)\|\,\mathrm{d}t \to 0$ holds for all such sequences and thus the procedure in the previous remark can be done with any such sequence.

Let X be *reflexive* and $u\colon [0,T] \to X$ be absolutely continuous. Then u is classically differentiable in $(0,T)$, u' is BOCHNER integrable and

$$u(t) = u(t_0) + \int_{t_0}^t u'(s)\,\mathrm{d}s$$

for all $t, t_0 \in [0,T]$.

Let $u\colon \mathbb{R} \to X$ be *B-integrable*, $u|_{\mathbb{R}\setminus[0,T]} \equiv 0$ Then a.e. in $[0,T]$ 1. $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} u(s)\,\mathrm{d}s = u(t)$, and 2. $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \|u(s) - u(t)\|\,\mathrm{d}s = 0$,

1 follows from 2: by the triangle inequality we have $\left\|\frac{1}{h}\int_t^{t+h} u(s)\,\mathrm{d}s - u(t)\right\| = \left\|\frac{1}{h}\int_t^{t+h} u(s) - u(t)\,\mathrm{d}s\right\| \leqslant \frac{1}{h}\int_t^{t+h} \|u(s) - u(t)\|\,\mathrm{d}s$.
2: We can't guarantee the measurability of $\|u(\cdot) - u(t)\|$, so we have to use an approximation step. By PETTIS' Theorem, u is *essentially separable valued.* Hence for almost all $t \in [0,T]$ there exists a sequence $(x_n^{(t)})_{n\in\mathbb{N}} \subset X$ converging to $u(t)$.
By the triangle inequality, we have $\frac{1}{h}\int_t^{t+h} \|u(s) - u(t)\|\,\mathrm{d}s \leqslant \frac{1}{h}\int_t^{t+h} \|u(s) - x_n^{(t)}\|\,\mathrm{d}s + \|u(t) - x_n^{(t)}\|$. Taking $n \to \infty$ and then $h \to 0$ guarantees the measurability $s \mapsto \|u(s) - x_n^{(t)}\|$.

Definition

Bochner space $L^p((0,T);X)$

Theorem w/o proof

Properties of the Bochner spaces

Definition & Remark

Weak time derivative

Theorem w/o proof

Characterisation of weak derivatives

Definition, Theorem w/o proof

$W^{1,1}((0,T);X)$ and absolutely continuous functions

Definition & Remark

Gelfand triple

Definition

The spaces $W(0,T)$ and $W_p(0,T)$

Theorem w/o proof

Properties of $W(0,T)$

Remark

Assumptions on the form a

Remark

Obtaining the operator A from the form a

For $p \in [1,\infty]$, $L^p((0,T);X)$ is a BANACH *space*. For $p \in [1,\infty)$, the *simple functions and* $\mathcal{C}([0,T];X)$ *are both dense* in $L^p((0,T);X)$ and $L^p((0,T);X)$ is *separable* if X is, too.
Let $u \in L^p((0,T);X)$ and $v \in L^q((0,T);X^*)$ where $p,q \in [1,\infty]$ are HÖLDER conjugates. Then $\langle v(\cdot),u(\cdot)\rangle \in L^1((0,T))$ and the HÖLDER *inequality* holds: $\left|\int_0^T \langle v(t),u(t)\rangle \,\mathrm{d}t\right| \leqslant \|v\|_{L^q((0,T);X^*)}\|u\|_{L^p((0,T);X)}$.
For $p \in (1,\infty)$, $L^p((0,T);X)$ is *reflexive* if X is, too. If X *is reflexive or* X^* *is separable*, then $(L^p((0,T);X))^* \cong L^q((0,T);X^*)$ via the *dual pairing* $\langle v,u\rangle_{(L^p((0,T);X))^* \times L^p((0,T);X)} := \int_0^T \langle v(t),u(t)\rangle_{X^*\times X}\,\mathrm{d}t$.
If $X = H$ is a HILBERT *space*, then $L^2((0,T);H)$ is HILBERT *space* with the *inner product* $\langle u,v\rangle_{L^2((0,T);H)} := \int_0^T \langle u(t),v(t)\rangle_H\,\mathrm{d}t$.
If $X \hookrightarrow Y$ are BANACH spaces, then $L^p((0,T);X) \hookrightarrow L^q((0,T);Y)$ *for all* $1 \leqslant q \leqslant p \leqslant \infty$.

For $p \in [1,\infty)$, $L^p((0,T);X)$ is the linear space of *equivalence classes* of BOCHNER measurable functions $u\colon [0,T] \to X$ with

$$\|u\|_{L^p((0,T);X)} := \left(\int_0^T \|u(t)\|^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \infty$$

and $L^\infty((0,T);X)$ is the linear space of equivalence classes of *bounded* BOCHNER measurable functions $u\colon [0,T] \to X$ with

$$\|u\|_{L^\infty((0,T);X)} := \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\| < \infty.$$

Let $u,v \in L^1_{\mathrm{loc}}((0,T);X)$. Then the following are equivalent

1. v is a *weak derivative* of u
2. there exists a $u_0 \in X$ such that
$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s$$
almost everywhere in $(0,T)$.
3. for all $f \in X^*$ the function $t \mapsto \langle f,u(t)\rangle$ has the weak derivative $t \mapsto \langle f,v(t)\rangle$.

Let $u,v \in L^1_{\mathrm{loc}}((0,T);X)$. Then v is the **weak time derivative** of u if

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = -\int_0^T v(t)\varphi(t)\,\mathrm{d}t \qquad \forall \varphi \in \mathcal{C}_0^\infty((0,T);\mathbb{R}).$$

or (dual characterisation) if for all $f \in X^*$

$$\left\langle f, \frac{u(t+h)-u(t)}{h} - v\right\rangle \xrightarrow{h\to 0} 0.$$

The FTOCOV and its corollary hold.

Let $(V,\|\cdot\|)$ be a *real reflexive separable* BANACH space, $(H,|\cdot|)$ a *real* HILBERT space and $V \overset{\mathrm{d}}{\hookrightarrow} H$. We identify $H \cong H^*$. Since V is reflexive, we get $H^* \overset{\mathrm{d}}{\hookrightarrow} V^*$. We call $V \subset H \subset V^*$ a **Gelfand triple**.
The space H is called *pivot* space.
The norm on V will be denoted by $\|\cdot\|$, the norm on H will be $|\cdot|$ and the norm on V^* will be $\|\cdot\|_*$. The dual pairing will be $\langle\cdot,\cdot\rangle_{V^*\times V}$ and the scalar product on H is $(\cdot,\cdot)$ such that we have $\langle g,v\rangle = (g,v)$ *for all* $g \in H$ *and* $v \in V$.

The space

$$W^{1,1}((0,T);X) := \{u \in L^1((0,T);X) : \exists u' \in L^1((0,T);X)\}$$

equipped with $\|u\|_{1,1} := \|u\|_1 + \|u'\|_1$ is a BANACH space. For every function $u \in W^{1,1}((0,T);X)$ we can find an absolutely continuous function, which is almost equal to u, that is, $W^{1,1}((0,T);X) \hookrightarrow \mathrm{AC}([0,T];X) \hookrightarrow \mathcal{C}([0,T];X)$.

The space $(W(0,T);\|\cdot\|_{W(0,T)})$ is a BANACH space.
$\mathcal{C}^\infty([0,T];V) \subset W(0,T)$ is *dense*.
We have $W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$.
The *integration-by-parts* formula holds: for $u,v \in W(0,T)$ and $0 \leqslant s \leqslant t \leqslant T$

$$\int_s^t \langle u'(\tau),v(\tau)\rangle + \langle v'(\tau),u(\tau)\rangle\,\mathrm{d}\tau = (u(t),v(t)) - (u(s),v(s)).$$

For $u \in W(0,T)$ we have $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 = \langle u'(t),u(t)\rangle$ a. e. in $(0,T)$.

Let $V \subset H \subset V^*$ be a GELFAND triple. We define

$$W(0,T) := \{u \in L^2((0,T);V) : \exists u' \in L^2((0,T);V^*)\}$$

and endow it with the norm

$$\|u\|_{W(0,T)} := \left(\|u\|^2_{L^2((0,T);V)} + \|u'\|^2_{L^2((0,T);V^*)}\right)^{\frac{1}{2}}$$

and analogously for $p \in [1,\infty]$ and $q := \frac{p}{p-1} \in [1,\infty]$

$$W_p(0,T) := \{v \in L^p((0,T);V) : \exists v' \in L^q((0,T);V^*)\}.$$

For all $t \in [0,T]$ and all $v \in V$, the map $a(t,v,\cdot)\colon V \to \mathbb{R}$ is *linear* and *bounded*. We define $A(t)v := a(t,v,\cdot) \in V^*$ which fulfils $\|A(t)v\|_* \leqslant \beta\|v\|$.
For all $t \in [0,T]$, $A(t) \in L(V,V^*)$ with $\|A(t)\|_{L(V,V^*)} \leqslant \beta$.
Finally, define $A\colon [0,T] \to L(V,V^*)$, $t \mapsto A(t)$.
The GÅRDING inequality now becomes $\langle (A(t)+\kappa I)v,v\rangle_{V^*\times V} \geqslant \mu\|v\|^2$, where $I\colon V \to V^*$ is the *embedding* via $(\cdot,\cdot)$: $\langle Iv,v\rangle_{V^*\times V} = (v,v) = |v|^2$. Hence A *with a positive shift* is *strongly positive*.

$V \subset H \subset V^*$ GELFAND triple, $a\colon [0,T]\times V\times V \to \mathbb{R}$ s.t.

A1 $a(\cdot,v,w)$ is LEBESGUE *measurable* on $[0,T]$ $\forall v,w \in V$,

A2 $a(t,\cdot,\cdot)$ is *bilinear* for all $t \in [0,T]$,

A3 the form a is *uniformly bounded* with respect to the first input variable, that is, there exists a $\beta > 0$ such that $|a(t,v,w)| \leqslant \beta\|v\|\|w\|$ for all $t \in [0,T]$ and all $v,w \in V$.

A4 the form a fulfills the GÅRDING *inequality*, that is, there exists a $\mu > 0$ and a $\kappa \geqslant 0$ such that $a(t,v,v) \geqslant \mu\|v\|^2 - \kappa|v|^2$ for all $t \in [0,T]$ and all $v \in V$. (For $\kappa = 0$, $a(t,\cdot,\cdot)$ is *strongly positive* for all $t \in [0,T]$.)

The linear problem

Differential Equations III

Remark

Weak formulation of the linear problem

Differential Equations III

Theorem w/o proof

Lions

Differential Equations III

Lemma w/o proof

Uniqueness and stability

Differential Equations III

Proof of Lions' Theorem

Existence via time discretisation: 1. Setup

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Proof of Lions' Theorem

Existence via time discretisation: 2. Unique solvability of the discretised problem

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Proof of Lions' Theorem

Existence via time discretisation: 4. Constructing the approximate solution using the solutions of the time-discretised problems

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Proof of Lions' Theorem

Existence via time discretisation: 5. Translating a priori estimates from before to for the approximate solutions

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Proof of Lions' Theorem

Existence via time discretisation: 6. Extracting subsequences

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Proof of Lions' Theorem

Existence via time discretisation: 9. Passing to the limit

Differential Equations III

As $L^2((0,T);V^*) = (L^2((0,T);V))^*$, $u' + \mathcal{A}\,u = f$ is equivalent to

$$\int_0^T \langle u'(t), v(t)\rangle + \langle(\mathcal{A}\,u)(t), v(t)\rangle \,\mathrm{d}t = \int_0^T \langle f(t), v(t)\rangle \,\mathrm{d}t \;\forall v \in L^2((0,T);V).$$

Since $C_0^\infty(0,T)\otimes V$ is dense in $\mathcal{C}_0^\infty((0,T);V) \xhookrightarrow{\mathrm{d}} L^2((0,T);V)$ (Exercise!), we can *restrict the test functions* to $v(t) = \varphi(t)w$ with $\varphi \in \mathcal{C}_0^\infty(0,T)$ and $w \in V$. Hence $(\star)$ is equivalent to

$$\int_0^T (\langle u'(t), w\rangle + \langle(\mathcal{A}\,u)(t), w\rangle)\,\varphi(t)\,\mathrm{d}t = \int_0^T \langle f(t), w\rangle \varphi(t)\,\mathrm{d}t.$$

for all $\varphi \in \mathcal{C}_0^\infty(0,T)$, $w \in V$. The Fundamental Theorem now implies

$$\langle u'(t), w\rangle + a(t, u(t), w) = \langle f(t), w\rangle \,\forall w \in V \qquad \text{a.e. in } (0,T).$$

$$\begin{cases} \text{To } u_0 \in H \text{ and } f \in L^2((0,T),V^*) \text{ find } u \in W(0,T) \text{ with} \\ u(0) = u_0 \text{ and} \\ u' + \mathcal{A}\,u = f \text{ in } L^2((0,T);V^*). \end{cases} \qquad (\star)$$

Since $u \in W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$, the initial condition has to be understood to be attained in H.
For $u \in W(0,T)$ we find $u' \in L^2((0,T);V^*) \hookrightarrow L^1((0,T);V^*)$. Since $u \in W^{1,1}((0,T);X)$, u is an absolutely continuous function $u\colon [0,T] \to V^*$. Since V^* is reflexive, by KOMURA u is *classically differentiable almost everywhere*. Hence $(\star)$ is equivalent to $u'(t) + Au(t) = f(t)$ in V^* almost everywhere in $(0,T)$.

Under the assumptions (A1) - (A4) the *a priori* estimate

$$|w(t)|^2 + \mu \int_0^t \|w(s)\|^2 \,\mathrm{d}s \leqslant c\left(|w_0|^2 + \|g\|_{L^2((0,T);V^*)}\right)$$

holds for every solution $w \in W(0,T)$ of

$$\begin{cases} w' + Aw = g & \text{in } L^2((0,T);V^*), \\ w(0) = w_0 & \text{in } H. \end{cases}$$

Under the assumptions (A1) - (A4), the problem (P) is well-posed in the sense of HADAMARD, that is, a unique solution exists and we have continuous dependence on the right side f.

Generalisation by TARTAR/TEMAM: in the above theorem we can allow $f \in L^2((0,T);V^*) \oplus L^1((0,T);H)$, i.e. $f = f_1 + f_2$ with $f_1 \in L^1((0,T);H)$ and $f_2 \in L^2((0,T);V^*)$.

The approximate system is well defined. For every $n \in \{1,\dots,N\}$ consider the problem in V^*

$$\left(\frac{1}{\tau}I + A\right)u^n = f^{(n)} + \frac{1}{\tau}Iu^{n-1}$$

The operator $\frac{1}{\tau}I + A$ *is a linear, bounded and strongly positive operator*: for the last property observe $\left\langle \left(\frac{1}{\tau}I + A\right)v, v\right\rangle = \frac{1}{\tau}|v|^2 + \langle Av, v\rangle \geqslant \frac{1}{\tau}|v|^2 + \mu\|v\|^2 \geqslant \mu\|v\|^2$. For $\kappa > 0$, choose τ small enough, i.e. $\tau < \frac{1}{\kappa}$, then $\frac{1}{\tau}I + A$ is strongly positive.
We have $u_0 \in H$ and $u^{n-1} \in V$ and hence $u^{n-1} \in V^*$. By the *Theorem of* LAX-MILGRAM, there exists a *unique* u^n for every $n \in \{1,\dots,N\}$, that is, a solution to the problem in V^*.
In the following we identify $Iu^n \leftrightarrow u^n$ *and don't write the* I *anymore.*

Let $N \in \mathbb{N}$, $\tau := \frac{T}{N}$ be the step size and $t_n := n\tau$ be equidistant time step for $n \in \{1,\dots,N\}$. Then we consider the *implicit* EULER *scheme*: for $n \in \{1,\dots,N\}$ let $u^n := u(t_n)$, $u'(t_n) \approx \frac{u^n - u^{n-1}}{\tau}$ and for the right hand side use $f^n := \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t \in V^*$. We consider the problem

$$\begin{cases} \text{To } u^{n-1} \text{ find } u^n \in V \text{ such that} \\ \frac{Iu^n - Iu^{n-1}}{\tau} + A(t_n)u^n = f^n, \qquad n \in \{1,\dots,N\}. \end{cases}$$

In the following we only consider $\kappa = 0$ and assume that A *is independent of* t (time), otherwise we would have to set $A(t_n) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} a(t,\cdot,\cdot)\,\mathrm{d}t$.

Let $N_\ell \to \infty$ for $\ell \to \infty$ with $N_\ell \in \mathbb{N}$ and $\tau_\ell := \frac{T}{N_\ell}$ and $\{u_{\tau_\ell}\}_\ell$, $\{\hat{u}_{\tau_\ell}\}_\ell$ and $\{f_{\tau_\ell}\}_\ell$ be constructed as above. For $\ell \in \mathbb{N}$, we choose a sequence $\{u_\ell^0\}_\ell \subset H$ such that $u_\ell^0 \to u_0$ as $\ell \to \infty$. We have $u_{\tau_\ell}^0 \to u_0$ as $\ell \to \infty$ in H and $u_{\tau_\ell}(0) = u_\ell^0$. We want to show that $\{f_{\tau_\ell}\}_\ell$ converges to f in $L^2((0,T);V^*)$. We have $\|f_{\tau_\ell}\|^2_{L^2((0,T);V^*)} \leqslant \|f\|^2_{L^2((0,T);V^*}$. The a priori estimates are independent of ℓ and we may deduce

$$\|u_{\tau_\ell}\|^2_{L^\infty((0,T);H)} = \max_{i=1}^{N_\ell} |u^i|^2 \leqslant |u_\ell^0|^2 + \frac{\tau_\ell}{\mu}\sum_{i=1}^N \|f^i\|_*^2,$$

$$\|u_{\tau_\ell}\|^2_{L^2((0,T);V)} = \tau_\ell \sum_{i=1}^{N_\ell} \|u_{\tau_\ell}\|^2 \leqslant \mu\left(|u_\ell^0|^2 + \frac{\tau_\ell}{\mu}\sum_{i=1}^N \|f^i\|_*^2\right),$$

$$\|\widehat{u_{\tau_\ell}}\|^2_{L^\infty((0,T);H)} = \max_{i=1}^{N_\ell} |u^i|^2 \leqslant |u_\ell^0|^2 + \frac{\tau_\ell}{\mu}\sum_{i=1}^N \|f^i\|_*^2.$$

For $t \in (t_{n-1}, t_n]$ define $u_\tau(t) := u^n$ and $u_\tau(0) = u^0$. Hence u_τ is piecewise constant. Let $\hat{u}_\tau(t) := u^{n-1} + (t - t_{n-1})\frac{u^n - u^{n-1}}{\tau}$ and $f_\tau(t) := f^n$ for $t \in (t_{n-1}, t_n]$. As $\hat{u}_\tau$ is piecewise linear, it is LIPSCHITZ continuous and hence weakly differentiable almost everywhere with derivative $\hat{u}_\tau'(t) = \frac{u^n - u^{n-1}}{\tau}$ for $t \in (t_{n-1}, t_n]$. Hence we can interpret the implicit EULER scheme via these functions: we may write $\hat{u}_\tau'(t) + Au_\tau = f_\tau$.

We have $f_{\tau_\ell} \to f$ in $L^2((0,T);V^*)$ (Exercise). We observe that $A\colon L^2((0,T);V) \to L^2((0,T);V^*)$ is linear and continuous. Hence A is *weak-weak-continuous* and thus $Au_{\tau_\ell} \rightharpoonup Au$ in $L^2((0,T);V^*)$. We find in $L^2((0,T);V^*)$

$$\hat{u}_{\tau_\ell}' + Au_{\tau_\ell} = f_{\tau_\ell} \qquad \text{in } L^2((0,T);V^*).$$

The three terms above converge weakly to u', Au and f in $L^2((0,T);V^*)$, respectively. This implies that u is a solution to the abstract equation.

As $(u_{\tau_\ell})_\ell$ is bd in $L^\infty((0,T);H) \cap L^2((0,T);V)$ and $(\hat{u}_{\tau_\ell})_\ell$ is bd in $L^\infty((0,T);H)$, $\exists u \in L^\infty((0,T);H) \cap L^2((0,T);V)$ such that $u_{\tau_\ell} \overset{*}{\rightharpoonup} u$ in $L^\infty((0,T);H) \cap L^2((0,T);V)$ and there exists a $\hat{u} \in L^\infty((0,T);H)$ such that $\hat{u}_{\tau_\ell} \overset{*}{\rightharpoonup} \hat{u}$. One has to observe that one can do this steps one after the other to infer that this holds afterwards for *one* subsequence. Since $L^2((0,T);V)$ is *reflexive* and $L^\infty((0,T);H) \cong (L^1((0,T);H))^*$ is the *dual of a separable* BANACH *space*, this follows from the *Theorem of* BANACH ALAOGLU.

Proof of Lions' Theorem

Existence via time discretisation: 10. Identify the initial condition.

Differential Equations III

Theorem w/o proof

Error estimates

Differential Equations III

Theorem w/o proof

Additional regularity $u \in W^{1,\infty}((0,T);H) \cap W^{1,2}((0,T);V)$

Differential Equations III

Remark

Hypothesis for the nonlinear case

Differential Equations III

Lemma w/o proof

Properties of $W_p(0,T)$: completeness, IBP rule, embedding

Differential Equations III

Theorem w/o proof

Main Theorem on Monotone nonlinear PDEs

Differential Equations III

Lemma & Proof

Uniqueness in the nonlinear case

Differential Equations III

Lemma w/o proof

Continuity of solution operator if $B = 0$

Differential Equations III

Theorem w/o proof

Continuity of solution operator if $\mathcal{A}$ is "g-monotone"

Differential Equations III

Theorem w/o proof

Continuity of solution operator if $\mathcal{A}$ is p-monotone

Differential Equations III

Let $u \in W(0,T)$ be the solution of (P) with $f \in L^2((0,T);V^*)$ and let additionally $(f-u')' \in L^2((0,T);V^*)$. Then the *error estimate*

$$|u(t_n)-u^n|^2 - \mu\tau \sum_{j=1}^{n} \|u(t_j)-u^j\|^2 \leqslant |u_0-u^0|^2 + \frac{\tau^2}{3\mu}\|(f-u')'\|^2_{L^2((0,T);V^*)}$$

for all $n \in \{1,\dots,N\}$ holds for the implicit EULER time discretisation given in the previous proof.

We have to show that $u(0)=u_0$ in H. We have $\hat{u}_{\tau_\ell} \rightharpoonup u$ in $W(0,T) \hookrightarrow \mathcal{C}([0,T];H)$. The embedding is linear and continuous and hence *weak-weak-continuous*, so the weak convergence is translated to a pointwise weak convergence on H. There exists a linear continuous trace operator $\Gamma\colon W(0,T) \to H$, $\Gamma(u) := \Gamma_u := u(0)$ in H. Hence $\hat{u}_{\tau_\ell} \rightharpoonup u$ in $W(0,T)$ and so $\hat{u}_{\tau_\ell}(0) \rightharpoonup u(0)$ in H. We had the condition that $\hat{u}_{\tau_\ell}(0) = u^0_\ell \to u_0$ in H. Hence the weak and strong limits have to coincide.

Let $p>1$, $\frac{1}{p}+\frac{1}{p'}=1$ and $V \subset H \subset V^*$ a GELFAND triple. Let $A_0, B\colon V \to V^*$ and $A := A_0 + B$ with $(\mathcal{A}_0 v)(t) := A_0 v(t)$, $(\mathcal{B}v)(t) := Bv(t)$ for $v\colon [0,T] \to V$ and $\mathcal{A} := \mathcal{A}_0 + \mathcal{B}$ with $\mathcal{A}_0\colon L^p((0,T);V) \to L^{p'}((0,T);V^*)$ *monotone, hemi-continuous*, $\mathcal{B}\colon L^p((0,T);V) \to L^{p'}((0,T);V^*)$ *strongly continuous*, $\mathcal{A}\colon L^p((0,T);V) \to L^{p'}((0,T);V^*)$ *coercive* with $\mu>0$, $\lambda \geqslant 0$ s.t. $\langle \mathcal{A}v, v\rangle = \int_0^T \langle \mathcal{A}v(t), v(t)\rangle \mathrm{d}t \geqslant \mu\|v\|^p_{L^p((0,T);V)} - \lambda$ for all $v \in L^p((0,T);V)$ and *bounded* with $\beta \geqslant 0$ such that $\|\mathcal{A}v\|_{L^{p'}((0,T);V^*)} \leqslant \beta(1+\|v\|^{p-1}_{L^p((0,T);V)})$ for all $v \in L^p((0,T);V)$.

Let $u_0 \in V$ and $f \in W^{1,2}((0,T);V^*)$ and the operator $A'(t)\colon V \to V^*$ be continuous and linear for all $t \in (0,T)$. If the *compatibility condition* $A(u_0) - f(0) \in H$ holds, then the solution $u \in W(0,T)$ to (P) admits the *additional regularity* $u \in W^{1,\infty}((0,T);H) \cap W^{1,2}((0,T);V)$. Especially, it holds

$$\begin{aligned} \hat{u}_n' &\rightharpoonup u' &&\text{in } L^\infty((0,T);H),\\ \hat{u}_n' &\rightharpoonup u' &&\text{in } L^2((0,T);V). \end{aligned}$$

Assuming the standard assumptions, for every $u_0 \in H$ and $f \in L^{p'}((0,T),V^*)$, there exists a solution $u \in W_p(0,T)$ with

$$\begin{cases} u' + A(u) = f & \text{in } L^{p'}((0,T);V^*),\\ u(0) = u_0, & \text{in } H. \end{cases}$$

Let $p \in (1,\infty)$. Then $W_p(0,T)$ equipped with the norm $\|u\|_{W_p(0,T)} := \|u\|_{L^p((0,T);V)} + \|u'\|_{L^{p'}((0,T);V^*)}$ is a BANACH *space*. We have $W_p(0,T) \hookrightarrow \mathcal{C}([0,T];H)$ and the rule of *integration by parts*:

$$\int_s^t \langle u'(\tau), v(\tau)\rangle + \langle v'(\tau), u(\tau)\rangle \mathrm{d}\tau = (u(t),v(t)) - (u(s),v(s))$$

for all $v,w \in W_p(0,T)$ and all $s,t \in [0,T]$. Finally, $\mathcal{C}^\infty([0,T];V) \overset{\mathrm{d}}{\hookrightarrow} W_p(0,T)$.

Assuming the standard assumptions and $B=0$, the solution operator of the nonlinear problem

$$L^{p'}((0,T);V^*) \times H \to \mathcal{C}([0,T];H), \qquad (f,u_0) \mapsto u$$

is *continuous on bounded sets.*

Assuming the standard assumptions and $B=0$, the solution of the nonlinear problem is *unique* and the *whole sequence of approximate solutions converges* to u.

Let $u,v \in W_p(0,T)$ be two solutions to the problems

$$\begin{cases} u' + \mathcal{A}(u) = f, & \text{in } L^{p'}((0,T);V^*),\\ u(0) = u_0 & \text{in } H \end{cases}, \quad \begin{cases} v' + \mathcal{A}(v) = f, & \text{in } L^{p'}((0,T);V^*),\\ v(0) = u_0 & \text{in } H \end{cases}$$

As $\mathcal{A}$ is *monotone* we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u-v|^2 &= \langle u'-v', u-v\rangle\\ &\leqslant \langle u'-v', u-v\rangle + \langle \mathcal{A}u - \mathcal{A}v, u-v\rangle = \langle f-f, u-v\rangle = 0 \end{aligned}$$

and hence (by integration) $|u(t)-v(t)|^2 \leqslant |u_0-u_0|^2 = 0$ for all $t \in [0,T]$.

Let the standard assumptions be fulfilled with $A = A_0\colon V \to V^*$ being *p-monotone*, that is, there exists a $\tilde{\mu} > 0$ such that

$$\langle \mathcal{A}v - \mathcal{A}w, v-w\rangle \geqslant \tilde{\mu}\|v-w\|^p \qquad \forall v,w \in V.$$

Then the solution operator of the nonlinear problem

$$L^{p'}((0,T);V^*) \times H \to \mathcal{C}([0,T];H) \cap L^p((0,T);V),\ (f,u_0) \mapsto u$$

is *continuous.*

Let the standard assumptions be fulfilled. Additionally, we require that $A\colon [0,T] \times V \to V^*$ fulfills

$$\langle \mathcal{A}(t)v - \mathcal{A}(t)w, v-w\rangle \geqslant -g(t)|v-w|^2$$

for $v,w \in V$ and $g \in L^1(0,T)$. The operator $\mathcal{A}\colon L^p((0,T);V) \to L^{p'}((0,T);V^*)$ is then given by $(\mathcal{A}u)(t) = Au(t)$. Then the solution operator of the nonlinear problem

$$L^2((0,T);H) \times H \to \mathcal{C}([0,T];H), \qquad (f,u_0) \mapsto u$$

is LIPSCHITZ-*continuous.*

Theorem w/o proof

Solution of the heat equation

Differential Equations III

Definitions & Remarks

Solenoidal spaces

Differential Equations III

Remark

Navier-Stokes equation

Differential Equations III

Weak formulation of Navier Stokes

Differential Equations III

We consider a *bounded* Lipschitz *domain* $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ and the incompressible Navier-Stokes equation

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = f, & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = 0 & \text{in } \Omega, \end{cases}$$

where $u\colon \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ is the *velocity field*, $p\colon \overline{\Omega} \times [0,T] \to \mathbb{R}$ is the *pressure* and ν is the *viscosity*. The time derivative of the velocity is the acceleration, the second (dissipative) term $\nu\Delta\mathbf{u}$ described how friction behaves in the fluid.

Let $\Omega \subset \mathbb{R}^3$ be a *bounded* Lipschitz *domain*. Assume that $0 < \beta < \beta'(r) \leqslant \bar{\beta} < \infty$ and $b_1, b_2 > 0$. Assume additionally that $\hat{c}^{-1}\colon \overline{\mathbb{R}} \to \mathbb{R}$ is a monotone function with $|(\hat{c}^{-1})'| \leqslant C$. Then there exists a weak solution to

$$\begin{cases} \partial_t u + \nabla\beta(u) = g & \text{in } \Omega \times (0,T), \\ \mathbf{n} \cdot \nabla\beta(u) + (b_1 + b_2|\hat{c}^{-1}(u)|^3)\hat{c}^{-1}(u) = h & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

As test functions we take $\mathbb{V} := \left\{\varphi \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^d) \mid \nabla\cdot\varphi \equiv 0 \text{ in } \Omega\right\}$. Since this is to regular for our purposes, we will take the closure with respect to the H_1-norm. Now the spaces $V := \operatorname{clos}_{\|\cdot\|_{H_0^1}} \mathbb{V}$ and $H := \operatorname{clos}_{\|\cdot\|_{L^2}} \mathbb{V}$ form a Gelfand-triple (compact embedding follows from Rellich-Kondrachov). $V \overset{c}{\hookrightarrow} H \cong H^* \hookrightarrow V^*$, where V is equipped with $\|\cdot\| := \|\cdot\|_{H_0^1}$ and the scalar-product $((u,v)) := \int_\Omega \nabla u : \nabla v \,\mathrm{d}x$ and H with $|\cdot| := \|\cdot\|_{L^2}$ and the scalar-product $(u,v) := \int_\Omega u\cdot v\,\mathrm{d}x$. One can show the characterisations $V = \left\{u \in H_0^1(\Omega)^d \mid \nabla\cdot u \equiv 0 \text{ in } \Omega\right\}$ and $H = \left\{u \in L^2(\Omega)^d \mid \nabla\cdot u \equiv 0 \text{ in } \Omega,\ n\cdot u \equiv 0 \text{ on } \partial\Omega\right\}$, where the condition of zero divergence means $\int_\Omega u\cdot\nabla\varphi = 0$ for a.a. $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and the vanishing on the boundary is to be understood in the sense of a certain trace.