Technical University Berlin

Lecture Notes

Discrete and Computational Topology

read by Dr. Frank Lutz in the summer semester of 2022

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Thanks to Mino for finding and correcting various typos and mistakes.

These lecture notes are neither endorsed by the lecturer nor the university and make no claim to accuracy or correctness.

Last edited on November 16, 2022.

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19.04.2022

The aim and plan of this semester is the study of discrete / discretized topological objects and their algorithmic processing. Some objects might be given to us in a discrete way (e.g. through combinatorial problems), sometimes we have to discretise surfaces.

Example .0.1 (Discretizing surfaces)
One discretisation of the round sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$ (this is an algebraic, not a topological description) is the tetrahedron, which is a discretized surface. This discretisation yields a homeomorphism. Its inverse can be described using a radial projection.

Using this discretisation we can compute EULER characteristic of $\mathbb{S}^2$:

$$\chi(\mathbb{S}^2) = \#\text{vertices} - \#\text{edges} + \#\text{edges} = 4 - 6 + 4 = 2. \qquad \diamond$$

Analogously, the EULER characteristic of the cube is $8 - 12 + 6 = 2$.

Example .0.2 (Network design)
Given some sensor network (e.g. smoke detectors, tsunami buoys, or an intrusion detection system), how "good" is the network?

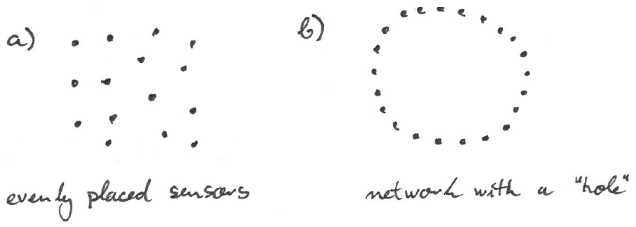


Fig. 1: Two very different network designs.

How can we detect "holes" in the network to place extra sensors? Visually, we can immediately detect the hole, but mathematically, this is not so easy. A mathematical procedure would be to place around each sensor a disk of fixed small radius. If the disks intersect, we connect the sensors to form a graph, from which we can extract more mathematical information. $\diamond$

This can be generalised: given some (finite) data set $S \subset \mathbb{R}^n$,

1. associate to S a topological space $K(S)$ (e.g. a simplicial or cubical complex),
2. compute (meaningful, computable) topological invariants $I(K(S))$ (e.g. algebraic ones like (the fundamental group or) (persistent) homology or other ones like the EULER characteristic) of $K(S)$,
3. use the results of 2 for the further analysis / understanding / interpretation of the S, e.g. via visualizations.

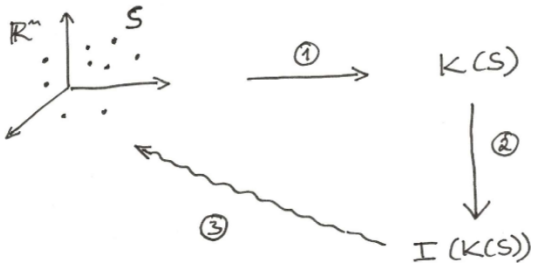


Fig. 2: For the third step experience is helpful.

Example .0.3 (Genome data for cancer research)
There are gene libraries available on the internet, the aim of the topological data scientist being to find correlations, e.g. "if there are defects at genes A and B, then there is an increased chance to develop a certain cancer". ◇

Example .0.4 (Image processing)
The data S can originate from a 2D or 3D scan, e.g. because a hand was placed on a scanner. Aiming to reduce noise, we first have to distinguish between features and noise. pics missing

One approach is to thicken contour lines by some $\varepsilon > 0$ so that small holes disappear. A more sophisticated method is to compute the k-dimensional homology groups for different choice of ε, along us to detect holes that go away quickly (noise) and holes that persist (features).

Persistent homology can also be applied e.g. in for sensor networks (cf. above), genome data, finance data or porosity of materials. ◇

Back to the discretisation of spaces / topological objects. The torus T^2 can be discretised as a CW complex (one square with pairwise identified edges), a Δ complex (consisting of simplices) or as a finite simplicial complex. pics missing

Can every topological object be decomposed (nicely) into finitely many pieces, that can be handles on a computer? No, this doesn't even hold for compact manifolds.

Theorem .0.1: Rado (1925) [Rad25], Moise (1952) [Moi52]

Every compact surface and every compact 3-manifold can be triangulated by a finite abstract simplicial complex.

Theorem .0.2: Freedman (1982) [Fre82], Perelman (2003) [Per03]

There are compact 4-manifolds that cannot be triangulated.

The following theorem is the disproof of the "triangulation conjecture".

Theorem .0.3: Manolescu (2013) [Man13]

For every $d \geqslant 5$, there are compact d-manifolds that cannot be triangulated.

Part I: Discrete Topology

The aim of this part is the study of discretizations of topological spaces.

Example I.0.1 (Common objects in topology)
Common objects in topology are the circle $\mathbb{S}^1$, the 2-dimensional sphere $\mathbb{S}^2$ ("2-sphere"), or, more generally, the d-sphere $\mathbb{S}^d$. Other common objects include the (2-)(dimensional) torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the d-torus

$$T^d := (\mathbb{S}^1)^{\times d} := \underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{d \text{ times}}$$

and orientable surfaces of higher genus (double, triple, ... torus) and non-orientable surfaces like $\mathbb{R}\mathrm{P}^2$, which is obtained by identifying antipodal points on $\mathbb{S}^2$, and the Klein bottle. Lastly, there are general compact spaces, the Euclidean plane $\mathbb{E}^2$ and higher-dimensional manifolds with or without boundary.

[TODO: lots of pictures] ◇

I.1 Metric and topological spaces

Recall from you calculus or topology class:

Definition I.1.1 (Metric space)
Let X be a set. A map $d\colon X \times X \to \mathbb{R}_{\geqslant 0}$ is a metric if

1. $d(x,y) = 0$ if and only if $x = y$, (positive definite)
2. $d(x,y) = d(y,x)$, (symmetry)
3. $d(x,z) \leqslant d(x,y) + d(y,z)$ (triangle inequality)

for all $x, y, z \in X$. The pair (X,d) is a metric space.

Example I.1.2 On $X = \mathbb{R}^n$ there is the Euclidean metric

$$d_E(x,y) := \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} = \sqrt{\langle x - y, x - y \rangle} = \|x - y\|_2$$

induced by the standard Euclidean norm $\|\cdot\|_2$. ◇

Definition I.1.3 (Euclidean space)
The metric space $\mathbb{E}^n := (\mathbb{R}^n, d_E)$ is the Euclidean n-space.

In the following, let (X,d) be a metric space.

Definition I.1.4 (Open disc / ball)
For $x \in X$ and $r \geqslant 0$, the open ball with center x and radius r is

$$\mathring{B}_r(x) := \{y \in X : d(x,y) < r\}.$$

Fig. 3: An open ball with radius r and center x and a $y \in \mathring{B}_r(x)$.

Definition I.1.5 (Open set)
A subset $O \subset X$ is open if for every $x \in O$ there is a $r > 0$ such that $\mathring{B}_r(x) \subset O$.

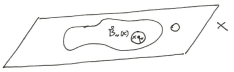


Fig. 4: An open set $O \subset X$ and $x \in O$ with $\mathring{B}_r(x) \subset O$.

Theorem I.1.1: Open ball is open

An open ball is open.

Example I.1.6 (Open sets in $\mathbb{E}^1$) The open sets in $\mathbb{E}^1$ are $\varnothing$, $\mathbb{E}^1$ and unions and finite intersections of open intervals. ◇

Theorem I.1.2: Properties of open sets

1. $\bigcup_{i \in I} O_i$ is open for any index set I if each O_i, $i \in I$, is open.
2. $\bigcap_{i \in I} O_i$ is open for any finite index set I if each O_i, $i \in I$, is open.
3. $\varnothing$, X are open.

Proof. (1) Left as an exercise.

(2) Let $O := \bigcap_{k=1}^{n} O_i$, where each $O_k \subset X$ is open, and $x \in O$. Then for every $k \in \{1, \dots, n\}$, there exist a $r_k > 0$ such that $\mathring{B}_{r_k}(x) \subset O_i$. Let $r := \min(r_1, \dots, r_n) > 0$. Then $\mathring{B}_r(x) \subset O$.

(3) Left as an exercise. □

Example I.1.7 The intersection of infinitely many open sets need not be open: for $x \in \mathbb{E}^1$,

$$\bigcap_{k=1}^{\infty} \left(x - \frac{1}{k}, x + \frac{1}{k} \right) = \{x\}$$

is not open. ◇

Next, we generalise metric spaces by only requiring the properties from theorem I.1.2.

Definition I.1.8 (Topology, topological space; open, closed set)
Let X be a set. A collection $\mathcal{O} \subset 2^X$ of subsets of X is a topology on X if

(1) $\varnothing, X \in \mathcal{O}$,

(2) $\bigcup_{i \in I} O_i \in \mathcal{O}$ for a index set I and $(O_i)_{i \in I} \subset \mathcal{O}$,

(3) $\bigcap_{i \in I} O_i \in \mathcal{O}$ for a finite index set I and $(O_i)_{i \in I} \subset \mathcal{O}$.

A subset $O \in \mathcal{O}$ is open and its complement $X \setminus \mathcal{O}$ is closed. The pair $(X, \mathcal{O})$ is a topological space.

Example I.1.9 (Topology) Let X be a set. Then $\mathcal{O} := \{\varnothing, X\}$ is the indiscrete topology on X and $\mathcal{O} := 2^X$ is the discrete topology on X. ◇

26.04.2022

Now we turn to the morphisms in the category of topological spaces: continuous maps. In Analysis (when considering metric spaces), one has the ε-δ and the sequential definition (which are equivalent), but here we use a different one not relying on a metric.

Definition I.1.10 (Continuous map)
Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map $f \colon X \to Y$ is continuous if $f^{-1}(O) \in \mathcal{O}_X$ for every $O \in \mathcal{O}_Y$, that is, if preimages of open sets are open.

Example I.1.11 The identity map $\mathrm{id} \colon (X, \{\varnothing, X\}) \to (X, 2^X)$ is not continuous, whereas $\mathrm{id} \colon (X, 2^X) \to (X, \{\varnothing, X\})$ is continuous. ◇

Remark I.1.12 (Notation) For a topological space $(X, \mathcal{O})$ we often simply write X if $\mathcal{O}$ is clear. ○

Definition I.1.13 (Homeomorphism)
Let X and Y be topological spaces. A bijective map $f \colon X \to Y$ is a homeomorphism if f and f^{-1} are continuous. Then X and Y are homeomorphic and we write $X \cong Y$.

Definition I.1.14 (Neighbourhood)
Let $(X, \mathcal{O})$ be a topological space and $x \in X$. A subset $N \subset X$ is a neighbourhood of x if there is an $O \in \mathcal{O}$ with $x \in O \subset N$.

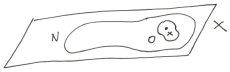
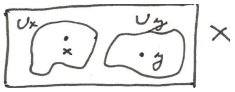


Fig. 5: A neighbourhood N of a point $x \in X$ and an open set $O \subset N$ containing x.

Being HAUSDORFF is a particular separation property.

DEFINITION I.1.15 (HAUSDORFF)
A topological space is HAUSDORFF if for any two points $x, y \in X$ there are open nonempty disjoint neighbourhoods U_x of x and U_y of y.

HAUSDORFF



DEFINITION I.1.16 (COMPACT, OPEN COVER, SUBCOVER)
A topological space $(X, \mathcal{O})$ is compact if every open cover of X has a finite subcover, that is, for every $(O_i)_{i\in I} \subset \mathcal{O}$ with $X = \bigcup_{i\in I} O_i$ there is a finite subfamily $J \subset I$ such that $X = \bigcup_{j\in J} O_j$.

DEFINITION I.1.17 (BASE)
Let B be a collection of open sets in a topological space $(X, \mathcal{O})$. If each open set in X is the union of open sets in B, then B is a base of $\mathcal{O}$ on X.

base

Example I.1.18 (Base for $\mathbb{E}^n$) The collection of open balls $\mathrm{B}_r(x)$ in $\mathbb{E}^n$ is a base for the EUCLIDEAN topology $\mathbb{E}^n$. ◇

DEFINITION I.1.19 (SECOND COUNTABLE)
A topological space is second countable if it has a countable base.

second countable

Example I.1.20 ((Not) second countable)
The line $\mathbb{E}^1$ is second countable since we can take as a base all open intervals with rational endpoints.

If we paste together countably many copies of the half open unit interval $(0, 1]$, we obtain a space still homeomorphic to a half line with countable base.

Glueing together uncountable many copies yields a space that is not second countable. ◇

DEFINITION I.1.21 (SUBSPACE TOPOLOGY)
Let $Y \subset (X, \mathcal{O})$ be a subset. Then

$$\mathcal{O}_Y := \{O \cap Y : O \in \mathcal{O}\}$$

is the subspace topology on Y induced by the topology on X.

subspace topology

I.2 Simplicial complexes

26.04.2022

Let $V := \{v_0, \dots, v_k\} \subset \mathbb{E}^n$ be a set of $k+1$ vertices.

Definition I.2.1 (Affine hull)
The affine hull of V is

$$\operatorname{aff}(V) := \left\{ \sum_{j=0}^{k} \lambda_j v_j : \sum_{j=0}^{k} \lambda_j = 1,\ (\lambda_j)_{j=0}^{k} \subset \mathbb{R} \right\}.$$

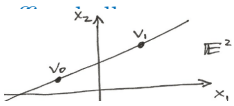


Fig. 6: The affine hull of two points $v_0, v_1 \in \mathbb{E}^2$ is the line in $\mathbb{E}^2$ through them and the affine hull of the three vertices of a triangle in the plane is the whole plane.

Remark I.2.2 The affine hull of V is an affine subspace of $\mathbb{E}^n$, that is, the solution set of a system of $\dim(\operatorname{aff}(V))$ not necessarily homogeneous linear equations with $k+1$ variables. ○

Definition I.2.3 (Dimension of affine hull)
If $k \geqslant 1$, then the dimension of the affine hull is

$$\dim(\operatorname{aff}(V)) := \dim(\operatorname{span}(\{v_1 - v_0, v_2 - v_0, \dots, v_k - v_0\})) \leqslant k.$$

The dimension of $\operatorname{aff}(\{v_0\}) = \{v_0\}$ is zero.

the dimension of $\operatorname{aff}(\varnothing) = \varnothing$ is -1?

Exercise: Prove the independence of the point v_0.

Definition I.2.4 (Affine independence)
The set V is affinely independent if $\operatorname{aff}(W) \subsetneq \operatorname{aff}(V)$ for any $W \subsetneq V$.

Definition I.2.5 (General position)
The set V is in general position if no $r \in \{2, \dots, n+1\}$ points lie in an $(r-2)$-dimensional affine subspace.

general position

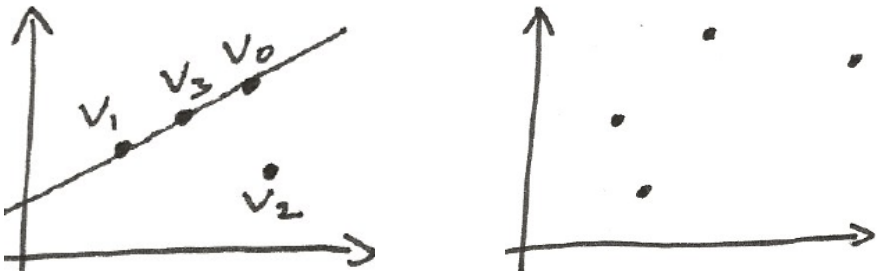


Fig. 7: The points on the left are not in general position, as the $r = 3 \leqslant 2+1$ points v_0, v_1 and v_3 lie in an $3-2 = 1$-dimensional affine subspace - a line - while the ones on the right are in general position, as no $r = 2$ resp. $r = 3$ points lie in an zero- resp. one-dimensional affine subspace.

Definition I.2.6 (Convex hull)
The convex hull of V is

convex hull

$$\operatorname{conv}(V) := \left\{ \sum_{j=0}^{k} \lambda_j v_j : \sum_{j=0}^{k} \lambda_j = 1,\ \lambda_j \geqslant 0\ \forall j \in \{0, \dots, k\} \right\}.$$

Remark I.2.7 (Since V is compact and $\mathbb{E}^n$ is finite-dimensional,) $\operatorname{conv}(V)$ is the compact solution set of a system of not necessarily homogeneous linear inequalities. ○

Example I.2.8 Consider a square as in figure 8, which is the convex hull of the points $(1,1)$, $(-1,1)$, $(1,-1)$ and $(-1,-1)$. It can also be described by the linear inequalities $(x_1, -x_1, x_2, -x_2) \leqslant 1$ (to be understood componentwise), which yield a dual description. ◇

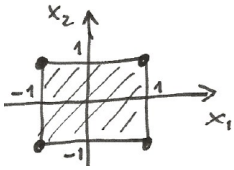


Fig. 8: The unit square can be interpreted as the convex hull of four points or as the solution of a system of four inequalities.

Remark I.2.9 (Since V is finite,) the set $\operatorname{conv}(V)$ is called a (convex) polytope. ○

Generally, one can not assign (consistently) a dimension to arbitrary topological spaces.

Definition I.2.10 (Dimension of $\operatorname{conv}(V)$)
Let $\varnothing \neq W \subset V$ be affinely independent. If W is of maximal cardinality with this property, then

$$\dim(\operatorname{conv}(V)) := \dim(\operatorname{aff}(W))$$

is the dimension of $\operatorname{conv}(V)$. Lastly, $\dim(\operatorname{conv}(\varnothing)) := -1$.

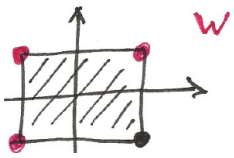


Fig. 9: The unit square and a maximal subset of its vertices, W, in general position.

Exercise: Prove the independence of this notion from the choice of W. Is setting the dimension of $\varnothing$ to be -1 consistent with the other part of the definition?

Example I.2.11 (Dimension of the square) Consider the subset W of the vertex set of the square in figure 9. Then W has maximal cardinality with respect to being a affinely independent vertex subset, so the cube $\operatorname{conv}(\{(\pm 1, \pm 1), (\pm 1, \mp 1)\})$ has dimension equal to

$$\dim\left(\operatorname{span}\left(\{(-1,1)-(-1,-1), (1,1)-(-1,-1)\}\right)\right) = \dim\left(\operatorname{span}\left((0,2),(2,2)\right)\right) = 2. \quad ◇$$

Definition I.2.12 (k-simplex)
If $V = \{v_0, \ldots, v_k\}$ is affinely independent and $k \leqslant n$, then $\operatorname{conv}(V)$ is a k-(dimensional) simplex.

doesn't the affine independence imply that $k \leqslant n$?

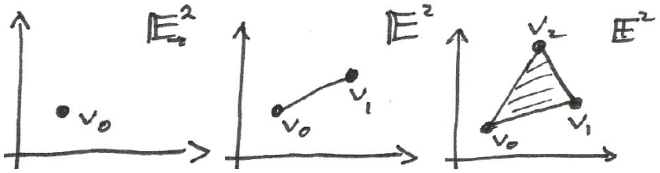


Fig. 10: Examples of k-simplices for $k \in \{0, 1, 2\}$. A 3-simplex is a tetrahedron.

Definition I.2.13 (Face)
Let $\sigma := \operatorname{conv}(V)$ be a simplex. For a subset $W \subset V$, $\tau := \operatorname{conv}(W)$ is a simplex, called a face of σ and we write $\tau < \sigma$.

face

Exercise. Prove that $<$ is reflexive, antisymmetric and transitive.

Example I.2.14 (Faces of a 2-simplex) For three affinely independent points v_0, v_1, v_2 consider the planar triangle (a 2-simplex) $\sigma := \operatorname{conv}(v_0, v_1, v_2) := \operatorname{conv}(\{v_0, v_1, v_2\})$. It has the faces

- σ,
- $\operatorname{conv}(v_0, v_1)$, $\operatorname{conv}(v_0, v_2)$, $\operatorname{conv}(v_1, v_2)$ (the edges),

- $\operatorname{conv}(v_0)$, $\operatorname{conv}(v_1)$, $\operatorname{conv}(v_2)$ (the vertices),
- $\operatorname{conv}(\varnothing) = \varnothing$. ◇

Definition I.2.15 (Proper face)
A face $\tau < \sigma$ is proper if $\dim(\tau) < \dim(\sigma)$.

Beware that in some texts, $\varnothing$ is not considered to be a face (due to reasons involving reduced homology).

Definition I.2.16 (Standard $(n-1)$-simplex)
Then standard $(n-1)$-simplex in $\mathbb{E}^n$ is $\Delta_{n-1} := \operatorname{conv}(e_1, \dots, e_n)$, where $e_k \in \mathbb{E}^n$ is the k-th unit vector.

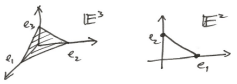


Fig. 11: The standard k-simplex for $k \in \{1, 2\}$.

Using Δ_n instead of an arbitrary n-simplex can make constructions simpler.

I.2.1 Geometric simplicial complexes

Definition I.2.17 ((Finite) geometric simplicial complex)
A (finite) geometric simplicial complex (GSC) $\mathscr{K}$ is a (finite) collection of (geometric) simplices in some $\mathbb{E}^n$ such that

1. if $\sigma \in \mathscr{K}$ and $\tau < \sigma$, then $\tau \in \mathscr{K}$,
2. if $\sigma, \tau \in \mathscr{K}$, then $(\sigma \cap \tau) < \tau$ and $(\sigma \cap \tau) < \sigma$.

geometric simplicial complex

We will explain geometric simplices later. For now it suffices to know that k-simplices are geometric simplices for any $k \in \mathbb{N}_0$.

when?

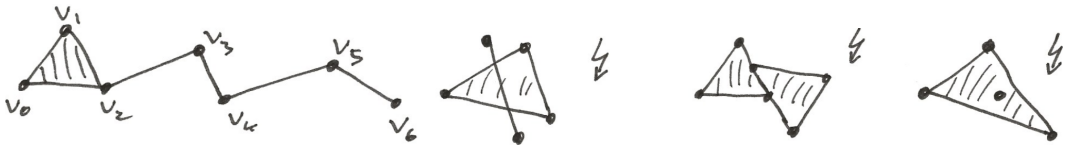


Fig. 12: The object of the left is a GSC, whereas all others are not, the second violating the first condition(???) and the two on the right both violating the second condition. For the rightmost one: intersecting the single vertex an the triangle yields that vertex, which is not a face of the triangle. The one left to it is similar, as there are line segments on the boundary of a triangle that are not faces of that triangle.

We want to make a (geometric) simplicial complex into a topological space.

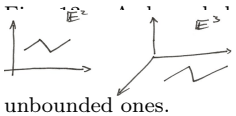
Definition I.2.18 (Polyhedron)
The polyhedron of a geometric simplicial complex $\mathscr{K}$ is

$$|\mathscr{K}| := \bigcup_{\sigma \in \mathscr{K}} \sigma.$$

polyhedron

Remark I.2.19 (Other definitions of polyhedra in other fields)
In polytope theory, polyhedra are finite intersections of half-spaces and are convex. ○

28.04.2022



unbounded ones.
Fig. 14: Uniqueness of topology on polyhedra.

Definition I.2.20 (Topology on polyhedra)
We equip a polyhedron $|\mathscr{K}| \subset \mathbb{E}^n$ (as a set) with the subspace topology inherited from $\mathbb{E}^n$.

Remark I.2.21 For a finite GSC, the inherited topology is "unique", that is, independent of the dimension of the ambient Euclidean space. ○

Remark I.2.22 The topology on $|\mathscr{K}|$ is induced by the base of open balls in $\mathbb{E}^n$. ○

Why not for infinite ones, too?

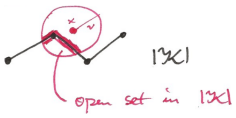


Fig. 15: An open set in the polyhedron is its intersection with an open ball in the ambient Euclidean space.

I.2.2 Abstract simplicial complexes

To store a GSC on a computer, we can store the coordinates of the vertex set V and the list (or: collection) of the (maximal) faces, cf remark I.2.47.

In a way, abstract simplicial complexes are GSCs where we forgot the coordinates.

Definition I.2.23 (Abstract simplicial complex, simplex, face)
An (finite) abstract simplicial complex (ASC for short) $K \subset 2^V$ on a (finite) vertex set V is such that if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$. The elements $\sigma \in K$ (with finite cardinality) are called simplices or faces of K. The dimension of $\sigma \in K$ is $\#\sigma - 1$ and the dimension $d \in \mathbb{N} \cup \{+\infty\}$ of K is the maximum dimension of its facets (we then say K is a d-complex).

abstract simplicial complex

Definition I.2.24 (Subcomplex)
Let L and K be abstract simplicial complexes with $L \subset K$. Then L is a subcomplex of K.

subcomplex

Remark I.2.25 (ASC underlying GSC) In this set-theoretic / combinatorial setting, the second condition for GSCs is always fulfilled, as the intersection of two finite sets is always a subset of both of them. Hence if we have a GSC $\mathscr{K}$, then the set of its faces (replacing actual coordinates by abstract vertex names) form an ASC K, its underlying ASC. ○

Remark I.2.26 We write $V(K)$ for the vertex set of K. We often choose the vertex set $V := \{1, \dots, k\}$. ○

Example I.2.27 (Abstract simplicial complex) Consider the ASC

$$K := \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$$ ◇

on the vertex set $V := \{1, \dots, 4\}$. How can we realise (cf. later) K in $\mathbb{R}^2$?

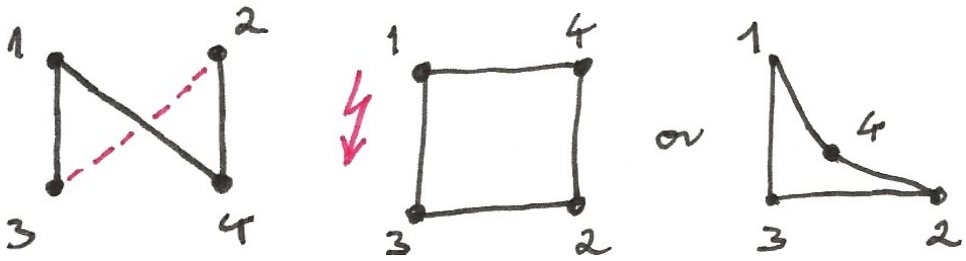
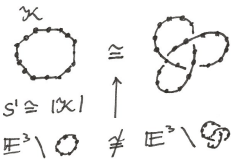


Fig. 16: The realisation on the left is bad because of the intersection of $\{2,3\}$ and $\{1,4\}$, while the other two realisations are without intersection.

Example I.2.28 (Knots) The subdivided unknot (the circle) and the subdivided trefoil knot (*dt.: Kleeblattschlinge*) are isomorphic as simplicial complexes (provided the number of subdivision points is equal), homeomorphic as polyhedra, but their complements in $\mathbb{E}^3$ are not homeomorphic. ◇



What does the knot example tell us?

Definition I.2.29 (Simplicial map)
Let K and L be (abstract) simplicial complexes with vertex sets $V(K)$ and $V(L)$. A map

$$\varphi\colon V(K) \to V(L)$$

is a simplicial map if for all $\sigma \in K$ we have $\tilde{\varphi}(\sigma) \in L$, where $\tilde{\varphi}(\sigma) := \{\varphi(v_{i_1}), \ldots, \varphi(v_{i_j})\}$ for $\sigma = \{v_{i_1}, \ldots, v_{i_j}\}$ (or $\sigma = \operatorname{conv}\{v_{i_1}, \ldots, v_{i_j}\}$ in the case of GSCs).

simplicial map

Remark I.2.30 Simplicial maps preserve simplices. The map $\tilde{\varphi}$ on the faces of K is induced by the vertex map φ. ○

Example I.2.31 Consider the two (geometric realisations of abstract) simplicial complexes in figure 17. The map $V(K) \to V(L)$, $v_k \mapsto v_k$, $k \in \{1,2,3\}$ is simplicial, while $V(L) \to V(K)$, $v_k \mapsto v_k$, $k \in \{1,3\}$ is not simplicial, as the edge $\{v_1, v_3\}$ of L is not mapped to a simplex in K.

Fig. 17: Geometric realisation of K and L.

Definition I.2.32 (Combinatorial / simplicial isomorphism)
Two simplicial complexes K and L are combinatorially isomorphic if there is a bijective simplicial map $\varphi\colon V(K) \to V(L)$ such that φ^{-1} is also simplicial.

combinatorially isomorphic

Counterexample I.2.33 (φ simplicial bijection, but φ^{-1} not simplicial)
Consider the simplicial complexes K and L in figure 18 Then $\varphi\colon V(K) \to V(L)$, $v_k \mapsto v_k$ is a bijective simplicial map, but φ^{-1} is not simplicial. ◇

Fig. 18: Geometric realisation of K and L.

Remark I.2.34 (Graphs as simplicial complexes) Combinatorial isomorphisms are generalisations of graph isomorphisms (cf. e.g. CoMa), since graphs can be thought of as one-dimensional simplicial complexes. Graph isomorphy is hard to test theoretically, but works pretty fast in practice. Hard instances are regular graphs, while easy ones are trees. ○

Geometric realisations

Definition I.2.35 (Geometric realisation of an ASC)
An ASC K has a GSC $\mathscr{K}$ as a geometric realisation, if K and the underlying ASC of $\mathscr{K}$ are combinatorially isomorphic.

As the next lemma shows, every ASC can be embedded (that is, realised geometrically) in a very high-dimensional space. One the other hand, deciding the lowest possible dimension it can embedded in is an NP-hard problem.

Lemma I.2.36
Every finite ASC has a realisation as a GSC.

This is also possible for ASCs with infinite vertex set, but more complicated.

Proof. Let K be an ASC with $V(K) = \{v_1, \ldots, v_k\}$. Define $\mathscr{K}$ to be the GSC in $\mathbb{E}^k$ (with vertices $e_1, \ldots, e_k$) such that $\mathscr{K}$ is a subcomplex of Δ_{k-1}, where $\operatorname{conv}(e_{i_1}, \ldots, e_{i_j})$ is a face of $\mathscr{K}$ whenever $\{v_{i_1}, \ldots, v_{i_j}\}$ is a face of K. □

I.2.3 | Triangulations of spaces

Definition I.2.37 (Polyhedron of an ASC)
The polyhedron of an ASC K is

$$|K| := |\mathscr{K}| = |\varphi(K)|$$

for a combinatorial isomorphism $\varphi\colon K \to \mathscr{K} \subset \Delta_{k-1}$.

polyhedron

Exercise: Prove that the polyhedron is independent of $\mathscr{K}$ resp. φ.

Definition I.2.38 (Triangulation)
An ASC K is a triangulation of a topological space X or: K triangulates X, if $|K| \cong X$.

triangulation

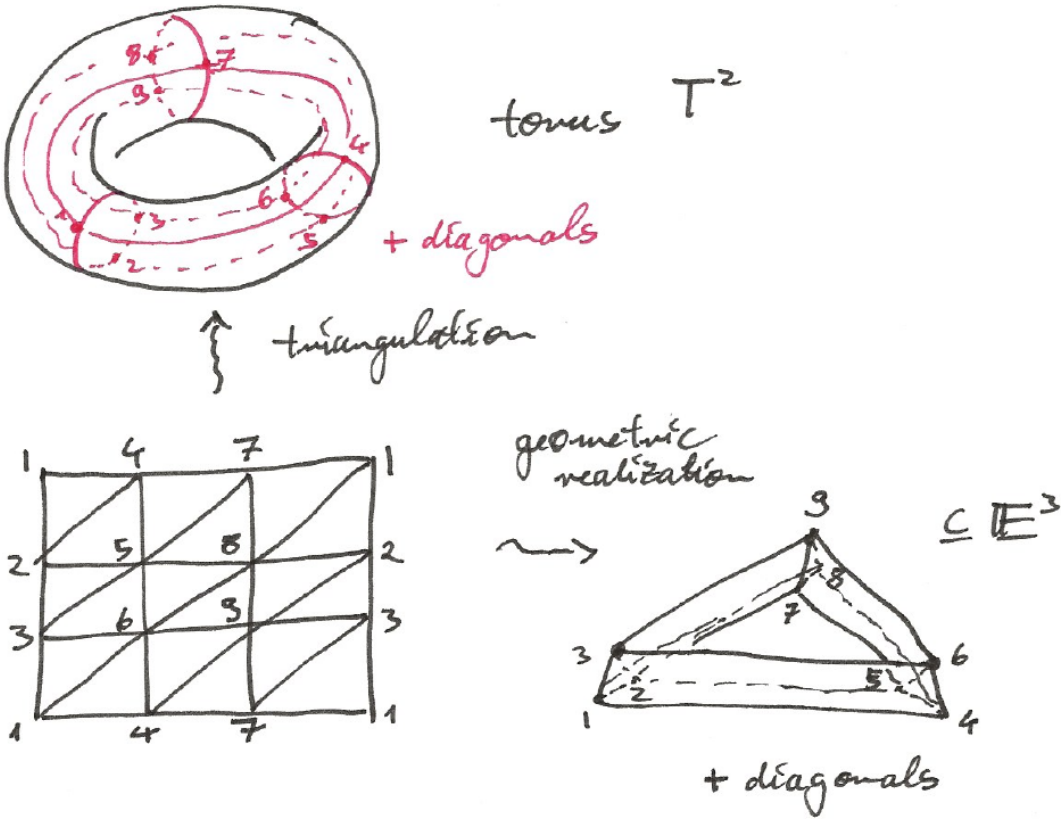


Fig. 19: The torus T^2 as a topological space, an ASC triangulating it and one of its geometric realisation in $\mathbb{E}^3$.

Infinite simplicial complexes

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Remark I.2.39 Not every topological space can be triangulated by a (finite) ASC, for example there are compact d-manifolds for any $d \geqslant 4$ that cannot be triangulated by theorem .0.2 and theorem .0.3. ○

We can generalise finite ASCs in order to triangulate non-compact spaces. For example, the plane $\mathbb{E}^2$ can be triangulated like in figure 20 using a finite description.

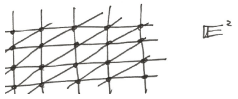


Fig. 20: A section of an infinite triangulation (more specifically: a tiling or tessellation) of $\mathbb{E}^2$.

Definition I.2.40 (Finite-dimensional, locally finite, infinite SC)
An infinite ASC K is

- of finite dimension if the dimension of the simplices of K is bounded.
- locally finite if every vertex of K is contained in finitely many simplices.

Counterexample I.2.41 (Infinite simplicial complexes)
Figure 20 shows a realisation of a locally finite, infinite two-dimensional simplicial complex of finite dimension. On the other hand, glueing a 2-simplex to a 1-simplex, then a 3-simplex to that 2-simplex at a different vertex and so on, we get an infinite, locally finite simplicial complex of unbounded dimension. If we instead use the same vertex, the complex is not even locally finite anymore. ◇

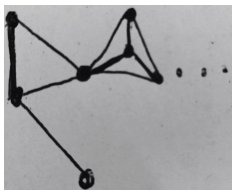


Fig. 21: An infinite, locally finite simplicial complex of unbounded dimension.

Remark I.2.42 All infinite simplicial complexes we consider are locally finite and of finite dimension. ○

Facet description of simplicial complexes

Definition I.2.43 (Facet)
A face σ of a geometric or abstract simplicial complex K is a facet of K if it is the unique face of the complex it is contained in.

facet

Fig. 22: This simplicial complex has the facets $\{1,2,3\}$, $\{3,4\}$, $\{4,5\}$ and $\{5,6\}$.

Remark I.2.44 Facets are faces that are maximal with respect to inclusion. ○

Remark I.2.45 As we can see in figure 22, facets can be of different dimension. ○

Definition I.2.46 (Pure simplicial complex)
A geometric or abstract simplicial complex K is pure if the facets of K are all of the same dimension.

pure

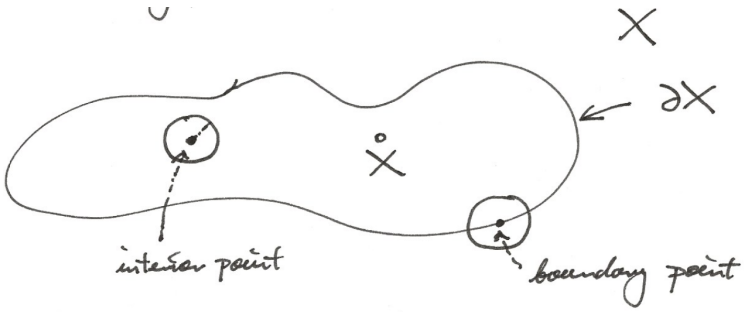
Remark I.2.47 (Facet description) The list of facets of a geometric or abstract simplicial complex contains all the information about it. Thus its facet description is a condensed format useful for storing it on a computer. ○

I.3 Realisability

Definition I.3.1 (Interior, boundary)
Let $X \subset \mathbb{E}^n$ be a subset.

- A point $x \in X$ is an interior point of X if there is an open ball $\mathring{B}_r(x)$ centered at x that is fully contained in X. interior point
- A point $x \in X$ (or $\overline{X}$) is a boundary point of X if $x \notin \mathring{X}$, where $\overline{X}$ is the closure of X. boundary point

The interior of X, $\mathring{X}$, is the set of all interior points of X and its boundary, ∂X, is the set of all boundary points of X. interior



Definition I.3.2 (Hyperplane)
A hyperplane is an $(n-1)$-dimensional affine subspace of $\mathbb{E}^n$.

Definition I.3.3 (Supporting hyperplane)
Let $X \subset \mathbb{E}^n$. A hyperplane H is supporting X if H bounds a half-space of $\mathbb{E}^n$ that contains X.

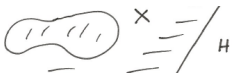


Fig. 23: A supporting hyperplane H for a set X.

Definition I.3.4 (Face, facet, boundary complex of a polytope)
An n-polytope $P \subset \mathbb{E}^n$ is the convex hull of finitely many points. polytope

- A face of P is the intersection of P with a supporting hyperplane H.
- A facet of P is an $(n-1)$-dimensional face.
- The boundary complex ∂P is the collection/union (this is different, but it is often clear from context which one is meant) of faces of P. boundary complex

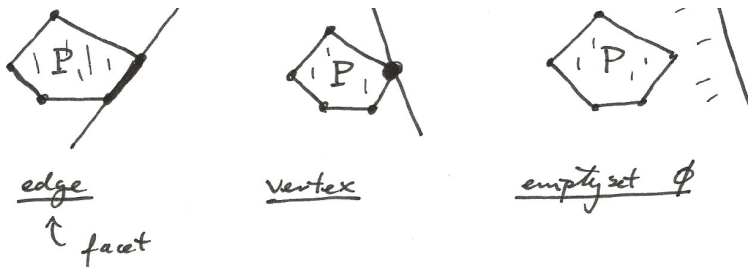


Fig. 24: An edge, a vertex and the empty face can all be obtained as intersection of the planar polytope P with a supporting hyperplane H.

Definition I.3.5 (Simplicial / cubical polytope)
A polytope is

- simplicial if all of its faces are k-simplices for varying $k \in \mathcal{N}$. simplicial
- cubical if all of its faces are cubes (of varying dimension), which are simplicial complexes, that are combinatorially isomorphic to the standard cube (having as vertices the vectors with entries from 0 and 1). cubical

Example I.3.6 (Simplicial and cubical polytopes)
Polygons (that is, 2-polytopes) are both simplicial and cubical be cause its faces are $\varnothing$, its vertices and its edges, which are k-simplices and k-cubes for $k \in \{0, 1\}$, respectively. A tetrahedron is simplicial and not cubical, because its triangular faces are 2-simplices, but not k-cubes for any $k \in \mathcal{N}$. ◇

Definition I.3.7 (Schlegel diagram)
A Schlegel diagram of a polytope P is the projection of P onto one of its facets through a point that lies just outside the facet. Schlegel diagram

Remark I.3.8 The expression "just outside of the facet" above can be interpreted as: choose as the projection point any point outside of the polytope but still inside the intersection of all halfspaces that arise from the supporting hyperplanes of all facets bordering the facet that you project onto. ○

Example I.3.9 (Schlegel diagram)
The Schlegel diagram of a two-polytope is a subdivided line segment.

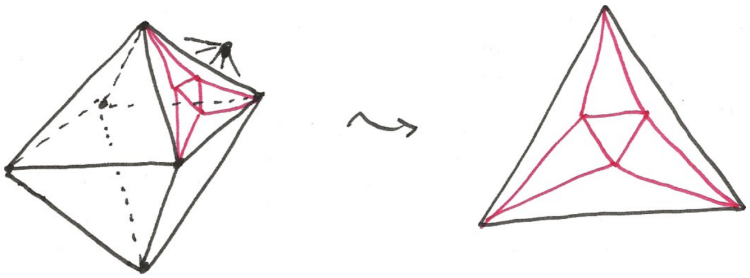


Fig. 26: The Schlegel diagram of an octahedron, which is a simplicial 3-polytope.

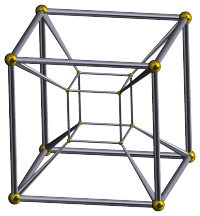


Fig. 25: The Schlegel diagram of a four-dimensional cube. One facet of such a cube is a three-dimensional cube (the larger one) and the smaller cube is the opposite facet.

Remark I.3.10 Let P be a simplicial 3-polytope, then

- ∂P is a simplicial complex triangulating $\mathbb{S}^2$,
- any Schlegel diagram of P along with the projection facet is combinatorially isomorphic ∂P (as an ASC). ○

Theorem I.3.1: Steinitz (1916) [Ste16]

Every ASC triangulating $\mathbb{S}^2$ can be realised as the boundary complex ∂P of a simplicial 3-polytope.

Any triangulation of $\mathbb{S}^1$ into line segments is the boundary of a simplicial 2-polytope. But this theorem does not remain true in higher dimensions.

Definition I.3.11 (Polytopal triangulation)
A triangulation K of $\mathbb{S}^n$ is polytopal if K can be realised as the boundary complex ∂P of a simplicial $(n+1)$-polytope. polytopal

Altschuler combinatorially enumerated triangulations of $\mathbb{S}^3$.

Theorem I.3.2: Bokowski, Garms (1987) [BG87]
Altschuler's 3-sphere M_{425}^{10} with ten vertices is not polytopal.

Corollary I.3.12
For $n \geqslant 3$ there are non-polytopal triangulations of $\mathbb{S}^n$.

Theorem I.3.3: [BS89, RG95, Mnë88]
It is NP-hard to decide whether a triangulation of $\mathbb{S}^n$ for $n \geqslant 3$ is polytopal.

Proof. Use matroid theory to reduce this problem to the boolean satisfiability problem 3-SAT. □

Remark I.3.13 (Realising graphs in $\mathbb{R}^2$ or $\mathbb{R}^3$) Every one-dimensional simplicial complex (graph) can be realised geometrically in $\mathbb{R}^3$. The graphs containing K_5 or $K_{3,3}$ as a minor are exactly the non-planar graphs, that is, the ones that cannot be realised geometrically in $\mathbb{R}^2$.

Testing planarity has linear running time in the number of vertices of the graph.

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The reason for a non-planar graph to be not realisable in $\mathbb{R}^2$ is that non-planar graphs are not embeddable in $\mathbb{R}^2$ (this is stronger!). Thus non-embeddability is an obstruction to realisability. ○

Definition I.3.14 (Embedding)
Let X and Y be topological spaces. An injective continuous map $f\colon X \to Y$ is an embedding if f is a homeomorphism between X and $f(X) \subset Y$, where the latter is equipped with the subspace topology.

Theorem I.3.4: Whitney embedding theorem (1936) [Whi36]
A smooth d-manifold can be smoothly embedded in $\mathbb{R}^{2d}$.

Remark I.3.15 (Sharpness of the bound) The real projective space $\mathbb{R}\mathrm{P}^d$ cannot be embedded in $\mathbb{R}^{2d-1}$ if $d = 2^k$ for $k \in \mathbb{N}$. For example, for $d = 1$, $\mathbb{R}\mathrm{P}^1 \cong \mathbb{S}^1$ cannot be embedded in $\mathbb{R}^{2\cdot 1-1} = \mathbb{R}$. ○

By restricting the class of manifolds under consideration, one can obtain improved results.

Theorem I.3.5: [HH63] for $d > 4$, Wall (1965) [Wal65] for $d = 3$
A smooth d-manifold with $d \neq 2^k$ can be smoothly embedded in $\mathbb{R}^{2d-1}$.

Theorem I.3.6: Haefliger, Hirsch (1963) [HH63]

A compact orientable (smooth) d-manifold can be embedded in $\mathbb{R}^{2d-1}$.

Example I.3.16 (Embeddability of standard surfaces) Compact orientable surfaces (e.g. T^2, $\mathbb{S}^2$) can be embedded in $\mathbb{R}^3$, whereas compact non-orientable surfaces (e.g. Klein bottle) can only be embedded in $\mathbb{R}^4$. Clearly, $\mathbb{S}^d$ embeds in $\mathbb{R}^{d+1}$. ◇

It is notoriously hard to determine the smallest dimension a d-manifold M embeds in.

Back to simplicial complexes

The following theorem improves upon theorem I.3.1. How so?

Theorem I.3.7: Archdeacon et al. (2007) [ABEM07]

Every triangulation of T^2 can be geometrically realised in $\mathbb{R}^3$.

This settled a conjecture by Duke / Grünbaum from 1970 / 1973.

Theorem I.3.8: Brehm, Schild (1995) [BS95]

Every triangulation of $\mathbb{R}\mathrm{P}^2$ can be geometrically realised in $\mathbb{R}^4$.

Theorem I.3.9: Bokowski, Guedes de Oliveira (2000) [BGdO00]

There is a 12-vertex (vertex-minimal by theorem I.6.3) triangulation of the orientable surface of genus 6 that is not realisable in $\mathbb{R}^3$.

The proof required 10 CPU years and makes use of oriented matroids (and thus computing determinants). Nowadays there is a 3-SAT formulation, which can be checked in half an hour.

Theorem I.3.10: Schewe (2007) [Sch07]

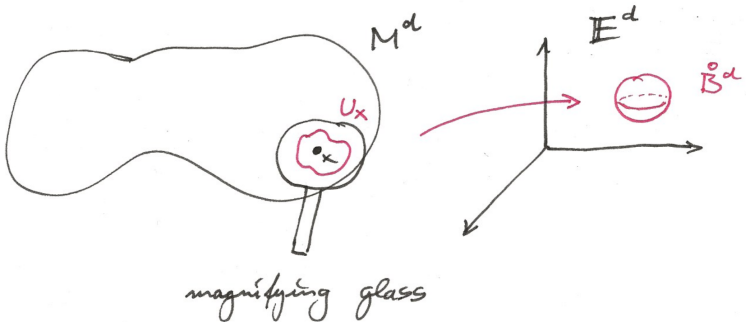
For every orientable surface of genus $g \geqslant 5$, there is a triangulation that is not realisable in $\mathbb{R}^3$.

Open problem: Can every triangulation of an orientable surface of genus $g \in \{2, 3, 4\}$ be realised in $\mathbb{R}^3$?

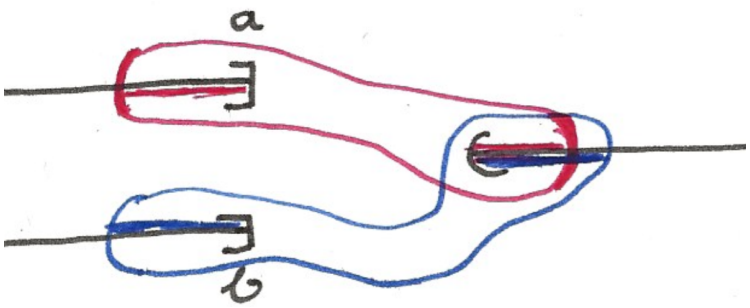
I.4 Manifolds and Triangulations

Generally speaking, manifolds are topological spaces that locally look like EUCLIDEAN space. Physicists would say: the space is homogeneous - wherever we are, space looks the same for us.

DEFINITION I.4.1 (TOPOLOGICAL d-MANIFOLD)
A topological space M is a d-dimensional (topological) manifold if M is HAUSDORFF and second countable and if for every $x \in M$ there is an open neighbourhood U_x of x such that U_x is homeomorphic to an open d-ball in $\mathbb{E}^d$. manifold



Counterexample I.4.2 (non-HAUSDORFF but locally $\cong$ to a subset of $\mathbb{E}^1$)
Let X be a set consisting of three rays, two closed and one open:



As a basis for X we take all open intervals on the three rays plus open "intervals" that connect the left rays with the right ray. Then X locally looks like $\mathbb{R}^1$, but a and b cannot be separated by disjoint open sets and thus X is not HAUSDORFF. ◇

Remark I.4.3 Second countable not necessarily HAUSDORFF manifolds can be embedded in finite dimensional EUCLIDEAN space. ○

I.4.1 Combinatorial properties of triangulated manifolds

Recall that by theorem .0.1, every compact surface is triangulable. We want to rule out undesirable (that is, non-surface-like) behaviour of simplicial complexes by assuming certain requirements.

Example I.4.4 (Simplicial complexes that do not triangulate manifolds)
The simplicial complex on the left is of mixed dimension and the one on the right is not everywhere locally homeomorphic to $\mathbb{R}^1$.

First requirement. Any ASC triangulating a (compact) manifold needs to be pure.

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Some part of a triangulation K of a surface hence will look like this:

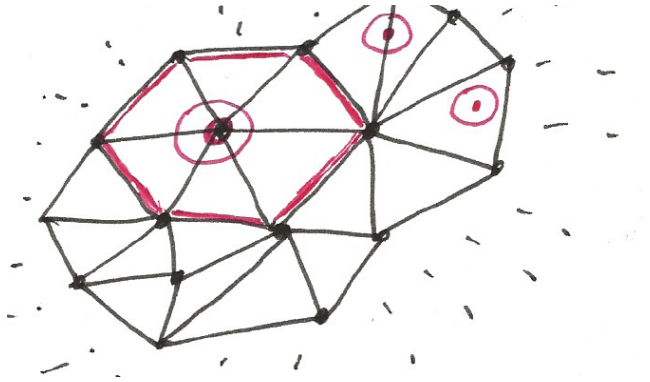


Fig. 27: A part of a triangulation K of a surface and the three different position of a point $x \in |K|$.

Every point $x \in |K|$ has a neighbourhood homeomorphic to an open disc $\mathring{B}^2$. In particular, x can lie (1) in the interior of a triangle, (2) on an edge or is (3) a vertex. Let us see what needs to be fulfilled in each three cases for this to happen.

(1) Interior points of triangles have small discs around them.

(2) For a point on an edge every incident triangle contributes a half-disc:

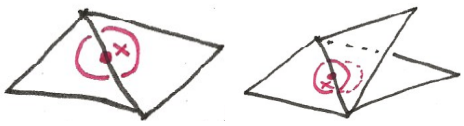


Fig. 28: The left picture resembles the case that the neighbourhood is locally homeomorphic to $\mathbb{R}^2$, whereas on the right, this is not the case.

Second requirement. Every edge in an ASC triangulating a (compact) manifold needs to be contained in exactly two triangles.

(3) For a vertex x, the incident triangles have to form a disc:

Definition I.4.5 (Open, closed star, link)
Let σ be a face of K. The open star of σ in K is

$$\mathring{\mathrm{star}}(\sigma) := \{\tau \in K : \sigma \subset \tau\},$$

the closed star of σ in K is

$$\overline{\mathrm{star}}(\sigma) := \{\tau \in K : \sigma \cup \tau \in K\},$$

the link of σ in K is

$$\mathrm{link}(\sigma) := \{\tau \in K : \sigma \cup \tau \in K,\ \sigma \cap \tau = \varnothing\}.$$

open star

closed star

link

Remark I.4.6 Both $\overline{\mathrm{star}}(\sigma)$ and $\mathrm{link}(\sigma)$ are subcomplexes of K (Exercise!), while $\mathring{\mathrm{star}}(\sigma)$ for $\sigma \neq \varnothing$ is not, because it does not contain $\varnothing$.

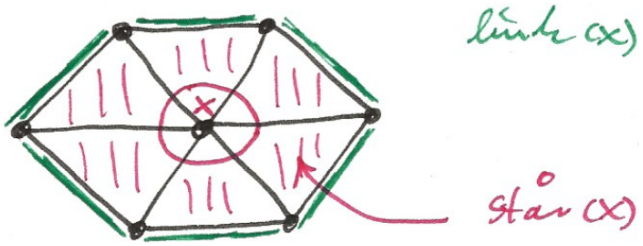


Fig. 29: A vertex x with a neighbourhood homeomorphic to $\mathbb{R}^2$, its link and its open star. (The incident edges of x and x itself are also part of its open star.)

Third requirement. The link of every vertex in an ASC triangulating a (compact) surface needs to be combinatorically isomorphic to $\mathbb{S}^1$.

We often use $\mathrm{star}(\sigma)$ to denote the closed star of σ.

Example I.4.7 (Link, star)
Consider the simplicial complex in figure 30. We have $\mathrm{link}(\{1,2\}) = \{\varnothing, \{3\}, \{4\}\}$ or in facet-description: $\mathrm{link}(12) = \{3,4\}$, $\mathrm{star}(12) = \{123, 124\}$. ◇

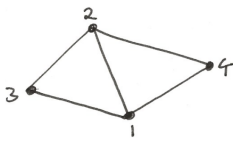


Fig. 30

Definition I.4.8 (Ridge)
A ridge is a $(d-1)$-dimensional (or: codimension-1) face of a pure simplicial d-complex.

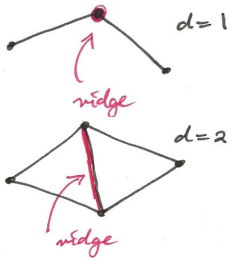


Fig. 31: Examples of ridges.

Lemma I.4.9 (link(ridge) $\cong \mathbb{S}^0$)
Let K be a triangulation of a (compact) d-manifold without boundary as a (finite) simplicial complex and let $r \in K$ be a ridge. Then

$$|\mathrm{link}(r)| \cong \mathbb{S}^0.$$

Fig. 32: Examples for lemma I.4.9.

Remark I.4.10 In a triangulation of a manifold (without boundary) a ridge lies in exactly two facets. ○

Fig. 33: What do we know about links of vertices? For $d \in \{1, \dots, 4\}$, we have $|\operatorname{link}(v)| \cong \mathbb{S}^{d-1}$, but for $d \geqslant 5$ the link is more complicated.

Definition I.4.11 (Path-connected)
A topological space X is path-connected if for any $x, y \in X$ there is a path connecting x and y.

path-connected

Theorem I.4.1: Connected manifolds

A connected manifold is path-connected.

Definition I.4.12 (Strongly connected)
A pure simplicial complex is strongly connected if for every pair of facets (σ, τ) there is a sequence

$$\sigma = \sigma_1 v_{1,2} \sigma_2 v_{2,3} \sigma_3 \dots v_{j-1,j} \sigma_j = \tau$$

of facets $\sigma_1, \dots, \sigma_j$ and ridges $v_{1,2}, \dots, v_{j-1,j}$ such that $v_{i-1\ i}$ lies in z_{i-1} and z_i, that is, there is a path of facets from σ to τ going via ridges.

strongly connected

Fig. 34: A strongly connected triangulation.

Lemma I.4.13
Path-connected triangulations of closed manifolds are strongly connected.

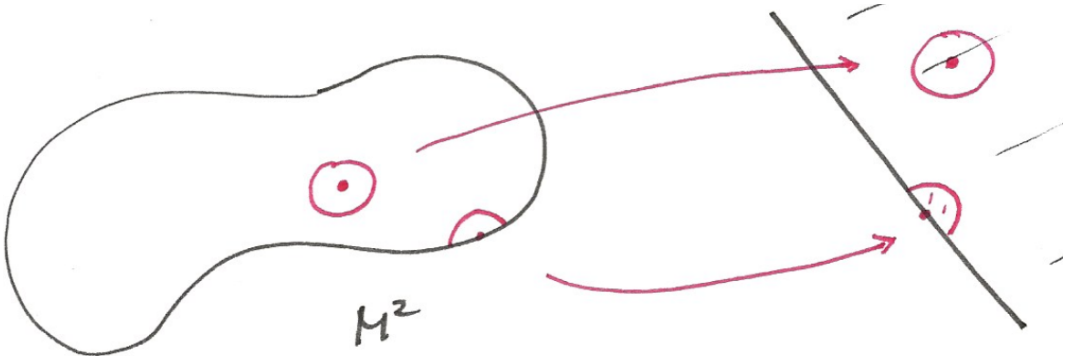
Proof. (Idea) A path in X is a continuous map $p\colon [0,1] \to X$. As I is compact and p is continuous, X is compact. A polygonal path p can revisit a facet of X only finitely many times.

Hence we can shorten resp. omit the revisits to obtain a path that goes via each facet only once and accomodate it that it goes via ridges. □

I.5 Classification of (closed) surfaces

In our zoo of surfaces (2-manifolds) we have e.g. $\mathbb{S}^2$, T^2, double-torus, triple-torus, ..., $\mathbb{R}\mathrm{P}^2$, the Klein bottle, non-closed manifolds like $\mathbb{E}^2$ or $\mathbb{E}^2 \setminus \overline{B}^2 \cong \mathbb{E}^2 \setminus \{\mathrm{pt}\}$, the doubly punctured plane, ..., the open cylinder $\cong \mathbb{E}^2 \setminus \{\mathrm{pt}\}$, the closed cylinder, the closed disk and the open disc $\cong \mathbb{E}^2$. As a model-space for manifolds with boundary we can take closed half spaces in $\mathbb{E}^n$.

Definition I.5.1 (Manifold with(out) boundary)
A (Hausdorff and second countable) space M is a d-manifold with or without boundary if for every $x \in M$ there is an open neighbourhood that is homeomorphic to an open d-ball in the closed half space $\mathbb{E}^d_+ := \mathbb{E}^{d-1} \times \mathbb{E}_{\geqslant 0}$ or to an open half-ball (= intersection of the half space with an open ball). In the first case, x is an interior point, in the latter, it is a boundary point.



Definition I.5.2 (Closed / open manifold)
A d-manifold without boundary is closed if it is compact and open else.

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From now on, we assume a manifold to be closed and connected (unless we consider special cases such as $\mathbb{S}^0$ or the boundary of a closed cylinder, $\mathbb{S}^1 \cup \mathbb{S}^1$).

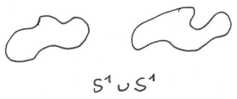

Fig. 35: The closed, disconnected manifold $\mathbb{S}^1 \cup \mathbb{S}^1$.

Theorem I.5.1: Rado (1925) [Rad25, § 4]
Closed 2-manifolds can be triangulated as finite ASCs.

From theorem I.4.1 and lemma I.4.13 we get the following theorem.

Theorem I.5.2: Triangulations of connected closed manifolds
Every triangulation of a connected closed manifold is strongly connected.

Example I.5.3 (Strongly connected SC not manifold-triangulation)
In figure 36 you can see two simplicial complexes that are strongly connected but not triangulations of a manifold, as the (realisation of the) link of their pinch point is homeomorphic to $\mathbb{S}^1 \sqcup \mathbb{S}^1$. Such triangulations are called pseudo-manifolds, where e.g every link only has to be a compact surface, not only a sphere. ◇

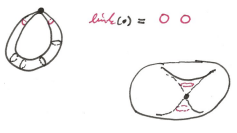

Fig. 36: Triangulations of pseudomanifolds. The link of "pinch points" is not a sphere.

Definition I.5.4 (Dual graph of manifold triangulation)
The dual graph of a triangulated (connected and closed) d-manifold M has as vertices the facets of M and edges whenever two facets have a common ridge.

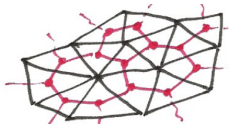

Fig. 37: The dual graph of a part of a triangulation of a connected and closed manifold.

Find example.

Remark I.5.5 The dual graph does *not* determine the triangulation, but the degree of a vertex in the dual graph reveals the dimension of the manifold. Distinct triangulations could have the same dual graph.

Remark I.5.6 (Determining isomorphic triangulations)
How can we determine whether two triangulated d-manifolds are isomorphic? We can start by checking if the number vertices and their degrees and the number of faces in each dimension are equal in both triangulations. By exhaustion, we can then check for each facet, if it can be mapped to a facet of the other triangulation, for which there are $\#$ facets $\cdot(d+1)!$ possible choices for each facet (this determines all possibilities by strong connectivity). Finally, neighbouring faces have to be mapped to neighbouring faces. ○

Remark I.5.7 (The automorphism group of a triangulation)
Note that there might be more than one isomorphism, e.g. if both triangulations are identical, and in that case one recovers the combinatorial automorphism group $\mathrm{Aut}(K)$ of the triangulation K. For example, the automorphism group of the n-polygon is the dihedral group D_n (which has order $2n$ for $n \geqslant 3$). The automorphism group of $\partial\Delta_n$ is the symmetric group S_{n+1} (which has order $(n+1)!$). ○

The following **Classification procedure (algorithm)** is due to [Bra21]. Its input is a triangulation of a closed and connected surface, its output is the "type" of the surface.

1. **Pick a spanning tree in the dual graph of the triangulation.**
   Remark I.5.8 The dual graph is a connected graph because the triangulation is strongly connected by theorem I.5.2. ○
2. **Cut open all edges in the triangulation that are not crossed by an edge of the spanning tree.** Equivalently: for every node in the dual graph, take the triangle it is contained in and also the neighbouring triangle if the dual graph crosses the edge both have in common, producing a "tree of triangles".

   We illustrate this procedure with a triangulation of the torus.

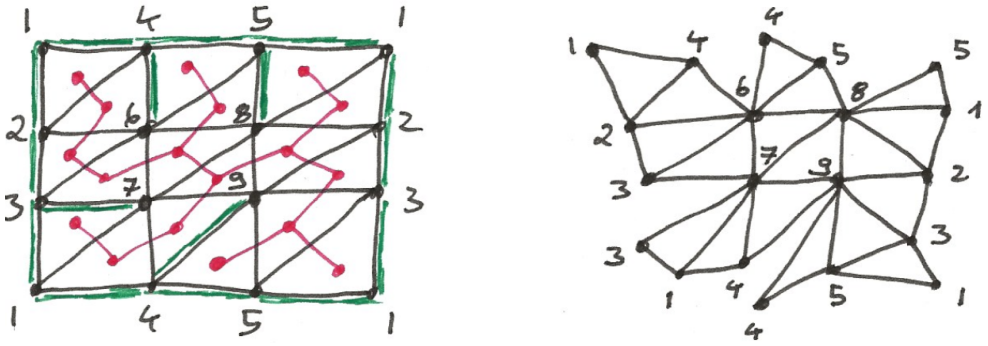

Fig. 38: **Left:** One spanning tree in red. The green edges are the ones not crossed by edges of the dual graph.
Right: This triangulation (a tree of triangles) is a triangulation of a disk with the extra property that the boundary edges are pairwise identified. Every edge on the "boundary" of this triangulation appears twice.

Definition I.5.9 (Polygonal decomposition / scheme)
A polygonal decomposition / scheme of a (closed and connected) surface is a decomposition into a (finite) family of polygons with pairwise identified oriented edges.

Example I.5.10 (Polygonal decomposition / scheme)
Starting out with nine disjoint squares, we can make up pairwise identifications of edges like in figure 39.

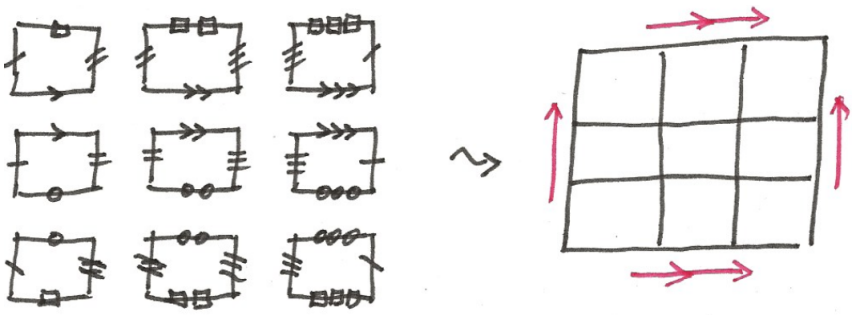

Fig. 39: Glueing together the squares according to their edge-identifications yields the composite figure on the right with identified edges.

Remark I.5.11 Polygons of a decomposition can be glued together along edges to form a single polygon with pairwise identified edges. ○

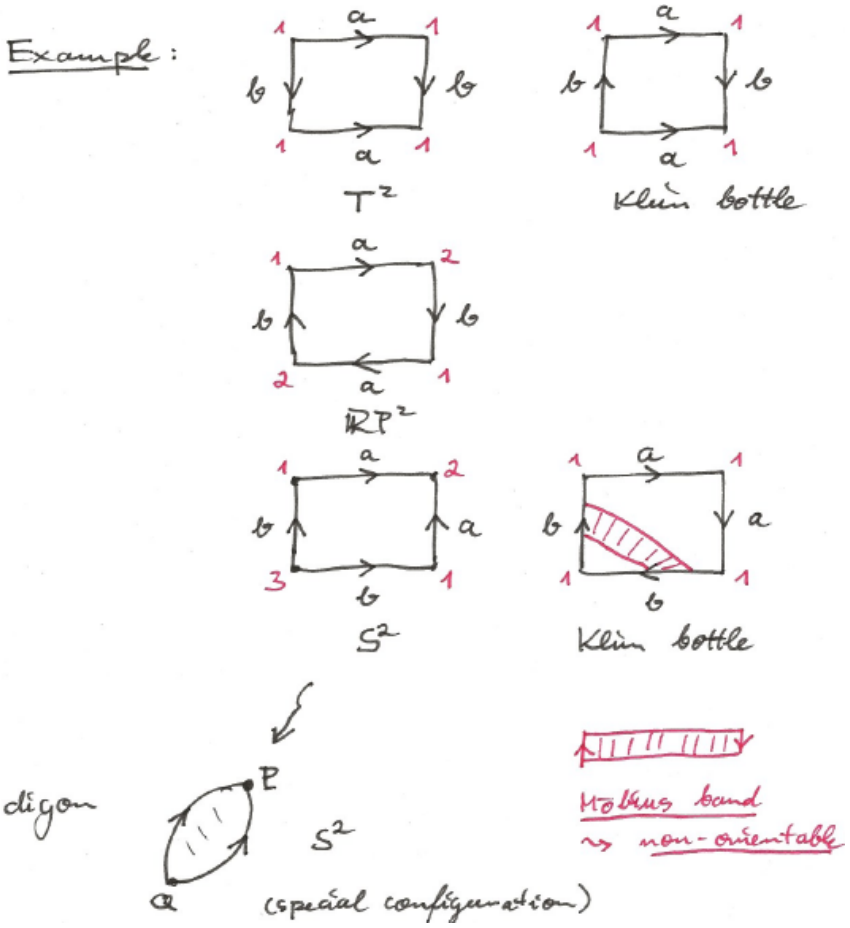

Fig. 40: Both $\mathbb{T}^2$ and the Klein bottle have one vertex, while $\mathbb{R}\mathrm{P}^2$ and $\mathbb{S}^2$ have two.

③ **For the previous tree of triangles, remove the internal edges to obtain a single polygon with pairwise identified edges.**

④ **Reduce such a scheme to a scheme with only one (or two) vertices. This yields a simplification of schemes.**

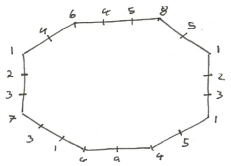

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Fig. 41: Removing the internal edges in the right diagram of figure 38 of the tree of triangles yields a single polygon (with the number of vertices reduced).

Lemma I.5.12 (Reducing the number of vertices)
Every scheme of a surface can be reduced to either the scheme aa^{-1} *of* $\mathbb{S}^2$ *or to a scheme with exactly one vertex.*

Proof. Suppose the scheme has at least two different equivalence classes of identified vertices.

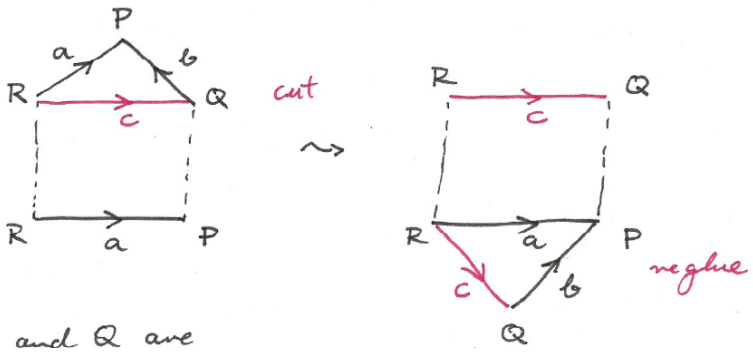


Fig. 42: On the left, P and Q are different vertices with m elements in the equivalence class of P.
On the right, there are now $m-1$ elements in the equivalence class of P, though the size of the class of Q has increased.

Through this cutting-and-reglueing procedure, P will eventually appear only once on the boundary of the (modified) scheme.

There are two cases (because edges are pairwise identified):

Fig. 43: Case 1: remove internal P and d. Case 2: there are only P and d.

Case ①: after deletion of further equivalence classes we obtain a scheme dd^{-1} of $\mathbb{S}^2$ or a scheme with only one vertex. □

⑤ **Transformation into canonical form.**

From now on the scheme has exactly one vertex. We simplify the scheme further by cross-cap normalisation, handle normalisation, or transforming handles into cross-caps (in the case that both are present).

Ⓐ **Cross-cap normalisation.**

Lemma I.5.13 (Cross-cap normalisation)
We can transform $\dots a \dots a \dots$ *into* $\dots a' a' \dots$*, that is, pairs of identified edges with same orientation on the boundary of a scheme can be transformed into neighbouring edges (cross-caps).*

Notation: $\dots a \dots a \dots$ is the sequence of edges we see when traversing the boundary of the scheme.

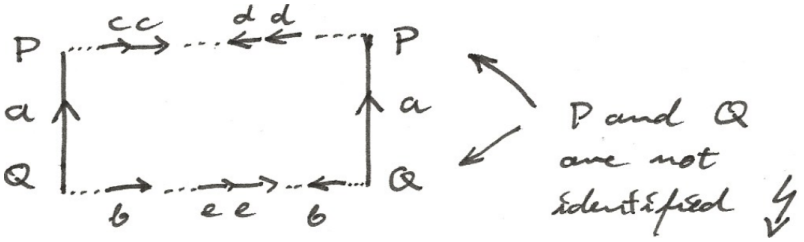
Proof. Cutting and reglueing a' yields the desired transformation:

Ⓑ **Handle normalisation.**

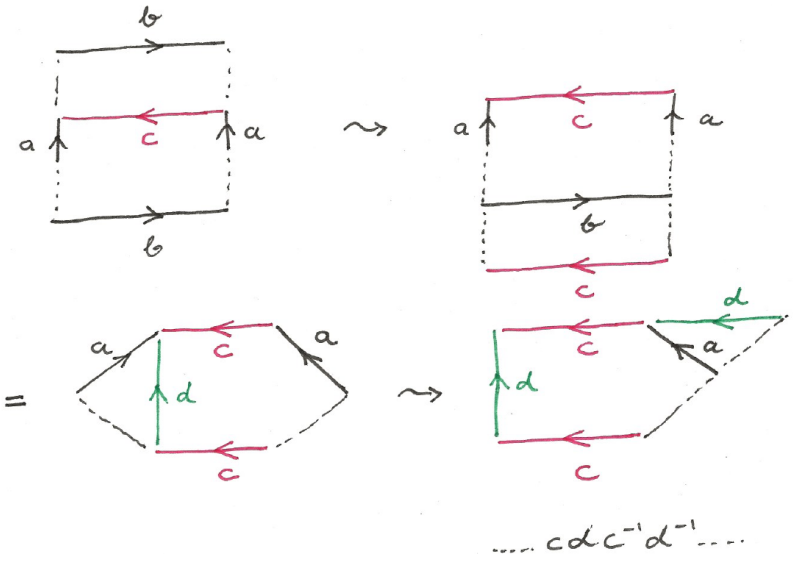
Lemma I.5.14 (Handle normalisation)
After crosscap normalisation

1. *pairs of oppositely oriented edges as crossed pairs* $\dots a \dots b \dots a^{-1} \dots b^{-1} \dots$
2. *and can be transformed into* $\dots cdc^{-1}d^{-1} \dots$.

Proof. 1. Let us assume that $\dots a \dots a^{-1} \dots$ is *not* separated by a pair $\dots b \dots b^{-1} \dots$. This is a contradiction to the assumption that we only have one equivalence class of vertices:



2. We start with $\dots a \dots b \dots a^{-1} \dots b^{-1} \dots$. Consider the following double cutting-and-reglueing procedure



Special configurations include $\mathbb{R}P^2$ (cf. figure 44) a cross-cap (cf. figure 45) or a handle (cf. figure 46), which gets its name from the following consideration:

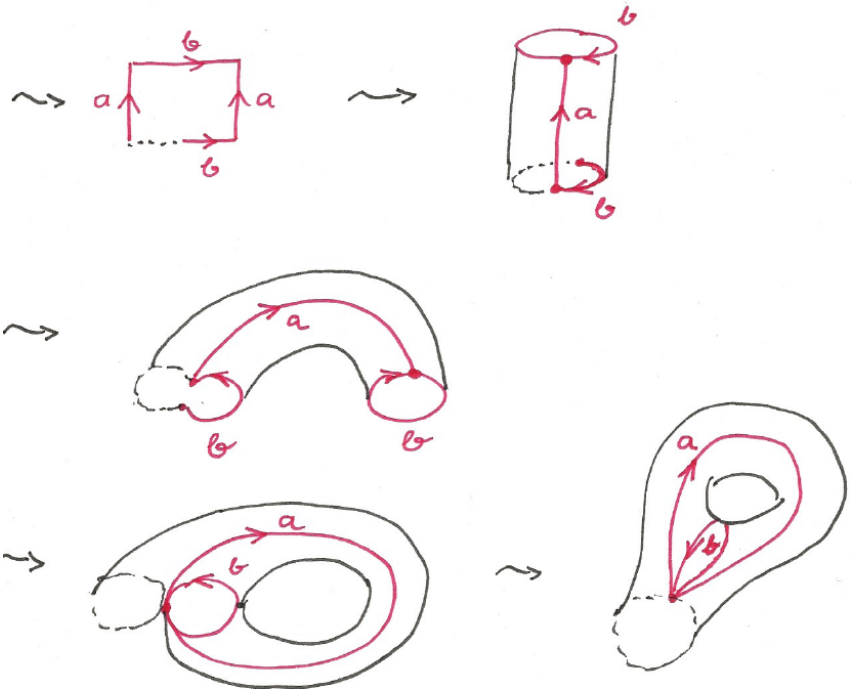


Fig. 44: $\mathbb{R}P^2$.

Fig. 45: A cross-cap.

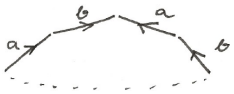


Fig. 46: A combinatorial handle.

Fig. 47: In the case there is a cross-cap, there is always a Möbius band which forces the surface to be non-orientable (though the handles themselves are orientable).

C Transformation of handles into cross-caps in the presence of both handles and cross-caps.

If there is at least one cross-cap and at least one handle, then the boundary of the polygon of the scheme has a subsequence $\ldots aabcb^{-1}c^{-1}\ldots$ (otherwise we would just have cross-caps or just handles - somewhere they need to be neighbours). The following triple cutting-and-regluing procedure leaves the …-part untouched and transforms the handle and cross-cap into three cross-caps.

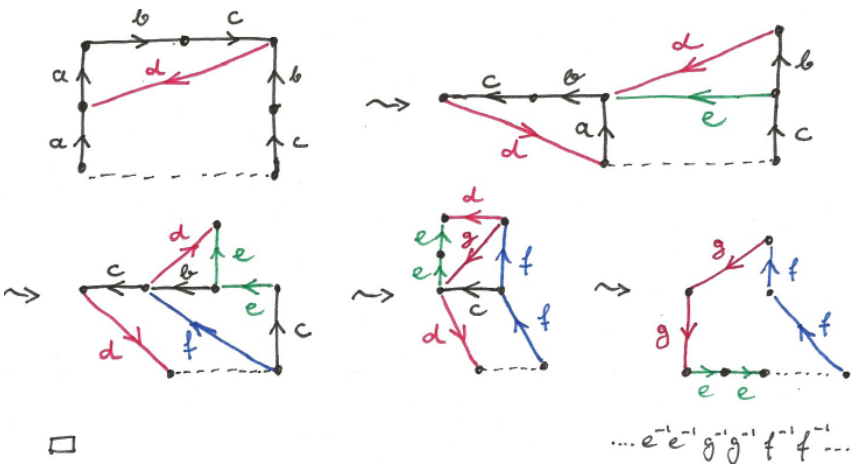


Fig. 48: First we re-glue along a, then along b, then along c.

This finishes the classification procedure.

Theorem I.5.3: Brahana (1921) [Bra21]

Every closed, connected surface can be transformed into one of the following schemes:

1. the sphere $\mathbb{S}^2$ (cf. figure 49), which is orientable,
2. the sphere with n handles (cf. figure 50), which is also orientable,
3. the sphere with n cross-caps (cf. figure 51), which is non-orientable.

Remark I.5.15 Such a scheme is the normal form of a surface. ○

Does theorem I.5.3 give us the classification of surfaces? Not yet! It could still be the case that some of the normal forms represent the same surface. Hence we need to distinguish between the different normal forms.

Definition I.5.16 (Topological invariant)
A topological invariant is a map I from (a subclass of) the category of topological spaces **Top** into e.g. the category or groups or $\mathbb{R}$, which assigns to each topological space $X \in$ **Top** some object, which can be a group or a number of something else. If a group resp. number is assigned, then I is a algebraic resp. numerical invariant.

It is required that if $X \cong Y$, then $I(X) \simeq I(Y)$ (where $\simeq$ denotes isomorphism).

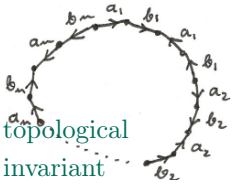
Example I.5.17 (Topological invariants)
The dimension is numerical invariant for manifolds and genus is a numerical invariant for surfaces. Connectivity and compactness can be modelled as maps from **Top** to $\{0, 1\}$. ◇

Definition I.5.18 (Complete classification)
A complete classification of a family of topological spaces $(X_j)_{j\in J}$, where J is any index set, is a partition into homeomorphic spaces, e.g. by specifying some list of topological invariants $I_1, \dots, I_k$ that together allow to distinguish between non-homeomorphic spaces in the family.

We will see that for surfaces

- Euler characteristic and orientability character (corollary I.5.22) or

Fig. 49: The scheme of the sphere $\mathbb{S}^2$.



topological invariant

Fig. 50: The scheme of the sphere with n handles.

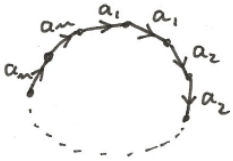


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Fig. 51: The scheme of the sphere with n cross-caps.

complete classification

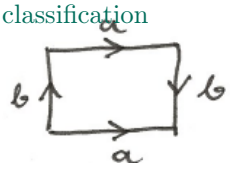


Fig. 52: Exercise: transform this into normal form.

- homology

give a complete classification.

Definition I.5.19 ((Non-)orientable)
A triangulated surface is orientable if the triangles of the triangulation can be oriented coherently such that each ridge inherits opposite orientations from its two neighbouring triangles. If a coherent orientation does not exist, the triangulated surface is non-orientable.

Fig. 53: A ridge receiving opposite orientations from its neighbouring triangles.

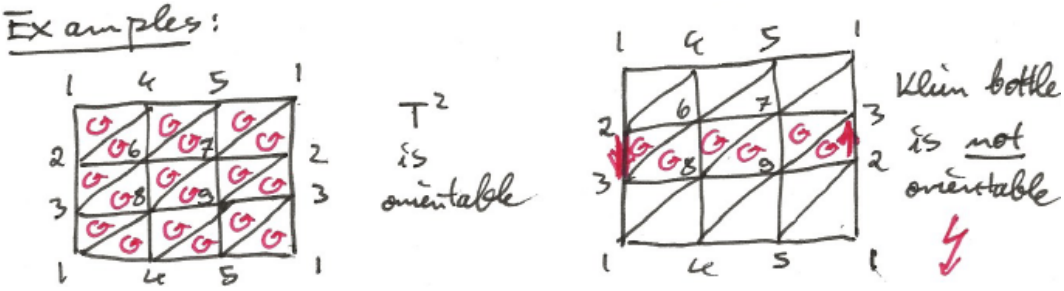


Fig. 54: The Möbius strip we found in the triangulation of the Klein bottle on the right is an obstruction to orientability.

Hence changing the the triangulation *locally* can make it non-orientable and we can detect this by checking locally (in the right place).

I.5.1 The Euler characteristic of a surface

Definition I.5.20 (Euler characteristic)
The Euler characteristic of a triangulation (or, more generally: polygonal decomposition) of a surface is

$$\chi := \#\text{vertices} - \#\text{edges} + \#\text{faces}.$$

Euler characteristic

Example I.5.21 (Euler characteristic)
The Euler characteristic of a tetrahedron is $4-6+5 = 2$ and of the cube it is $8-12+6 = 2$.◇

Schläfli gave a proof in ca. 1890, implicitly assuming ?, in 1971 by Bruggesser and Mani found a rigorous proof in [BM71].

Theorem I.5.4: Euler's polyhedron formula

A 3-polytope has Euler characteristic 2.

Let a surface M have triangulations K and K'. Do we have $\chi(K) = \chi(K')$, that is, is χ a topological invariant?

Theorem I.5.5: Kerékjártó (1923) [Ker23]

Let K and K' be triangulations of a (closed, connected) surface M. Then K and K' have a common subdivision K'' obtained from K respectively K' by sequences of

face of edge subdivisions.

Using that the EULER characteristic is invariant under subdivision, this shows that the EULER characteristic is an topological invariant.

This theorem is true for $d = 3$, but not for $d \geqslant 4$. There are topological d-manifolds for $d \geqslant 4$ for which there are non-equivalent PL (piecewise-linear) structures.

Open question (Smooth POINCARÉ conjecture in dimension 4). Does $\mathbb{S}^4$ have exotic structures?

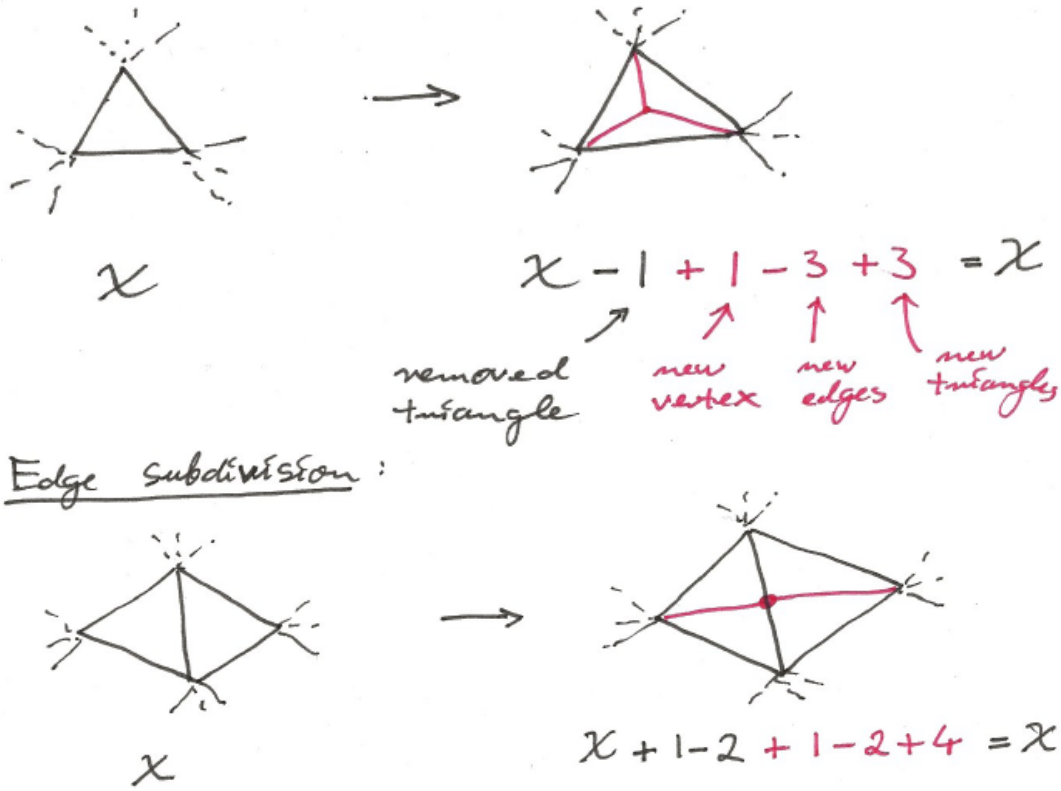


Fig. 55: Subdivision of edges and faces does not change the EULER characteristic. It should be $\chi + 1 - 2 + 1 - 4 + 4 = \chi$ in the second part.

Furthermore, subdivision does not change the orientability character of a surface.

Corollary I.5.22 (of theorem I.5.5)
The orientability character and the EULER *characteristic are topological invariants for surfaces.*

From topology we know the (non-reduced integral) homology

$$H_*(M;\mathbb{Z}) = \big(H_0(M;\mathbb{Z}), H_1(M;\mathbb{Z}), H_2(M;\mathbb{Z})\big) = \left(\mathbb{Z}^{\beta_0}, \mathbb{Z}^{\beta_1} \oplus T, \mathbb{Z}^{\beta_2}\right),$$

where $\beta(M) := (\beta_0, \beta_1, \beta_2)$ is the BETTI vector containing the BETTI numbers β_j and T is a torsion group. The number of connected components of M is β_0. If $\beta_0 = 1$, then β_1 is related to the genus of M and $\beta_2 = 0$ if M is non-orientable and 1 else.

BETTI number

Theorem I.5.6: Classification of surface

Every closed connected surface is one of the following types:

1. $\mathbb{S}^2$ (cf. figure 49), which is orientable, with $\chi = 2-1+1 = 2$ and $H_* = (\mathbb{Z}, 0, \mathbb{Z})$,
2. $\mathbb{S}^2$ with n handles (cf. figure 50), which is also orientable, with $\chi = 1-2n+1 = 2-2n$ and $H_* = \left(\mathbb{Z}, \mathbb{Z}^{2n}, \mathbb{Z}\right)$ and
3. $\mathbb{S}^2$ with n cross-caps (cf. figure 51), which is non-orientable, with $\chi = 1-n+1 = 2-n$ and $H_* = \left(\mathbb{Z}, \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2, 0\right)$.

In case 3, we have $H_*(\mathbb{Z}_2) = (\mathbb{Z}_2, \mathbb{Z}_2^n, \mathbb{Z}_2)$ and $H_*(\mathbb{Z}_3) = (\mathbb{Z}_3, \mathbb{Z}_3^{n-1}, 0)$ (cf. universal coefficient theorem), so we can't read from it the orientation for $\mathbb{Z}_2$. This shows what is enough for complete classification and what is not.

Remark I.5.23 Hence the Euler characteristic and orientability character give a complete classification of closed connected surfaces. Alternatively, we can use homology with $\chi = \beta_0 - \beta_1 + \beta_2$. ○

Example I.5.24

The projective plane $\mathbb{R}\mathrm{P}^2$ (cf. figure 44) has $\chi = 1 - 1 + 1 = 1$ and $H_* = (\mathbb{Z}, \mathbb{Z}_2, 0)$.

The torus T^2 has $\chi = 1 - 2 + 1 = 0$ and $H_* = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$.

The Klein bottle has $\chi = 1 - 2 + 1 = 0$ and $H_* = \left(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, 0\right)$.

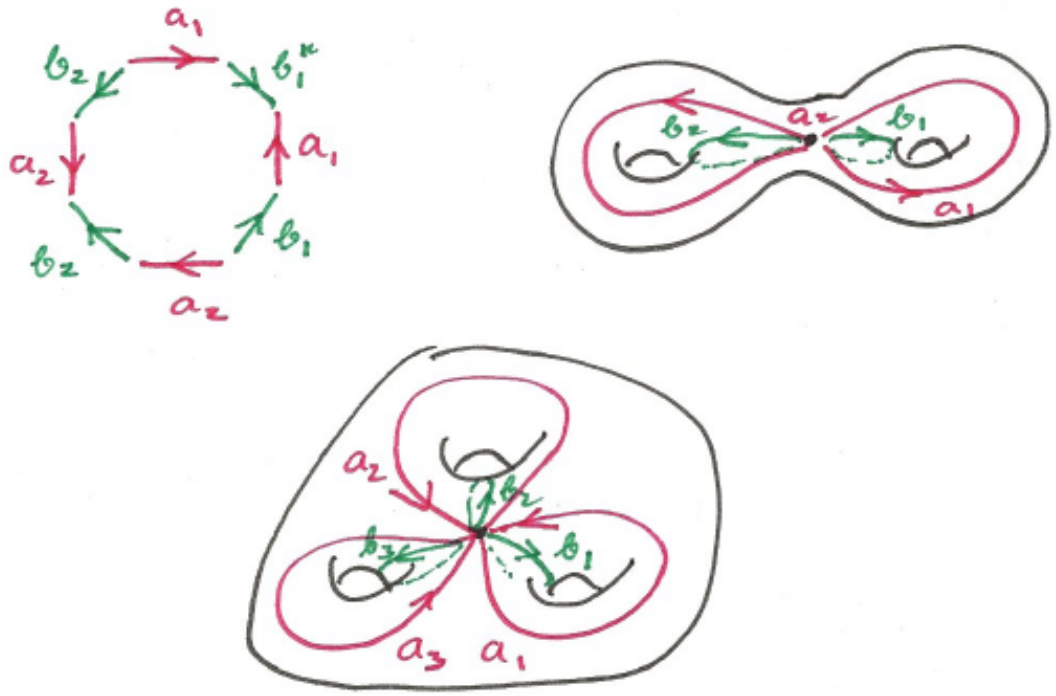


Fig. 56: **Top left:** The polygonal decomposition of the double torus and **Top right:** its geometric realisation. **Bottom:** Similarly, we can get a triple torus.

Definition I.5.25 (Genus)

For orientable resp. non-orientable surfaces with Euler characteristic $\chi = 2 - 2n$ resp. $2 - n$ the genus is n.

Remark I.5.26 (Classifying surfaces in Differential Geometry) In Differential Geometry there are three types of surfaces:

- spherical surfaces (with $\chi > 0$): $\mathbb{S}^2$, $\mathbb{R}\mathrm{P}^2$ (as quotients of $\mathbb{S}^2$),
- flat surfaces (with $\chi = 0$): T^2, Klein bottle (as quotients of $\mathbb{E}^2$),
- hyperbolic surfaces (with $\chi < 0$): all others (as quotients of the hyperbolic space $\mathbb{H}^2$).

For 2-manifolds there are three model geometries: $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$ (where the last two both are $\mathbb{R}^2$ as a set, but equipped with a different metric). For 4-manifolds there are eight model geometries: $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, ... The geometrisation of 3-manifolds is due to Thurston and Perelman. ○

I.6 Vertex-minimal triangulations of surfaces

24.05.2022

How many vertices do we need to triangulate a surface (as a simplicial complex)?

Example I.6.1 (How many vertices needed to decompose $\mathbb{S}^2$?)
For $\mathbb{S}^2$ we need at most four vertices if we triangulate it as the tetrahedron.

We only need three vertices if we also allow pseudo-simplicial complexes, where all faces are simplices, cf. figure 57.

We only need two vertices if we also allow schemes, like in figure 58.

We only need one vertex if we also allow CW-complexes, like in figure 59.

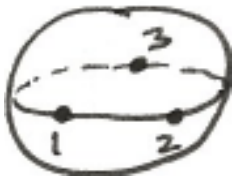


Fig. 57: A pseudo-simplicial-complex-triangulation of $\mathbb{S}^2$ with three vertices.

Fig. 58: A polygonal scheme of $\mathbb{S}^2$ with two vertices.

Fig. 59: A decomposition of $\mathbb{S}^2$ as a CW complex.

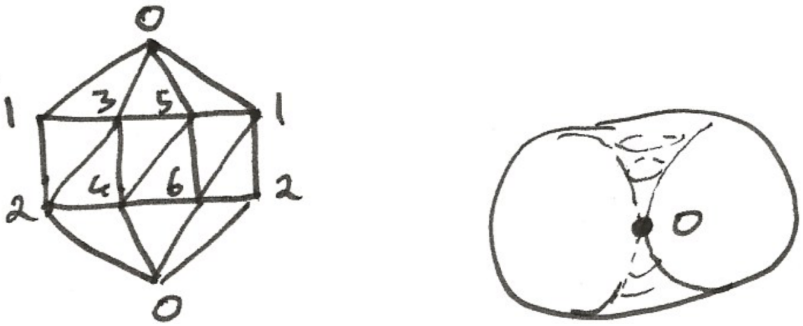


Fig. 60: Here, the identification of vertices induces identification of edges. For polygonal schemes, identification of edges induces an identification of vertices.

Definition I.6.2 (f(ace)-vector, Euler characteristic)
The f-vector (or face vector) of a simplicial d-complex K is

$$f = (f_0, f_1, \dots, f_d),$$

where f_k for $k \in \{0, \dots, d\}$ is the number of k-dimensional faces of K. The Euler characteristic of K is

$$\chi(K) := \sum_{k=0}^{d} (-1)^k f_k.$$

From Topology we know that χ is a topological invariant. Furthermore, if M is an odd-dimensional manifold, then $\chi(M) = 0$, so the Euler characteristic is not helpful for studying e.g. 3-manifolds.

Remark I.6.3 (f-vector of a surface) If K triangulates a surfaces with n vertices, then its f-vector is

$$(V, E, F) = \left(n, f_1, \frac{2}{3} f_1\right).$$

But since $V - E + F = \chi$, given a surface, we can't choose n and f_1 freely, but only one of them. ○

Definition I.6.4 (Hasse diagram)
The Hasse diagram is the (layered) graph consisting of

- the faces of K as its vertices,

Hasse diagram

- edges whenever an $(k-1)$-face is contained in an k-face.

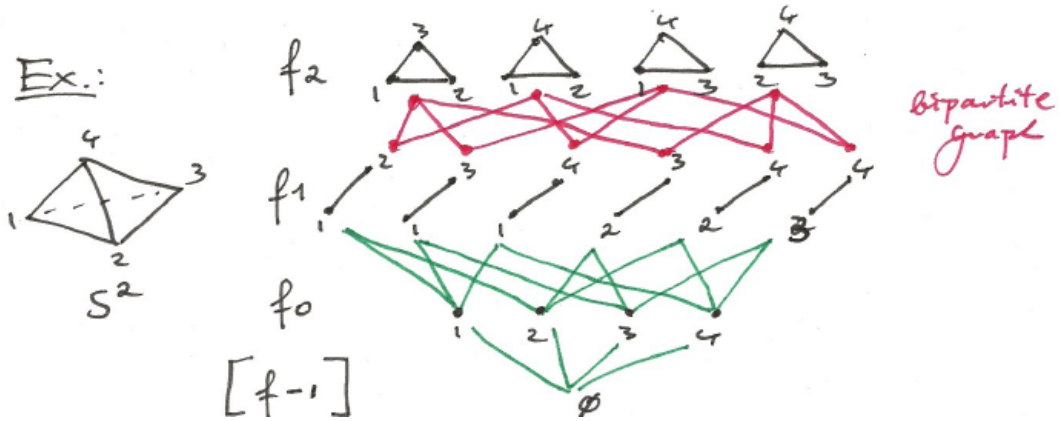


Fig. 61: The HASSE diagram for $\mathbb{S}^2$ with the convention that the only -1-dimensional face is the empty set. In every layer, we get a bipartite graph.

As every edge is contained in exactly two triangles and every triangles has exactly three edges, there are two ways to count the edges of the red graph: "from below" and "from above". This yields $2f_1 = 3f_2$.

For surfaces we have

$$\begin{cases} \chi = n - f_1 + f_2, \\ 2f_1 = 3f_2, \end{cases}$$

which are two equations for the three unknowns n, f_1, f_2.

They yield $f_2 = \frac{2}{3}f_1$ and thus

$$\chi = n - f_1 + f_2 = n - f_1 + \frac{2}{3}f_1 = n - \frac{1}{3}f_1.$$

Thus

$$(f_1, f_2) = (3n - 3\chi, 2n - 2\chi),$$

so the f-vector is

$$f = (n, 3n - 3\chi, 2n - 2\chi).$$

Example I.6.5 For a 9-vertex-triangulation of T^2 we have $f = (9, 27, 18)$ and thus $\chi = 0$, as expected. (The seven-vertex triangulation can be achieved using bi-stellar flips.) ◇

Let K be a triangulation of a surface M with EULER characteristic $\chi(M)$ on n vertices. We have just shown that then

$$f(K) = \big(n, 2n - \chi(M), 2n - 2\chi(M)\big).$$

If the surface can be triangulated with n vertices, then it can also be triangulated with $n+1$ vertices, as illustrated in figure 62. The new triangulation then has the f-vector

$$f = \big(n+1, 3n - 3(\chi - 1), 2n - 2(\chi - 1)\big) = \big((n+1), 3(n+1) - 3\chi, 2(n+1) - 2\chi\big). \quad (1)$$

Thus if we know the minimal number $n_{\min}$ of vertices to triangulate a surface M, this completely determines the set of f-vectors of M. (This is harder in higher dimensions.)

Fig. 62: By subdividing one face of a triangulation, we obtain a triangulation with one additional vertex.

What kind of bounds do we know for n, f_1 or f_2? A graph with n vertices can have at most $\binom{n}{2}$ edges, so we get

$$3n - 3\chi = f_1 \leqslant \binom{n}{2} = \frac{n(n-1)}{2},$$

which we can reformulate as

$$n^2 - 7n + 6\chi \geqslant 0.$$

In the case of equality we have

$$n_\pm = \frac{1}{2}(7 \pm \sqrt{49 - 24\chi}).$$

Note that as $\chi \leqslant 2$, we have $49 - 24\chi \geqslant 1$. Furthermore,

$$n_- = \frac{1}{2}\left(7 - \sqrt{49 - 24\chi}\right) \leqslant 3.$$

We can discard this solution, since at least 4 vertices are needed to triangulate a surface (we need at least one triangle, which gives three vertices, and since every edge needs to be contained in two triangles, we need another vertex). We have just proven the following theorem.

Theorem I.6.1: Headwood's bound (1890) [Hea90]

Let M be a surface with Euler characteristic $\chi(M)$. Then a triangulation of M needs at least

$$\left\lceil \frac{1}{2}(7 + \sqrt{49 - 24\chi}) \right\rceil$$

vertices.

Remark I.6.6 (How good is Headwood's bound?)
If $\chi = 2$ (and thus $M \cong \mathbb{S}^2$), we need at least four vertices, so the bound is sharp. If $\chi = 1$, then $n \geqslant 6$. Identifying antipodal vertices on the icosahedron (which as 12 vertices) yields a six-vertex triangulation of $\mathbb{RP}^2$ (cf. figure 89), so the bound is again optimal.

If $\chi = 0$, then ($M \cong T^2$) and Headwood's bound is $n \geqslant 7$. You can see the unique (that is, up to relabelling of the vertices[1]) seven-vertex triangulation of T^2 in figure 63. ○

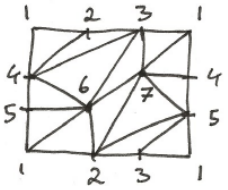


Fig. 63: The unique seven-vertex triangulation of T^2.

However, for the Klein bottle, which also has $\chi = 0$, Headwood's bound is not sharp (the only other surfaces being the orientable surface of genus 2 and the non-orientable surface of genus 3 by theorem I.6.3):

Theorem I.6.2: Franklin (1934) [Fra34]

There is *no* 7-vertex triangulation of the Klein bottle.

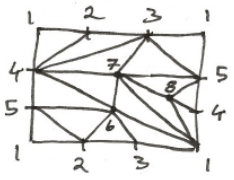


Fig. 64: An eight-vertex triangulation of the Klein bottle.

[1] The triangulation of $\mathbb{S}^2$ and $\mathbb{RP}^2$ shown here are unique as well.

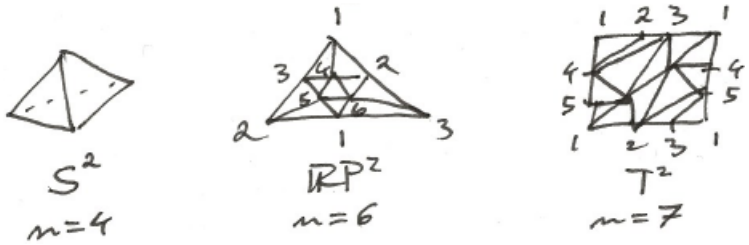
An alternative way to write HEADWOOD's bound is

$$\begin{aligned} &\iff && \binom{n}{2} && \geqslant 3n - 3\chi = f_2 \\ &\iff && \frac{n(n-1)}{2} - 3n + 6 && \geqslant 6 - 3\chi \\ &\iff && \frac{n^2 - 7n + 12}{2} && \geqslant 3(2-\chi) \\ &\iff && \binom{n-3}{2} && \geqslant 3(2-\chi), \end{aligned}$$

which equality if and only if $f_1 = \binom{n}{2}$.

DEFINITION I.6.7 (NEIGHBOURLY SURFACE)
A triangulated surface with $f_1 = \binom{n}{2}$ is neighbourly.

Example I.6.8 (Neighbourly triangulations)
The edges of the following triangulations form complete graphs K_n:



The following theorem was proven in [Rin55] for non-orientable surfaces and in [JR80] for orientable surfaces. The last sentence is due to [Hun78].

THEOREM I.6.3: VERTEX-MINIMAL TRIANGULATIONS OF SURFACES

Let M be a surface that is not a orientable surface of genus 2, the KLEIN bottle or a non-orientable surface of genus 3. Then there is a triangulation of M with n vertices if and only if

$$\binom{n-3}{2} \geqslant 3\big(2 - \chi(M)\big) \tag{2}$$

with equality if and only if the triangulation is neighbourly.
For the three omitted cases, $n-3$ has to be replaced by $n-4$, that is, one extra vertex is needed.

Characterising neighbourly triangulations of surfaces

31.05.2022

Equality in (2) can be characterised as

$$\binom{n-3}{2} = 3\big(2 - \chi(M)\big) \quad \iff \quad \chi = 2 - \frac{1}{3}\binom{n-3}{2} = 2 - \frac{(n-3)(n-4)}{6}. \tag{3}$$

As χ and 2 are integers, so is $\frac{(n-3)(n-4)}{6} \in \mathbb{N}_0$. This means we have $n \equiv 0, 1, 3, 4 \bmod 6$, that is, $n \equiv 0, 1 \bmod 3$ for $n \geqslant 4$.

For orientable surfaces χ is even, so $\frac{(n-3)(n-4)}{6}$ must be even by the same reasoning as above and thus $n \equiv 0, 3, 4, 7 \bmod 12$ for $n \geqslant 4$.

Corollary I.6.9 (Neighbourly triangulations)
The following are equivalent for a triangulated surface M with n vertices:

(1) *M is neighbourly.*
(2) $\binom{n-3}{2} = 3\big(2-\chi(M)\big)$.
(3) $\chi(M) = \frac{n(7-n)}{6}$.
(4) $n = \frac{1}{2}\left(7+\sqrt{49-24\chi(M)}\right)$.

Existence: *If M is a surface and not the* Klein *bottle and n is a number satisfying (2), (3) or (4), then M has a neighbourly triangulation with n vertices.*

Proof. (2) $\iff$ (3) is precisely (3) and the rest follows from theorem I.6.3. □

Further literature on this topic includes [Sti93, Küh06, Rin12].

I.7 Map colouring

This chapter follows [Rin55, Chp. 2].

In 1852 Francis Guthrie posed the "four colour problem": "Can every map be coloured with four colours [such that neighbouring countries have different colour]?".

Do we only consider connected closed surfaces here?

Definition I.7.1 (Map of a surface (first definition))
A map on a surface M is a decomposition of M into a finite collection of cells (i.e. a cell complex) consisting of vertices (0-cells), edges (1-cells), polygons (2-cells).

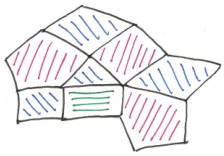


Fig. 65: In this "map", three colours suffice, but two do not due to the presence of a cycle with an odd number of faces.

Definition I.7.2 (Map of a surface (refined definition))
A map on a surface M is a decomposition of M into polygons (a finite cell complex) via a graph $G = (V; E)$ such that

- $\deg(v) \geqslant 3$ for all vertices $v \in V$,
- every vertex $v \in V$ with $\deg(v) = k$ is incident with k different polygons.

The polygons of a map are its countries (in particular, they are connected) and two countries are adjacent if they share an edge.

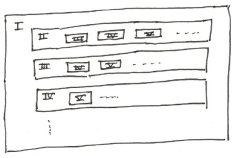


Fig. 66: The number of different colours is *not* bounded if the countries are not required to be connected (that is, if exclaves are allowed). As every country borders every other country, the number of colours required is equal to the number of countries.

Counterexample I.7.3 (Map of a surface)
The following decompositions are excluded by Definition I.7.2:

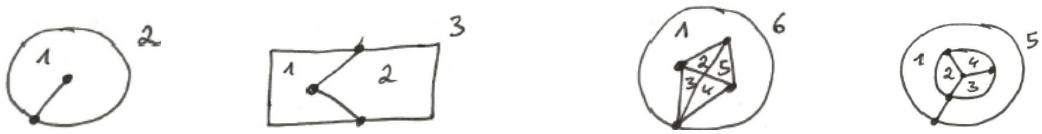


Fig. 68: For the first two decompositions, vertices of degree one or two can be removed to yields a simplified map and for the other two decompositions (which violate the second condition), if a country touches itself, this splits the surface into independent parts.

Example I.7.4 (Map of a surface)
The following decompositions *are* maps on a surface in the sense of Definition I.7.2:

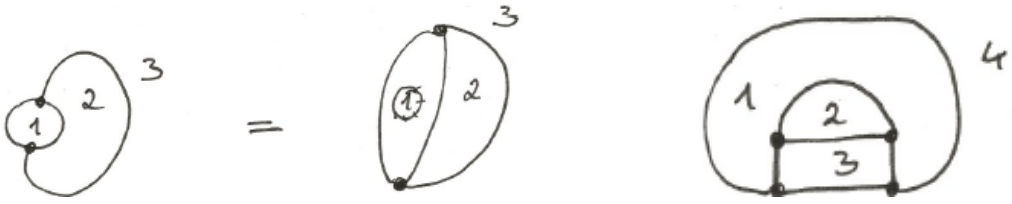


Fig. 69: Two neighbouring countries can share multiple edges (rightmost example).

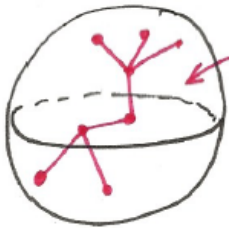


Fig. 67: A decomposition of $\mathbb{S}^2$ into eight vertices, seven edges and one 14-gon with pairwise identified edges (ignore the arrow).

I.7.1 The chromatic number of a surface

Let P be a map on a surface M.

Definition I.7.5 (Colouring of a map, Country graph)
The colouring of a map is a a colouring of the countries of a map such that adjacent countries have different colours. colouring of a map

A country graph G_P is the graph with a vertex for every country and an edge if two countries are adjacent. country graph

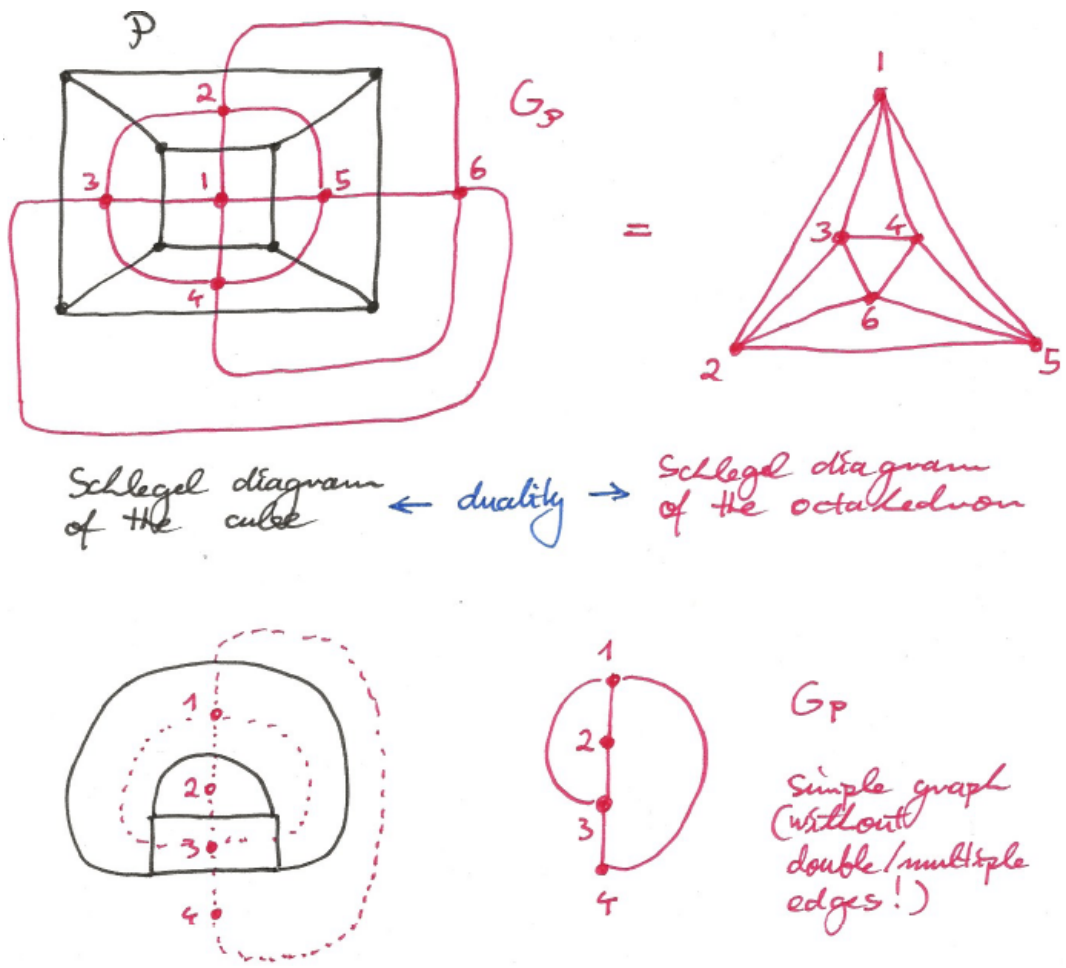


Fig. 70: The country graph of the Schlegel diagram of the cube is the Schlegel diagram of the octahedron.

Definition I.7.6 (Chromatic number of a map / surface)
The chromatic number $\chi_{\mathrm{CH}}(P)$ of a map P is the minimal number of colours that are needed to colour P. The chromatic number of a surface M is chromatic number

$$\chi_{\mathrm{CH}}(M) := \max_{P:P \text{ map on } M} \chi_{\mathrm{CH}}(P)$$

The maximum exists and the chromatic number is known for all surfaces.

For $\mathbb{S}^2$, a computer proof of the four colour theorem was done in [AH76] with 1476 cases and in [RSST96] with 633 cases.

For surfaces $M \neq \mathbb{S}^2$, [RY68] proved the map colour theorem.

Our aim is to prove the map colour theorem, which will take several steps.

Theorem I.7.1: $G_P \hookrightarrow M$

The (polyhedron of the) country graph G_P of a map P on a surface M can be embedded on M.

Proof. First, place capitals inside countries. Then subdivide the edges by placing one subdivision node on each edge. Then connect the capitals with the subdivision nodes of the adjacent edges and remove the subdivision nodes, that is, the make one edge out of an edge going from a capital via a subdivision node to another capital. Then G_P is a subgraph of the resulting (multi-)graph, because there is a one-to-one correspondence between capitals and countries and in the country graph adjacent countries share an edge, which is also the case the in constructed (multi-)graph. □

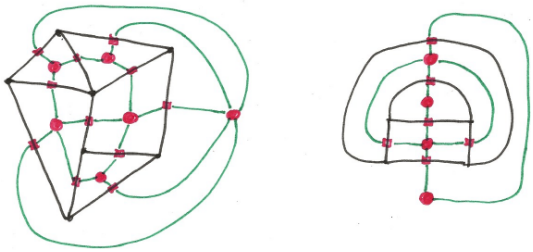


Fig. 71: Two examples of the procedure used in the proof of theorem I.7.1.

Remark I.7.7 In case two adjacent countries of P share exactly one edge, the resulting multi-graph is simple and isomorphic to G_P. In this case, the embedding of G_P on M defines the dual map to P. ○

02.06.2022

Embeddings of a graph into a surface can be very wild. The following theorem simplifies the situation.

Theorem I.7.2: $\exists P$ such that $G \cong G_P$

Let G be a graph that can be embedded on a surface M. Then there is a map P on M such that G is isomorphic to a subgraph of G_P.

Proof. First, thicken (this is the imprecise part of the proof) the graph G on M. Then cut each thickened edge. The resulting patches around the nodes of G on M yield countries with country graph G. Lastly, divide the rest of the surface into further countries yielding G_P. □

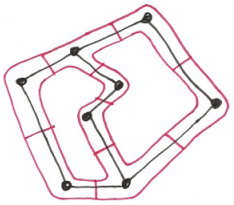


Fig. 72: The proof of theorem I.7.2 exemplified.

For every map P on M the country graph G_P can be embedded on M by theorem I.7.1 and $\chi_{\mathrm{CH}}(P) = \chi_{\mathrm{CH}}(G_P)$ (the chromatic number of a graph is defined analogously via vertex colourings). Thus

$$\chi_{\mathrm{CH}}(M) = \max_{P:P \text{ map on } M} \chi_{\mathrm{CH}}(P) \leqslant \max_{G \hookrightarrow M} \chi_{\mathrm{CH}}(G)$$

under the assumption that the maxima exist, which we will prove by giving an upper bound on the right hand side.

If conversely $G \hookrightarrow M$, then there is a map P on M such that G is isomorphic to a subgraph of G_P by theorem I.7.2, that is, $\chi_{\mathrm{CH}}(G) \leqslant \chi_{\mathrm{CH}}(P)$.

Together we get

$$\boxed{\chi_{\mathrm{CH}}(M) = \max_{P:P \text{ map on } M} \chi_{\mathrm{CH}}(P) = \max_{G \hookrightarrow M} \chi_{\mathrm{CH}}(G)} \tag{4}$$

still under the assumption that both maxima exist.

I.7.2 Some results from (topological) graph theory

Definition I.7.8 (Critical graph)
A graph G is critical if every proper subgraph has smaller chromatic number. critical

Example I.7.9 (Critical graph) A triangle is a critical graph. More generally, the complete graphs K_n yield an infinite sequence of critical graphs, as do the odd cycles C_{2n+1}. ◇

Theorem I.7.3: $\deg(v) \geqslant \chi_{\mathrm{CH}}(G) - 1$ for G critical

If G is critical, then $\deg(v) \geqslant \chi_{\mathrm{CH}}(G) - 1$ for all vertices v of G.

Proof. Let v be a vertex with $\deg(v) < \chi_{\mathrm{CH}}(G) - 1$.

Then the subgraph of G consisting of all vertices and edges of G except v and the edges containing v, called $G \backslash v$, is colourable with $\chi_{\mathrm{CH}}(G) - 1$ colours, as G is critical.

The vertex v has at most $\chi_{\mathrm{CH}}(G) - 2$ neighbours (as $\deg(v) < \chi_{\mathrm{CH}}(G) - 1$ by assumption), so that one of the $\chi_{\mathrm{CH}}(G) - 1$ colours is free for v, which is contradiction. □

For colourings, vertices of degree zero or one are not interesting, since they can be coloured as one wants, so they can be removed in preprocessing.

Theorem I.7.4: Topological bound

Let G be a graph with α_0 vertices and α_1 vertices such that $\deg(v) \geqslant 2$ for every vertex v. If G can be embedded on a surface M, then

$$\alpha_1 \leqslant 3\alpha_0 - 3\chi(M).$$

Remark I.7.10 Recall that for the 1-skeleton of a triangulation we proved $\alpha_1 = 3\alpha_0 - 3\chi(M)$, so this is a generalisation of this result for graphs embeddable in a surface. ○

Proof. If $G \hookrightarrow M$, then by theorem I.7.2 G is isomorphic to a subgraph G' of the 1-skeleton $\mathrm{Skel}_1(P)$ of a map P on M which has the f-vector $f(P) = (f_0, f_1, f_2)$.

Let P be cut via G' into r partial polyhehdra (partial map) $P_1, \ldots, P_r$ with f-vectors $f^{(k)} = \left(f_0^{(k)}, f_1^{(k)}, f_2^{(k)}\right)$ for $k \in \{1, \ldots, r\}$.

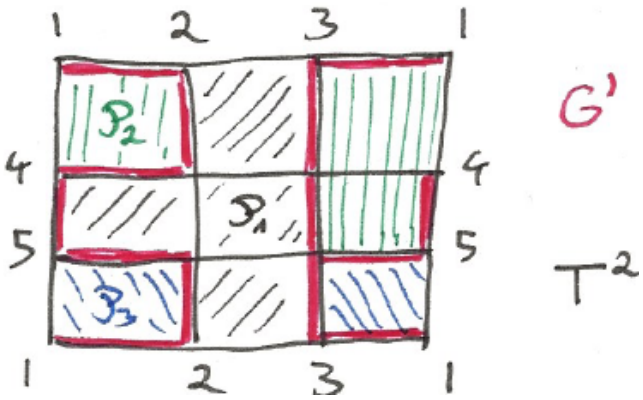


Fig. 73: The map on the torus is the grid consisting of nine squares. The partial maps have f-vectors $f^{(1)} = (8, 12, 4)$, $f^{(2)} = (6, 8, 3)$ and $f^{(3)} = (6, 7, 2)$.

For P we have

$$f_2 = \sum_{k=1}^{r} f_2^{(k)}$$

since every face of P is contained in exactly one of the P_k and

$$f_1 = \sum_{k=1}^{r} f_1^{(k)} - \alpha_1',$$

since the α_1' edges of G' are counted twice. Lastly,

$$f_0 = \sum_{k=1}^{r} f_0^{(k)} - \sum_{v \in G'} \deg(v) + \alpha_0'$$

since a vertex v of P that is not in G' is counted exactly one in $\sum_{k=0}^{r} f_0^{(k)}$ and a vertex of P that is in G' is counted $\deg(v)$ times in $\sum_{k=0}^{r} f_0^{(k)}$ and there are α_0' vertices in G'.

For G' we have by double counting

$$\sum_{v \in G'} \deg(v) = 2\alpha_1'. \tag{5}$$

Hence the Euler characteristic of M is

$$\begin{aligned}\chi(M) = \chi(P) = f_0 - f_1 + f_2 &= \sum_{k=1}^{r} f_0^{(k)} - \sum_{v \in G'} \deg(v) + \alpha_0' - \sum_{k=1}^{r} f_1^{(k)} + \alpha_1' + \sum_{k=1}^{r} f_2^{(k)} \\ &\overset{(5)}{=} \sum_{k=1}^{r} \left(f_0^{(k)} - f_1^{(k)} + f_2^{(k)} \right) - \alpha_1' + \alpha_0' \leqslant r - \alpha_1' + \alpha_0',\end{aligned}$$

since for any polyhedron we have $\chi(P_k) \leqslant 1$ (Exercise, corollary of $\chi \leqslant 2$!). Since G' has no nodes of degree at most 1, each P_k has at least three edges. Further, every edge of G' lies in either one or two of the P_k, so $3r \leqslant 2\alpha_1'$.

In summary,

$$3\chi(M) \leqslant 3r - 3\alpha_1' + 3\alpha_0' \leqslant 3\alpha_0' - \alpha_1'.$$

□

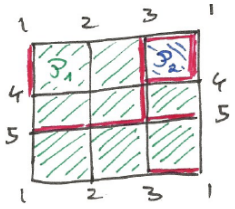


Fig. 74: TODO

For the case of equality we get a triangulation.

Theorem I.7.5: $\alpha_1 = 3\alpha_0 - 3\chi(M)$

Let G be a graph with α_0 vertices and α_1 edges such that $\deg(v) \geqslant 2$ for every vertex v of G. If G can be embedded on a surface M and $\alpha_1 = 3\alpha_0 - 3\chi(M)$, then there is a map P on M such that $G \cong \mathrm{Skel}_1(P)$ (isomorphic) and all faces of P are triangles.

The converse is also true.

Theorem I.7.6

If a graph G has a triangular embedding (that is, each face is a triangle) on a surface M, then $\alpha_1 = 3\alpha_0 - 3\chi(M)$.

Our aim now is to show that

$$\max_{G \hookrightarrow M} \chi_{\mathrm{CH}}(G)$$

exists.

Let G be a graph embedded on a surface M with $f_0 = n$ vertices and f_1 edges. Then by theorem I.7.4

$$f_1 \leqslant 3n - 3\chi(M). \tag{6}$$

Theorem I.7.7: χ_{CH} in terms of n and f_1

If G is critical, then

$$\big(\chi_{\mathrm{CH}}(G) - 1\big)n \leqslant 2f_1. \tag{7}$$

Proof. As G is critical, every vertex v of G has $\deg(v) \geqslant \chi(G) - 1$ by theorem I.7.3. Hence

$$\big(\chi_{\mathrm{CH}}(G) - 1\big)n \overset{(5)}{\leqslant} \sum_{k=1}^{n} \deg(v_k) = 2f_1,$$

where the second equality is the degree-sum formula (sometime referred to as handsking lemma). □

Lastly, combining both results yields

$$\big(\chi_{\mathrm{CH}}(G) - 1\big)n \overset{(7)}{\leqslant} 2f_1 \overset{(6)}{\leqslant} 6n - 6\chi(M),$$

that is,

$$\chi_{\mathrm{CH}}(G) - 1 \leqslant 6 - \frac{6}{n}\chi(M). \tag{8}$$

07.06.2022

Case 1. $\chi(M) \leqslant 0$. Since $\chi_{\mathrm{CH}}(G) \leqslant n$ (the upper bound is e.g. attained for complete graphs), we obtain from (8)

$$\chi_{\mathrm{CH}}(G)^2 - \chi_{\mathrm{CH}}(G) \leqslant 6\chi_{\mathrm{CH}}(G) - 6\chi(M),$$

which is equivalent to

$$\left(\chi_{\mathrm{CH}}(G) - \frac{7 + \sqrt{49 - 24\chi(M)}}{2}\right) \underbrace{\left(\chi_{\mathrm{CH}}(G) - \frac{7 - \sqrt{49 - 24\chi(M)}}{2}\right)}_{\geqslant 0 \text{ as } \chi(M) \leqslant 0, \chi_{\mathrm{CH}}(G) \geqslant 1, \text{ so } \sqrt{49-24\chi(M)} \geqslant 7} \leqslant 0.$$

Hence the first factor has to be nonpositive, so

$$\boxed{\chi_{\mathrm{CH}}(G) \leqslant \frac{7+\sqrt{49-24\chi(M)}}{2}.}$$

This also shows that the maximum in (4) above exists.

Case 2: $\chi(M) = 1$**.** Then $M \cong \mathbb{R}\mathrm{P}^2$. From (8) we get

$$\chi_{\mathrm{CH}}(G) - 1 \leqslant 6 - \frac{6}{n} < 6.$$

As $\chi_{\mathrm{CH}}(G)$ has to be an integer, this implies that

$$\chi_{\mathrm{CH}}(G) \leqslant 6 = \frac{7+\sqrt{49-24\cdot 1}}{2},$$

so we get the same bound as before.

Case 3: $\chi(M) = 2$**.** Then $M \cong \mathbb{S}^2$. From (8) we obtain

$$\chi_{\mathrm{CH}}(G) - 1 \leqslant 6 - \frac{12}{n} < 6,$$

so $\chi_{\mathrm{CH}}(G) \leqslant 6$. This is the six colour theorem for maps on the sphere or the plane.

Case 1 and 2: $\chi(M) \leqslant 1$**.** Since

$$\chi_{\mathrm{CH}}(G) \leqslant \frac{7+\sqrt{49-24\chi(M)}}{2},$$

holds for every *critical* graph embedded on M, this bounds holds for *all* $G \hookrightarrow M$. Thus

$$\chi_{\mathrm{CH}}(M) = \max_{P:P \text{ map on } M} \chi_{\mathrm{CH}}(P) = \max_{G \hookrightarrow M} \chi_{\mathrm{CH}}(G)$$

is well-defined.

Theorem I.7.8: Heawood's bound for colourings (1890) [Hea90]

Let M be a surface with $\chi(M) \neq 2$ (that is, $M \ncong \mathbb{S}^2$). Then

$$\chi_{\mathrm{CH}}(M) \leqslant \left\lfloor \frac{7+\sqrt{49-24\chi(M)}}{2} \right\rfloor.$$

Heawood conjectured that this bound was optimal, but couldn't prove it ("Heawood's conjecture").

We have also proved the following theorem.

Theorem I.7.9: Six colour theorem

We have $\chi_{\mathrm{CH}}(\mathbb{S}^2) \leqslant 6$.

But we can do even better.

Theorem I.7.10: Five colour theorem

We have $\chi_{\mathrm{CH}}(\mathbb{S}^2) \leqslant 5$, that is, every planar graph G can be coloured with five colours.

Proof. Let G be a planar graph, without loss of generality we can assume that G is connected has at least $n \geqslant 5$ vertices.

Proceeding by induction, we assume that every planar graph with fewer than n vertices can be coloured with five colours.

For the average vertex-degree of G we have

$$d(G) = \frac{2f_1}{n} \overset{I.7.4}{\leqslant} \frac{2(3n-6)}{n} = 6 - \frac{12}{n} < 6.$$

Hence there exists a vertex v of G with degree at most five. Then $H := G - v$ has a five-colouring $c\colon V(H) \to \{1, \dots, 5\}$ by the induction hypothesis.

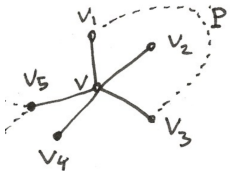


Fig. 75: In case 2, there is a vertex v in the planar connected graph G with degree five.

Case 1: The neighbours of v are coloured with at most four colours. We can use the free colour to colour v.

Case 2: v has five neighbours $(v_i)_{i=1}^5$ that are coloured differently. For $i, j \in \{1, \dots, 5\}$, let $H_{i,j}$ be the subgraph induced by the colours i and j (that is, the subgraph of H containing all vertices coloured i or j and the edges connecting them). Let C_1 be the component of $H_{1,3}$ than contains v_1.

Case 2.1. $v_3 \notin C_1$. Then we can swap colours 1 and 3 in the component C_1. Then v_1 and v_3 have colour 3 and v can be coloured with 1.

Case 2.2. $v_3 \in C_1$. As $H_{1,3}$ contains some v_1-v_3-path P. By Jordan's curve theorem, the circle vv_1Pv_3v separates v_2 and v_4; they lie in different components of H_{24}. As in Case 2.1, we can now swap colours in one of the components. □

How good is theorem I.7.8 for a surface $M \ncong \mathbb{S}^2$? It is the best possible, except for the Klein bottle.

For what kind of critical graphs? For complete graphs K_n.

The thread problem (dt. Fadenproblem) is: given n, what is the smallest number $\gamma(n)$ such that on an (orientable) surface of genus $\gamma(n)$ there are n points that can pairwisely be projected by curves that do not intersect each other. A reformulation is: what is $\gamma(n)$ so that K_n embeds on the (orientable) surface of genus $\gamma(n)$.

thread problem

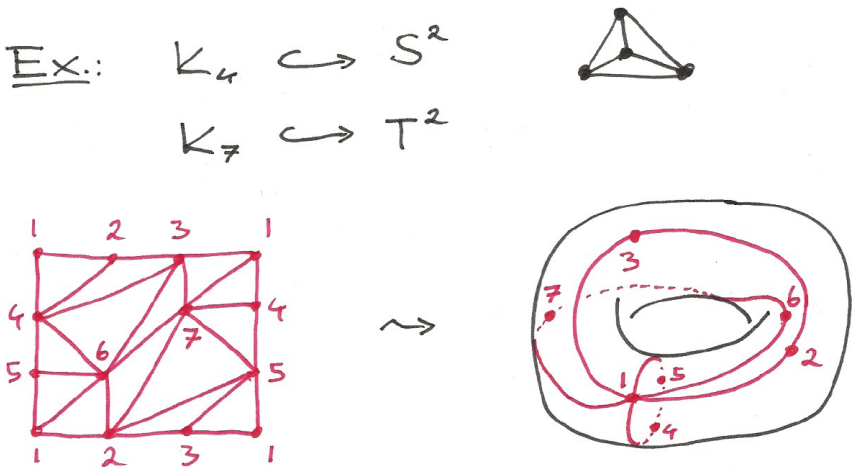


Fig. 76: We have $\gamma(4) = 0$, as $K_4 \hookrightarrow \mathbb{S}^2$ and $\mathbb{S}^2$ has genus zero. The bottom image shows that $K_7 \hookrightarrow T^2$ and this configuration induces the unique minimal triangulation of T^2. (In the bottom right image, the interior, non-diagonal edges are not included, for sake of readability.)

Theorem I.7.11: Ringel, Youngs (1968) [RY68]

We have

$$\gamma(n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

for $n \geqslant 3$.

For example, $\gamma(5) = \gamma(6) = 1$ and $\gamma(4) = \gamma(3) = 0$.

Theorem I.7.12: Map colour theorem (1968) [RY68]

For any surfaces $M \notin \{\mathbb{S}^2, \text{Klein bottle}\}$ the following are equivalent

1. There is an embedding $K_n \hookrightarrow M$.
2. $\chi(M) \leqslant \frac{n(7-n)}{6}$.
3. $n \leqslant \frac{7+\sqrt{49-24\chi(M)}}{2}$.
4. $\binom{n-3}{2} \leqslant 3\big(2-\chi(M)\big)$.

(For the Klein bottle, 1 is equivalent to $n \leqslant 6$.) If equality holds in the above, then the embedding $K_n \hookrightarrow M$ defines a neighbourly triangulation of M.

In summary: we have two Heawood bounds, one for vertex-minimal triangulations: $n \geqslant \left\lceil \frac{7+\sqrt{49-24\chi(M)}}{2} \right\rceil$ and one for colourings of surfaces: $\chi_{\text{CH}}(M) \leqslant \left\lfloor \frac{7+\sqrt{49-24\chi(M)}}{2} \right\rfloor$ (for $M \neq \mathbb{S}^2$; it also holds for $M = \mathbb{S}^2$ by the Four Colour Theorem), which more generally holds for embeddings $K_n \hookrightarrow M$.

In the case of equality we have

$$n = \frac{7+\sqrt{49-24\chi(M)}}{2} \iff \chi(M) = \frac{n(7-n)}{6}.$$

We have neighbourly triangulations with

Case 1: M is orientable. $n \equiv 0, 3, 4, 7 \mod 12$

Case 2: M is non-orientable. $n \equiv 0, 1 \mod 3$.

09.06.2022

How can we obtain (an infinite series of) vertex-minimal triangulations?

- Enumeration.
- Local modification.
- Construction.

One type of local modification is bistellar flips (or: bistellar moves). bistellar flips

Bistellar flips

Bistellar flips is a tool to locally "improve" a triangulation, e.g. to get rid of flat tetrahedra or non-stable triangles. They are also relevant in physics (quantum gravity), because if the number of triangulations of the sphere is exponential or higher than exponential, it is relevant for the convergence of some methods. It can also be interesting to consider if a sequence of bistellar flips gives a "path" in the space of triangulations from one to another. What are those?

Bistellar flips are local modifications of a surface that do not change the topological type:

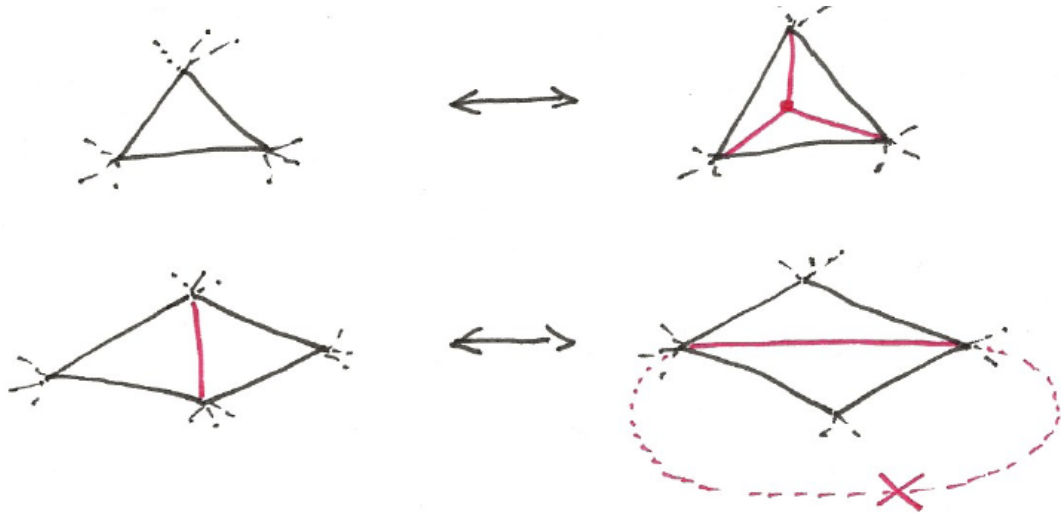


Fig. 77: **Top:** Adding a midpoint of a triangle is a bistellar operation, but removing it is not possible if the surface is a tetrahedron. **Bottom:** Flipping a diagonal is a bistellar move, provided the dotted edge does not already exist in the surface.

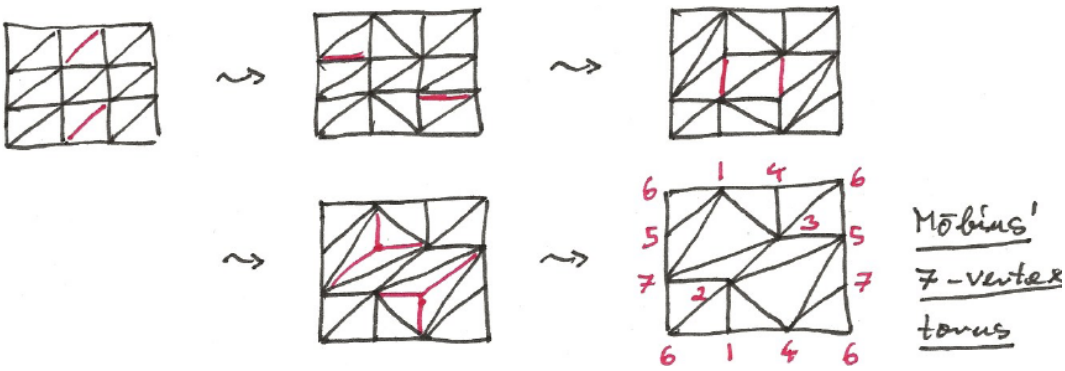


Fig. 78: For the nine-vertex triangulation we can preform flips to obtain MÖBIUS' seven-vertex triangulations. Our aim is to reduce the degree of some vertex from six to three, so we can perform the first step from above.

which vertex has degree 3?

There was a competition of the Paris Academy in 1858 on "perfectionner en quelque point impontant la théoremé géometricuqe des polyèdres" (perfect the geometric theory of polyhedra).

MÖBIUS' contribution on surfaces and polyhedra from 1861 contains the seven-vertex triangulation of the torus, which is combinatorially more symmetric than the nine-vertex triangulation: while the nine-vertex triangulation can be drawn on a square grid, the seven-vertex triangulation can be drawn on the triangular grid:

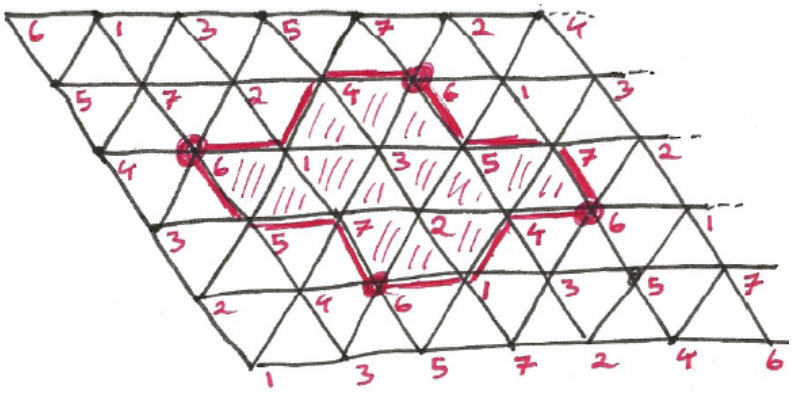


Fig. 79: This triangulation is invariant under the cyclic shift $(1,2,3,4,5,6,7)$, e.g. $(1,2,3,4,5,6,7)\cdot[1,2,4] = [2,3,5]$, where $[1,2,4]$ and $[2,3,5]$ are triangles, $(1,2,3,4,5,6,7)\cdot[2,3,5] = [3,4,6]$ and so on. The actual combinatorial automorphism group of the triangulation has size 42.

An infinite series of vertex-minimal triangulations for $n \equiv 7 \mod 12$

This subsection follows [Rin55, Sec. 2.3].

Our aim is to find examples for which the vertex-stars are mapped onto each other by a cyclic shift. (This is possible if and only if $n \equiv 7 \mod 12$.) why?

Construction principle. Start with a digraph (= directed graph with oriented edges ("arcs")) with oriented nodes.

The first diagram fully characterises the seven-vertex triangulation of the torus. Note that every vertex has degree three. Starting at the white node and following the turning directions, we obtain an induced cycle (in red). Every element $1, 2, \ldots, 6k+3$ of $\mathbb{Z}_{7+12k}$ is used as flow capacity for one of the arcs (in particular, there are $6k+3$ arcs and $2+4k$ nodes). KIRCHHOFF's law (flow conservation) holds: the sum of incoming flows (2+1 for the black one) is equal to the sum of the outgoing flows (3 for the black one).

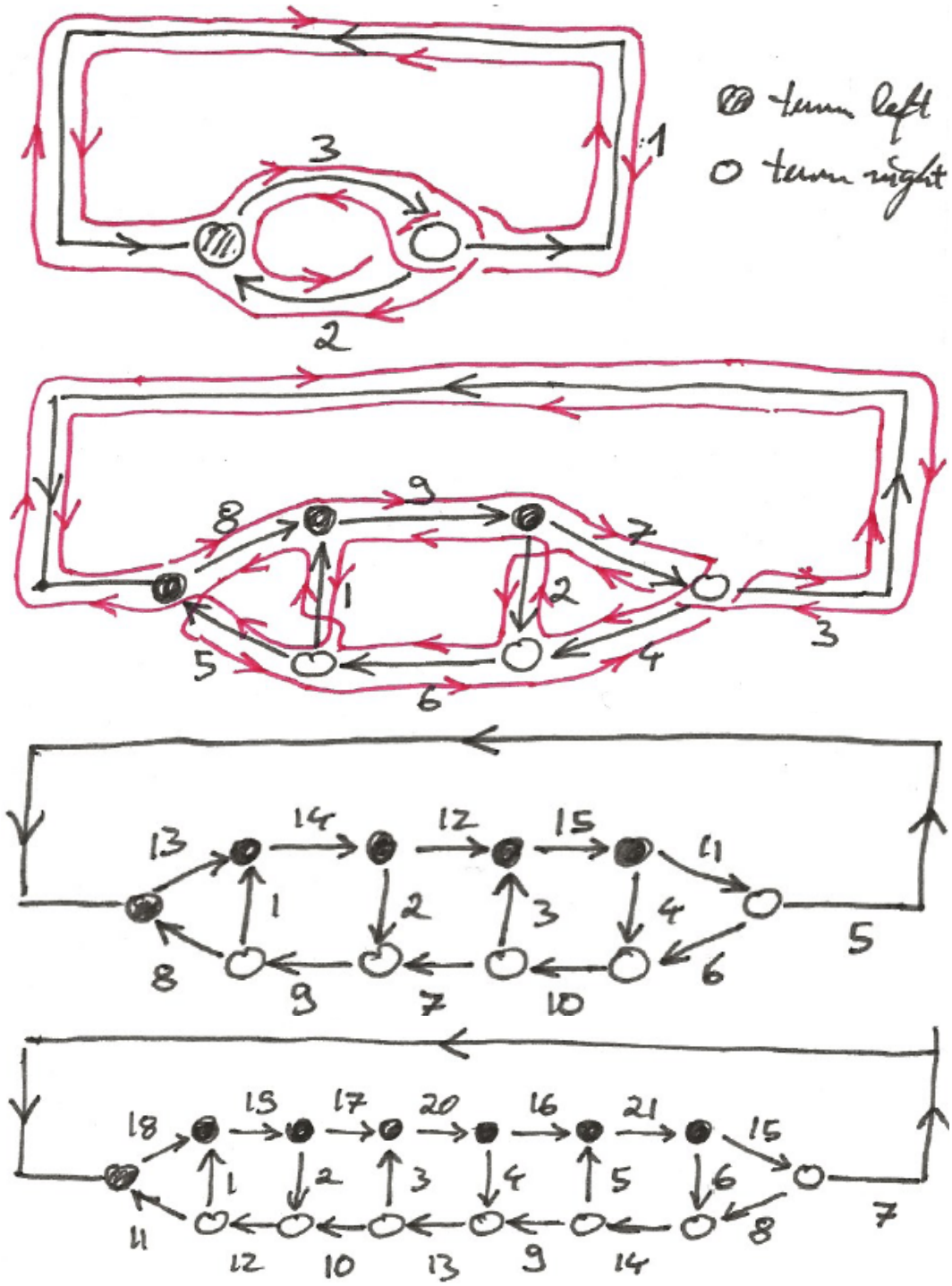


Fig. 80: Ladder-like graphs for $n \in \{7, 19, 31, 43\}$, that is, $k \in \{0, 1, 2, 3\}$. Note the alternating pattern of the arrows between the top black nodes and the white bottom nodes. The enumeration of the arrows follows a "leave one out" method, first traversing the lower part back and forth and then the top part.

Each diagram records/encodes the star of vertex 0 for a triangulated surface via the arc labellings of the induced red cycle. By the cyclic-shift property, knowing one star gives the entire triangulation.

For $n = 7$, the star of vertex zero can be deduced from the diagram. The red cycle traverse

the edges in the order (up to cyclic permutation) 1 3 2 $\overline{1}$, $\overline{3}$, $\overline{2}$, where $\overline{j}$ records a red arc that has opposite direction with respect to the underlying black arc. We interpret $\overline{j}$ as the the group inverse element of $j \in \mathbb{Z}_{7+12k}$. We then obtain the stars

0 : 1 3 2 6 4 5
1 : 2 4 3 0 5 6
2 : 3 5 4 1 6 0
3 : 4 6 5 2 0 1
4 : 5 0 6 3 1 2
5 : 6 1 0 4 2 3
6 : 0 2 1 5 3 4

Notice that the rows are cyclic shifts of each other and thus result in the seven-vertex triangulation

How?

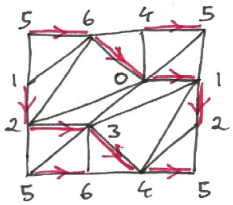


Fig. 81: The star of the vertex one in red.

but it is not...

Theorem I.7.13

This produces an infinite series of ($n \equiv 7 \mod 12$)-vertex triangulations of the torus.

of the torus??

The next simplest series uses $\mathbb{Z}_2 \times \mathbb{Z}_{2+6k}$ as symmetry group.

Part II: Computational Topology

II.1 The nerve of a covering

16.06.2022

Let S be a finite (or discrete) set of points in some metric space (X, d). Our aim is to associate a simplicial complex $K(S)$ with S.

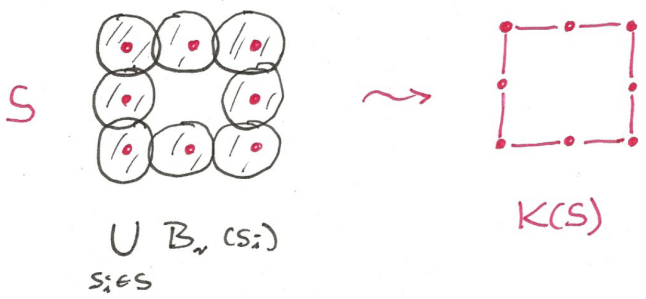


Fig. 82: We could for example place closed balls $B_r(s_k)$ with radius $r > 0$ around each of the points $s_k \in S$ and connect s_i with s_j if $B_r(s_i) \cap B_r(s_j) \neq \emptyset$, to obtain a simplicial complex $K(S)$.

In the following, let (X, T) be a topological space. We associate a collection of open sets U with the nerve complex $N(U)$, an ASC.

Definition II.1.1 (Nerve [Ale28])
The nerve (complex) of an open cover $U := (U_i)_{i \in I}$ is the ASC

nerve (complex)

$$N(U) := \left\{ J \subset I : |J| < \infty, \bigcap_{j \in J} U_j \neq \emptyset \right\}.$$

Remark II.1.2 (Finiteness of $N(U)$)
The nerve complex of a finite open cover is a finite ASC. Generally, $N(U)$ need not be a finite simplicial complex.

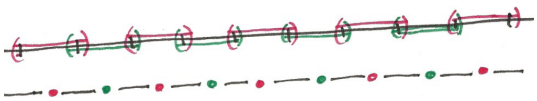


Fig. 83: The open cover $\big((k - \varepsilon, k + 1 + \varepsilon)\big)_{k \in \mathbb{N}}$ of $\mathbb{R}$ for some $\varepsilon \in (0, \frac{1}{2})$ and the resulting nerve complex.

Remark II.1.3 (Topology of $|N(U)|$)
The topology of $|N(U)|$ can be different from the topology of $X = \bigcup_{i\in I} U_i$.

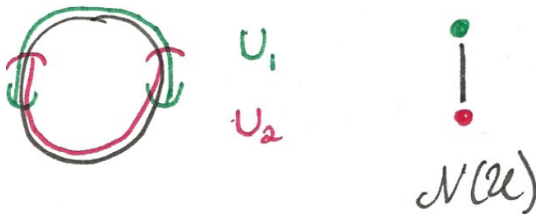


Fig. 84: If $X = \mathbb{S}^1$ and $\{U_1, U_2\}$ is the open cover shown here (left), then its nerve is $N(U) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$ (right).

Remark II.1.4 The nerve construction can be generalised to an arbitrary family of sets $(U_i)_{i\in I}$ not necessarily open. ○

We want $N(U)$ to "capture" the topology of U.

Definition II.1.5 (Homotopy (relative to a set))
Let $f, g\colon X \to Y$ be continuous maps. If there is a continuous map

$$F\colon X \times [0,1] \to Y$$

with $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$, then f is homotopic to g and we write $f \simeq_F g$ and call F a homotopy between f and g.

If $A \subset X$ and $F(a, \cdot) \equiv f(a) = g(a)$ for all $a \in A$, then F is a homotopy relative to A and we write $f \simeq_F g$ rel A.

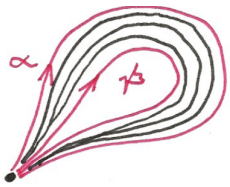
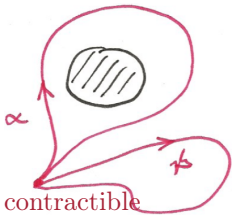


Fig. 85: A continuous deformation of curves $\alpha \simeq \beta$ relative to a point.

homotopy equivalent

Definition II.1.6 (Homotopy equivalent)
Topological spaces X and Y are homotopy equivalent and we write $X \simeq Y$ if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example II.1.7 (Homotopy equivalent spaces)
A disk is homotopy equivalent to a point and an annulus is homotopy equivalent to $\mathbb{S}^1$. ◇



contractible

Fig. 86: The curves α and β are *not* homotopic relative to their shared start- and end point, due to the hole in X.

Definition II.1.8 (Contractible)
A space X is contractible if $X \simeq \{\bullet\}$.

Definition II.1.9 (Good cover)
A cover $\{U_i\}_{i\in I}$ is good if for every $J \subset I$ the intersection $\bigcap_{j\in J} U_j$ is either empty or contractible.

Theorem II.1.1: Nerve lemma / theorem

Let U be a good open cover of X. Then $|N(U)| \simeq X$.

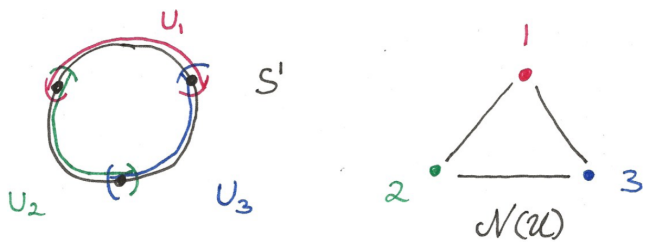


Fig. 87: A good open cover $\{U_1, U_2, U_3\}$ of $\mathbb{S}^1$ and the corresponding nerve complex $N(\{U_1, U_2, U_3\})$.

The version for simplicial complexes of this theorem is as follows.

Theorem II.1.2: Discrete nerve theorem (1948) [Bor48]

Let $K_1, \ldots, K_n$ be finite abstract simplicial complexes and $K := \bigcup_{k=1}^n K_k$. Let $A_k := |K_k|$ be their realisations for $k \in \{1, \ldots, n\}$. If the intersection $\bigcap_{j\in J} A_j$ is either empty or contractible for every $J \subset \{1, \ldots, n\}$, then

$$|N(\{A_1, \ldots, A_n\})| \simeq |K|.$$

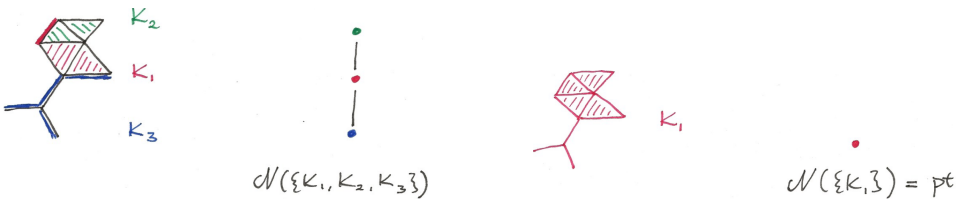


Fig. 88: Two examples for theorem II.1.2: subcomplexes $(K_j)_{j=1}^n$ and the corresponding nerve complex $N(\{K_1, \ldots, K_n\})$.

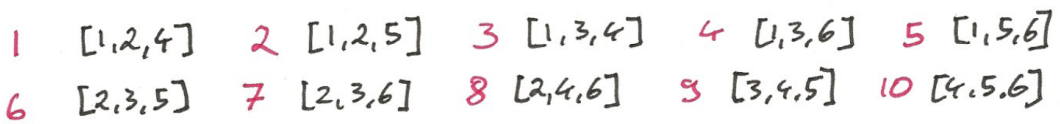
Definition II.1.10 (Standard cover)
Let K be a simplicial complex. The standard cover of K is the cover $\mathcal{F} = (F_i)_{i\in J}$ of K by its facets. We write $N(K) := N(\mathcal{F})$.

standard cover

By theorem II.1.2 we obtain

Corollary II.1.11 (Discrete nerve theorem)
We have $|N(K)| \simeq |K|$.

Example II.1.12 ($\mathbb{RP}^2_6$)
Consider the six-vertex triangulation of $\mathbb{RP}^2$ in figure 89. Its facets are



Then $N(\mathbb{RP}^2_6)$ is a four-dimensional simplicial complex on 10 vertices with the following six facets

1 [1,2,3,4,5] 2 [1,2,6,7,8] 3 [3,4,6,7,9]
4 [1,3,8,9,10] 5 [2,5,6,9,10] 6 [4,5,7,8,10]

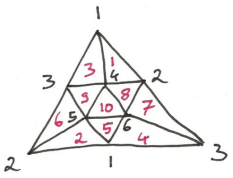


Fig. 89: A six-vertex triangulation of $\mathbb{RP}^2$.

Remark II.1.13 The nerve $N(K)$ can have larger dimension than K. ○

Repeated application of N

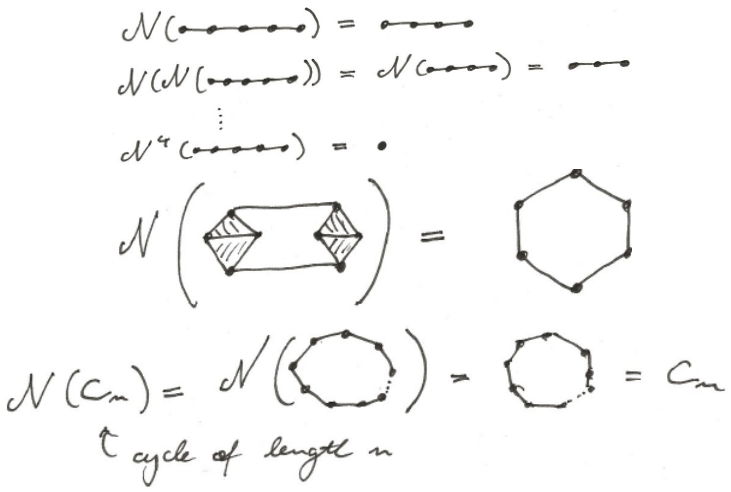


Fig. 90: Applying the nerve complex to a good cover of a simplicial complex over and over again simplifies a given complex, but progress can be slow and at a high computational cost.

21.06.2022

Definition II.1.14 (Taut complex (1970) [Grü70, p. 64, 70])
A simplicial complex is taut if every vertex is the intersection of the facets containing it.

taut

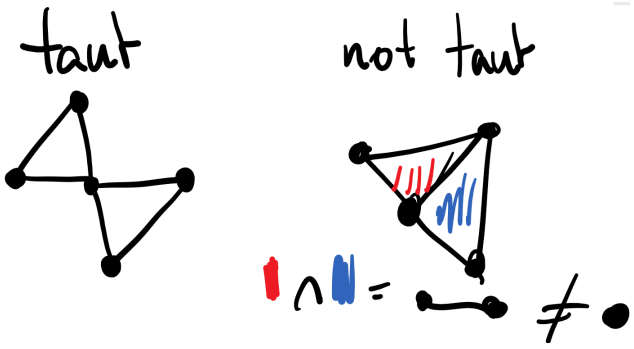


Fig. 91: A taut complex and a non-taut complex.

Lemma II.1.15 (Duality [Grü70, Thm. 4])
If K is taut, then $N(K)$ is taut and

$$K \cong N\big(N(K)\big),$$

where $\cong$ indicates (combinatorial?) isomorphy.

Hence if K is taut, then K and $N(K)$ are in duality in the sense that each is isomorphic to the the nerve of the other.

Proof. Let $\mathcal{F} = (F_j)_{j\in J}$ be the standard cover of K. The nerve $N := N(K)$ has one vertex j for every facet F_j of K, for $j \in J$.

For a vertex $v \in V(K)$, let $\Delta_v = \{j \in J : v \in F_j\}$. By the tautness of K we have

$$\bigcap_{j\in\Delta_v} F_j = \{v\}$$

and Δ_v is a facet of N. On the other hand let $j \in J$ be a facet index. Then $j \in \bigcap_{v\in V, j\in\Delta_v} \Delta_v$. Suppose there is some $j' \in J\backslash\{j\}$ such that $j' \in \bigcap_{v\in V, j\in\Delta_v} \Delta_v$. Then $F_j \subsetneq F_{j'}$, contradicting the maximality of F_j. Therefore, $\bigcap_{v\in V, j\in\Delta_v} \Delta_v = \{j\}$, so the nerve N is taut and thus $N\big(N(K)\big) = K$. □

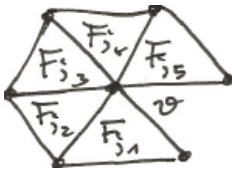


Fig. 92: Facets F_j containing a vertex v in a part of a simplicial complex, which is taut at v.

II.2 The Čech complex

21.06.2022

Let $S \subset X$ be a finite (discrete) set in some metric space (X, d). Our aim is to associate a simplicial complex $K(S)$ to S.

Definition II.2.1 (Čech complex)
For $S \subset X$ and $r > 0$ let $\left(B_r(s_j)\right)_{j \in J}$ be the collection of closed balls with radius r around the points $s_j \in S$. The Čech complex of S with radius r is

Čech complex

$$\check{\mathrm{C}}\mathrm{ech}_r(S) := \left\{ \sigma \subset S : \bigcap_{s \in \sigma} B_r(s) \neq \varnothing \right\}.$$

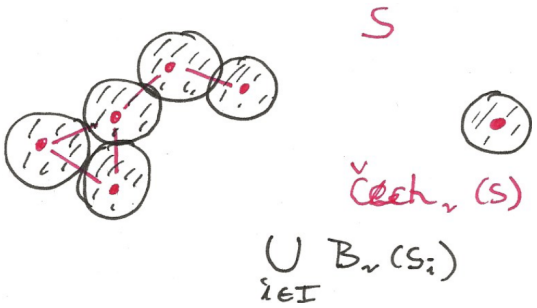


Fig. 93: The Čech complexes for two sets S.

Remark II.2.2 The Čech complex is the nerve complex of the cover $\left(B_r(s_j)\right)_{j \in J}$ by closed balls of their union $\bigcup_{j \in J} B_r(s_j)$. ○

Lemma II.2.3
We have $\bigcap_{s \in \sigma} B_r(s) \neq \varnothing$ if and only if $\sigma \subset S$ lies in a ball of radius r.

Proof. " $\Longrightarrow$ ": Let $x \in \bigcap_{s \in \sigma} B_r(s) \neq \varnothing$. Then $d(s, x) \leqslant r$ for all $s \in \sigma$, so $s \in B_r(x)$ for all $s \in \sigma$.

" $\Longleftarrow$ ": If there exists a $x \in X$ such that $\sigma \subset B_r(x)$, then $d(s, x) \leqslant r$ for all $s \in \sigma$ and thus $x \in \bigcap_{s \in \sigma} B_r(s)$, so $\bigcap_{s \in \sigma} B_r(s) \neq \varnothing$. □

Remark II.2.4 We have $\check{\mathrm{C}}\mathrm{ech}_{r_1}(S) \leqslant \check{\mathrm{C}}\mathrm{ech}_{r_2}(S)$ for $0 < r_1 \leqslant r_2$. ○

Remark II.2.5 For sufficiently large $r > 0$ we have $\check{\mathrm{C}}\mathrm{ech}_r(S) = \Delta_{|S|-1}$. In particular, for $S \subset \mathbb{R}^d$, $\dim(\check{\mathrm{C}}\mathrm{ech}_r(S))$ can be larger than d. ○

Remark II.2.6 If we *continuously* increase r from 0 to ∞, then we get a *discrete* family of nested Čech complexes. ○

II.2.1 Algorithmic construction of Čech$_r(S)$

The inputs are $S \subset \mathbb{R}^d$ and $r > 0$ and the output should by $\check{\mathrm{C}}\mathrm{ech}_r(S)$, where we call the decision problem

Data: $S \subset \mathbb{R}^d$, $r > 0$, $\sigma \subset S$
Result: yes or no (depending on whether $\sigma \in \check{\mathrm{C}}\mathrm{ech}_r(S)$).
Algorithm 1: Member$(\sigma, \check{\mathrm{C}}\mathrm{ech}_r(S))$

As $\sigma \in \check{\text{C}}\text{ECH}_r(S)$ if and only if σ is contained in a ball of radius r by lemma II.2.3, we need

Data: Finite set of points $\sigma \subset \mathbb{R}^d$
Result: Smallest ball B enclosing σ.
Algorithm 2: Miniball(σ) [Wel91]

Remark II.2.7 (Some first observations) The boundary of B, ∂B, contains at least two points of σ:

- If $\partial B \cap \sigma = \varnothing$, then one can shrink the radius.
- If $\partial B \cap \sigma = \{s_i\}$, then we can move the centre and then shrink that radius.

For $d = 2$ we have two cases:

1. if the boundary contains two antipodal points, we cannot shrink the ball any further.
2. if the boundary contains two non-antipodal points, we can move the balls centre such that those two points become antipodal and stay on the boundary and shrink the ball until the boundary contains a third point (three points uniquely determine a circle).

Hence Miniball is determined by the $k \in \{2, \dots, d+1\}$ points that lie on the boundary. In the case that $|\sigma| \gg d$, only few of the points of σ belong to the determining subset of k points, thus most of the points can be discarded. ○

First idea: obtain Miniball by randomised incremental construction

Here, incremental means that one adds points one by one and random means that one adds points in a random order, say, in the order $s_1, \dots, s_m$.

Let B_k be the smallest enclosing ball for $s_1, \dots, s_k$. Suppose we know B_{k-1} and we want to add s_k.

- If $s_k \in B_{k-1}$, then we set $B_k := B_{k-1}$.
- If $s_k \notin B_{k-1}$, then B_k must have s_k on its boundary.

The problem is that we can not just use the earlier boundary points to update, the points could look like in the rightmost picture below.

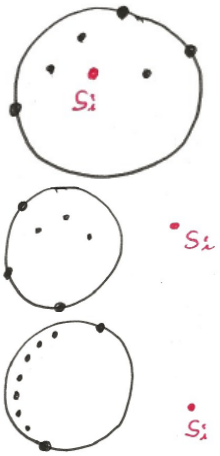


Fig. 94: The two cases.

As a solution, Miniball is randomised and recursive, taking two disjoint subset τ and v of

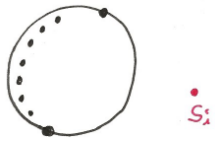
σ and returning the smallest enclosing ball containing all points of τ to be discarded and all points of τ to be discarded and all points of v as defining points on the boundary. It is initialised by calling Miniball$(\sigma, \varnothing)$.

if $\tau \neq \varnothing$ **then**
  compute the miniball B of v directly from the $k \in \{2, \dots, d+1\}$ points of v;
else
  choose random point $u \in \tau$;
  $B =$Miniball$(\tau \backslash \{u\}, v)$;
  if $u \notin B$ **then**
    $B =$Miniball$(\tau \backslash \{u\}, v \cup \{u\})$
  end
end
return B

Algorithm 3: Miniball(τ, v) [Wel91]

The expected run time is $O(n)$.

Example II.2.8 Consider the following data points and ball.



We start by calling Miniball$(\{s_1, \dots, s_5\}, \varnothing)$. As $s_1 \notin$ Miniball$(\{s_2, \dots, s_5\}, \varnothing) = \varnothing$, we call Miniball$(\{s_2, \dots, s_5\}, \{s_1\})$. As $s_2 \notin$ Miniball$(\{s_3, s_4, s_5\}, \{s_1\}) = \{s_1\}$, we call Miniball$(\{s_3, s_4, s_5\}, \{s_1, s_2\})$, which is the ball with s_1 and s_2 on the boundary, the correct solution.
As $s_3 \in$Miniball$(\{s_4, s_5\}, \{s_1, s_2\})$, we call Miniball$(\{s_4, s_5\}, \{s_1, s_2\})$. Since $s_4 \in$Miniball$(\{s_5\}, \{s_1, s_2\})$, we call Miniball$(\{s_5\}, \{s_1, s_2\})$. As $s_5 \in$Miniball$(\varnothing, \{s_1, s_2\})$, we call Miniball$(\{\varnothing\}, \{s_1, s_2\})$. $\diamond$

Bernd Gärtner's implementation of this algorithm in CGAL (the computational geometry algorithms library) takes 0.05 seconds for 10^6 points in $\mathbb{E}^5$.

Given $r > 0$, we have to decide which subsets $\sigma \in 2^S$ belong to $\check{\text{C}}\text{ECH}_r(S)$.

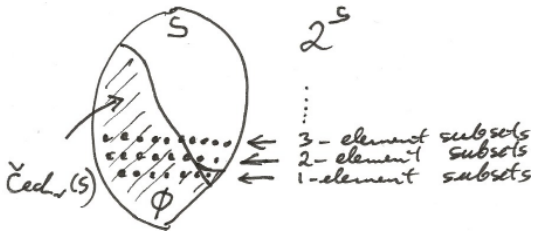


Fig. 95: ?? TODO

If we want to compute $\check{\text{C}}\text{ECH}_r(S)$ for all $r > 0$, then we compute Miniball$(\sigma, \varnothing)$ for all $\sigma \subset S$ and then order the face by the radii of their miniballs to obtain a discrete family of nested complexes.

If we want to compute $\check{\text{C}}\text{ECH}_r(S)$ for some fixed radius of for a bounded range of radii $r \leqslant R$, then we proceed incrementally by first deciding 2-element subsets, 3-element subsets, ...

II.3 Vietoris-Rips complex

28.06.2022

Leopold Vietoris (04.06.1891 - 09.04.2002) was the oldest living person in Austria (110 years and 10 months).

Let $S \subset \mathbb{R}^d$ be a finite set of data points and $\sigma \subset S$.

Definition II.3.1 (Diameter)
The diameter of σ is

$$\operatorname{diam}(\sigma) = \max\{|\sigma_1 - \sigma_2| : \sigma_1, \sigma_2 \in \sigma\}.$$

Example II.3.2 (Diameter)
The diameter of a set of three points is the length of the longest side of the triangle formed by these points. ◇

Definition II.3.3 (Vietoris-Rips complex)
Let $S \subset \mathbb{R}^d$ and $r > 0$. The Vietoris-Rips complex of S with respect to r is

$$\mathrm{VR}_r(S) := \{\sigma \subset S : \operatorname{diam}(\sigma) \leqslant 2r\}.$$

Vietoris-Rips complex

We can compute $\mathrm{VR}_r(S)$ as follows: for $n := |S|$ first compute all $\binom{n}{2}$ distances for pairs of vertices and then add two- and higher-dimensional simplices whenever $\operatorname{diam}(\sigma) \leqslant 2r$. (One can compute the list of facets in $O(n^2)$.)

Remark II.3.4 For sufficiently large $r > 0$, we have $\mathrm{VR}_r(S) = \Delta_{|S|-1}$, which is high-dimensional with with $2^{|S|}$ faces. ○

Definition II.3.5 (Minimal non-face)
Let K be an ASC. A subset $\sigma \subset \mathrm{Vert}(K)$ with $\#\sigma \geqslant 2$ is a minimal non-face / empty face / missing face of K if $\partial\sigma \subset K$ but $\sigma \notin K$, where $\partial\sigma := \{\mu \subset \sigma : \dim(\mu) + 1 = \dim(\sigma)\}$ (or $\partial\sigma$ is the ASC with those facets).

Example II.3.6
Consider a triangle (as an ASC K) without the "face". This face is a missing triangle of K. ◇

Definition II.3.7 (Flag complex)
An ASC is a flag complex if the minimal non-faces of K have only two elements, i.e. if $\sigma \subset \mathrm{Vert}(K)$ with $\#\sigma \geqslant 3$ such that $\partial\sigma \subset K$, then $\sigma \in K$.

flag complex

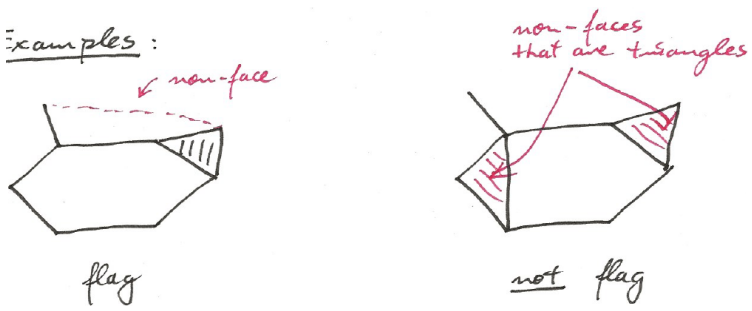


Fig. 96: The only non-edges on the left are edges, so the complex is flag. On the right, faces ares missing, so this complex is not flag.

Definition II.3.8 (Clique)
Let $G = (V, E)$ be a graph. A clique of G is a complete (every pair of distinct vertices is connected by a unique edge) subgraph of G.

clique

Definition II.3.9 (Clique complex)
The clique complex $C(G)$ of a graph $G = (V, E)$ is the ASC on V consisting of all subsets of V that are cliques of G.

clique complex

Since a subgraph of a clique is a clique, the clique complex is indeed an ASC.

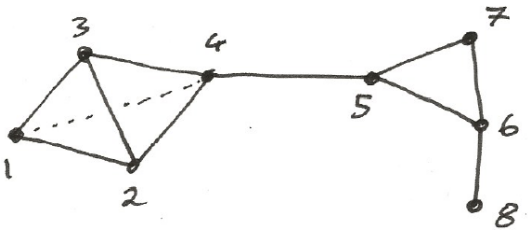


Fig. 97: Let G be the above graph. The facets of $C(G)$ are he maximal cliques of G, $[1, 2, 3, 4]$, $[5, 6, 7]$, $[4, 5]$ and $[6, 8]$.

It is not known whether finding maximal cliques has polynomial run time.

Lemma II.3.10
The clique complex of a graph is flag.

Proof. Let $G = (V, E)$ be a graph and $K := C(G)$ its clique complex. Let $\sigma \subset \mathrm{Vert}(K) = V$ with $k := \#\sigma \geqslant 3$ such that $\partial\sigma \subset K$. If all subsets $\mu \subset \sigma$ with $\#\mu = k - 1$ are contained in K, that is, cliques of G, then also their union $\sigma = \bigcup_{\mu\subset\sigma, \dim(\mu)<k} \mu$ is a clique, so $\sigma \in K$. □

Lemma II.3.11
A Vietoris-Rips *complex is flag.*

Proof. For a finite set $S \subset \mathbb{R}^d$ and $r > 0$, let $K := \mathrm{VR}_r(S)$. Take $\sigma \subset \mathrm{Vert}(K) = S$ with $\#\sigma \geqslant 3$ and $\partial\sigma \in K$. Then every proper subset $\mu \subsetneq \sigma$ has $\mathrm{diam}(\mu) \leqslant 2r$. Hence $|\mu_1 - \mu_2| \leqslant 2r$ for all $\mu_1, \mu_2 \in \sigma$, so $\mathrm{diam}(\sigma) \leqslant 2r$, so $\sigma \in K$. □

Remark II.3.12 All information of a flag complex is already contained in its 1-skeleton. ○

Lemma II.3.13 (Čech ⩽ VR)
We have $\check{\mathrm{C}}\mathrm{ech}_r(S) \leqslant \mathrm{VR}_r(S)$.

Proof. The complexes $\check{\mathrm{C}}\mathrm{ech}_r(S)$ and $\mathrm{VR}_r(S)$ have the same 1-skeleton because there is an edge between v_1 and v_2 in $\check{\mathrm{C}}\mathrm{ech}_r(S)$ if and only if $B_r(v_1) \cap B_r(v_2) \neq \emptyset$, that is, $|v_1 - v_2| \leqslant 2r$. Due to the flag property of $\mathrm{VR}_r(S)$, the latter contains all possible faces, yielding the statement. □

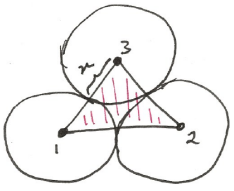


Fig. 98: The three balls intersect pairwise, but not all, so the face is a missing triangle. Since the sidelengths of the triangle are $2r$, this triangle is included in $\mathrm{VR}_r(S)$.

Lemma II.3.14 (Vietoris-Rips-Lemma)
We have $\mathrm{VR}_r(S) \leqslant \check{\mathrm{C}}\mathrm{ECH}_{\sqrt{2}r}(S)$.

Proof. Consider the standard d-simplex Δ_d in $\mathbb{R}^{d+1}$. The edges of Δ_d have length $\sqrt{2}$. Its barycentre $b := \frac{1}{d+1}(1,\dots,1) \in \mathbb{R}^{d+1}$ has $\|b\|_2 = \frac{1}{\sqrt{d+1}}$. The smallest d-sphere enclosing Δ_d is centred at b and has radius r_d with $r_d^2 = 1 - \|b\|_2^2 = \frac{d}{d+1}$. Hence $r_d = \sqrt{\frac{d}{d+1}} < 1$ (with $r_d \xrightarrow{d\to\infty} 1$).

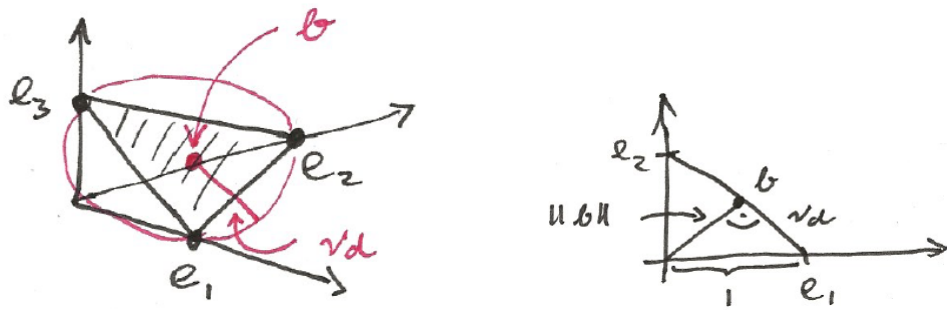


Fig. 99: **Left:** The barycentre b of the triangle $e_1e_2e_3$ and a circle in red with centre b and radius r_d. **Right:** The same situation in two dimensions.

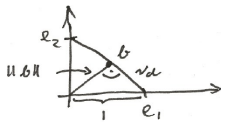


Fig. 100: If the triangle were equilateral, we get $\sqrt{2}$, otherwise it will be larger.

Any set $\sigma \subset S$ of $d+1$ or fewer points for which a d-ball of radius r_d is the smallest enclosing ball, has a pair of points of distance $\sqrt{2}$ or larger, that is, $\mathrm{diam}(\sigma) \geqslant \sqrt{2}$.

Every $\sigma \subset S$ with $\mathrm{diam}(\sigma) \leqslant \sqrt{2}$ has an enclosing ball with radius $\tilde{r} \leqslant r_d$, so $\sigma \in \check{\mathrm{C}}\mathrm{ECH}_{r_d}(S)$ by lemma II.2.3, which we can write as $\mathrm{VR}_{\frac{1}{\sqrt{2}}}(S) \leqslant \check{\mathrm{C}}\mathrm{ECH}_{r_d}(S)$. By multiplying with $\sqrt{2}r$ we get that $\mathrm{VR}_r(S) \leqslant \check{\mathrm{C}}\mathrm{ECH}_{\sqrt{2}rr_d}(S) \leqslant \check{\mathrm{C}}\mathrm{ECH}_{\sqrt{2}r}(S)$, where the last $\leqslant$ is due to $r_d < 1$ and remark II.2.4. □

Together we get

$$\check{\mathrm{C}}\mathrm{ECH}_r(S) \leqslant \mathrm{VR}_r(S) \leqslant \check{\mathrm{C}}\mathrm{ECH}_{\sqrt{2}r}(S).$$

$\check{\mathrm{C}}\mathrm{ECH}_r(S)$ is a nerve complex. $\mathrm{VR}_r(S)$ is flag and it is the clique complex of the 1-skeleton of $\check{\mathrm{C}}\mathrm{ECH}_r(S)$, that is

$$\mathrm{VR}_r(S) = C\big(\mathrm{Skel}_1\big(\check{\mathrm{C}}\mathrm{ECH}_r(S)\big)\big), \tag{9}$$

so empty faces of $\check{\mathrm{C}}\mathrm{ECH}_r(S)$ are filled in to yield $\mathrm{VR}_r(S)$.

By lemma II.3.14, both complex types roughly contain the same topological information about S.

The computational bottleneck is: for sufficiently large $r > 0$ both complex types become high-dimensional with up to $2^{|S|}$ faces, which makes it infeasible to set up the complexes and also to further compute topological invariants.

II.4 Voronoi diagrams and DELAUNAY triangulations

30.06.2022

Let $S \subset \mathbb{R}^d$ be a finite or discrete set of points.

DEFINITION II.4.1 (VORONOI CELL)
The VORONOI cell of a point $s_i \in S$ is the set of points $x \in \mathbb{R}^d$ closer to s_i that to any $s_j \neq s_i$:

$$V_{s_i} := \{x \in \mathbb{R}^d : \|x - s_i\| \leqslant \|x - s_j\|,\ s_j \in S\}$$

VORONOI cell

Due to the continuity of the norm, V_s is closed for any $s \in S$. Furthermore, it is convex as the intersection of (convex) halfspaces, whose boundary perpendicular to the bisector of s with any other $s_0 \in S$.

DEFINITION II.4.2 (VORONOI DIAGRAM)
The VORONOI diagram of S is $(V_s)_{s\in S}$, the collection of VORONOI cells of its points.

VORONOI diagram

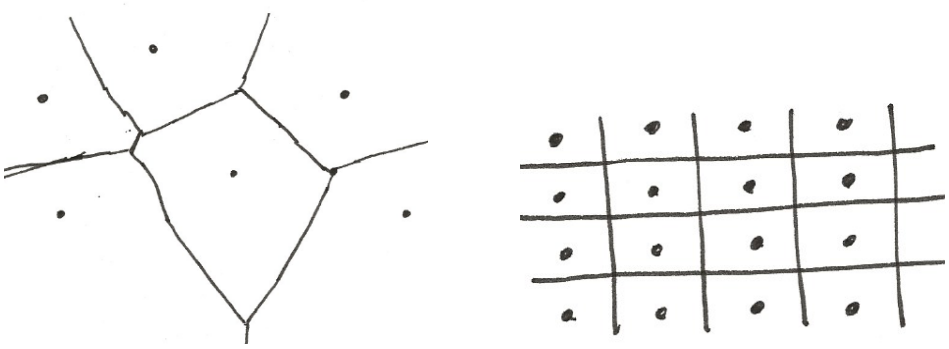


Fig. 101: **Top:** VORONOI diagram of finitely many points. **Bottom:** VORONOI diagram of a countable set of points.

DEFINITION II.4.3 (DELAUNAY COMPLEX)
The DELAUNAY complex of S is the nerve complex of the VORONOI diagram of S:

$$\mathrm{Del}(S) = \left\{\sigma \subset S : \bigcap_{s\in\sigma} V_s \neq \varnothing\right\}.$$

DELAUNAY complex

VORONOI cells are polyhedra (and thus convex) and thus their intersection is (convex and thus) contractible. Hence this is a good cover.

If four points lie on one circle, then all their VORONOI cells intersect and hence the resulting VORONOI graph is the complete graph K_4, which is not planar.

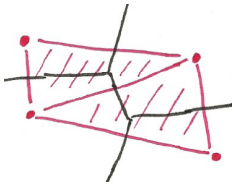


Fig. 102: DELAUNAY triangulation dual to the VORONOI diagram of four points consists of two triangles intersecting in an edge.

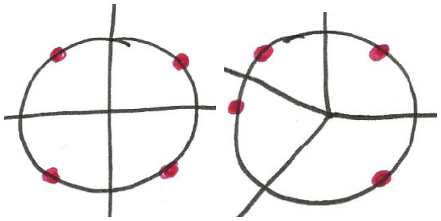


Fig. 103: Four points that line on a common circle.

We want to exclude this case (also in higher dimensions).

Definition II.4.4 (General position)
The set $S \subset \mathbb{R}^d$ is in general position if no $d+2$ of its points lie on a common $(d-1)$-sphere (or in a hyperplane).

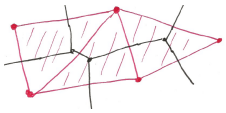
general position

For $d = 2$ this means that no four points are allowed to lie on a circle or on a line.

Lemma II.4.5 (General position $\implies$ $\dim(\mathrm{Del}(S)) \leqslant d$)
If S is in general position, then $\dim(\mathrm{Del}(S)) \leqslant d$.

Definition II.4.6 (Delaunay triangulation)
If S is in general position, then the Delaunay triangulation of S is the geometric realisation of $\mathrm{Del}(S)$ as the GSC on the points $S \subset \mathbb{R}^d$ consisting of the simplices of $\mathrm{Del}(S)$.

Delaunay triangulation



In this case, we have an embedding in the ambient $\mathbb{R}^d$.

Remark II.4.7 For a finite set $S \subset \mathbb{R}^d$ in general position, the Delaunay triangulation of S is a triangulation of $\mathrm{conv}(S)$. ○

The number of different triangulations of n-polygons (without adding new points) is C_n, the n-th Catalan number.

II.4.1 | Weighted Voronoi diagrams

We can modify the model to describe varying strengths of influence. One model is additively weighted Voronoi diagram: instead of $\|x - s\|$ we use $d_s(x) := \|x - s\| - w$ with positive weights $w > 0$. For points x of the boundary of two regions we have $d_{s_i}(x) = d_{s_j}(x)$, that is $\|x - s_i\| - \|x - s_j\| = w_i - w_j$. For $w_i \neq w_j$ this is the equation of a hyperbola as "bisector" between regions.

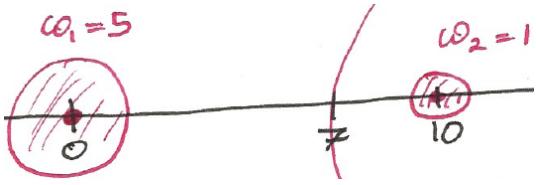


Fig. 104: For the weights $w_1 := 5$ and $w_2 := 1$ at the points $s_1 := (0,0)$ and $s_2 := (0,10)$, we have $d_{s_1}(x) = d_{s_2}(x)$ if and only if $\|x - s_1\| - \|x - s_2\| = w_1 - w_2 = 4$, that is $\sqrt{x_1^2 + x_2^2} - \sqrt{x_1^2 + (x_2 - 10)^2} = 4$, which describes the hyperbola with x-axis intersection at $(0, x_2)$, where x_2 fulfills $|x_2| - |x_2 - 10| = 4$, so $x_2 = 7$.

Remark II.4.8 If all weights are equal, then the additively weighted VORONOI diagram coincides with the standard VORONOI diagram. ○

The power distance is $d_s(x) := \|x - s\|^2 - w$ for positive weights $w > 0$ and the corresponding weighted VORONOI diagram is called power diagram or LAGUERRE-VORONOI diagram.

Example II.4.9 The weighted point s_i can be interpreted as a d-ball with radius $\sqrt{w_i}$.

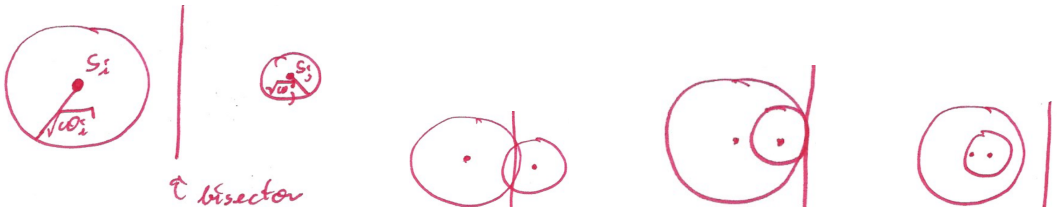
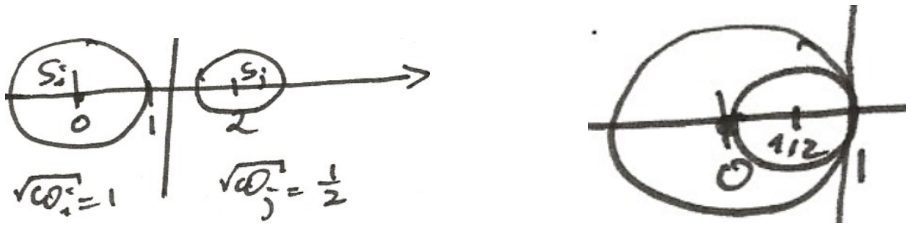


Fig. 105: Examples for where the bisector can lie.

We can also do explicit computations:



solving $\|x-0\|^2 - 1 = \|x-2\|^2 - \frac{1}{4}$ yields $x = \frac{19}{16}$ (left) and solving $\|x-0\|^2 - 1 = \|x-\frac{1}{2}\|^2 - \frac{1}{4}$ yields $x = 1$ (right). ◇

For multiplicative weighted VORONOI diagrams we define $d_{s_i}(x) := \frac{1}{w_i}\|x - s_i\|$ for $w_i > 0$. Then, bisectors are circular arcs (or line segments in the case that the two weights coincide).

DEFINITION II.4.10 (WEIGHTED VORONOI DIAGRAM, DELAUNAY COMPLEX) The weighted VORONOI cell of a point $s \in S$ is the set of points $x \in \mathbb{R}^d$ closest to s,

$$V_s := \{x \in \mathbb{R}^d : d_s(x) \leqslant d_{s_j}(x), s_j \in S\}.$$

The weighted VORONOI diagram of S is the collection of weighted VORONOI cells of its points.

The weighted DELAUNAY complex of S is the nerve complex of the weighted VORONOI diagram of S.

Remark II.4.11 In the case the all weights coincide, the multiplicatively weighted VORONOI diagram coincides with the standard one. ○

II.4.2 | Alpha complexes

05.07.2022

Alpha complexes are a nested family of subcomplexes of the DELAUNAY complex. They are similar to ČECH complexes but they are bounded in dimension and have a canonical realisation *if* S is in general position.

DEFINITION II.4.12 (ALPHA COMPLEX)
For $s \in S$ and $r > 0$ let $R_r(s) = B_r(s) \cap V_s$. For $r > 0$, the r-alpha complex of S is $\mathrm{Alpha}_r(S) := N\big((R_r(s))_{s\in S}\big)$. alpha complex

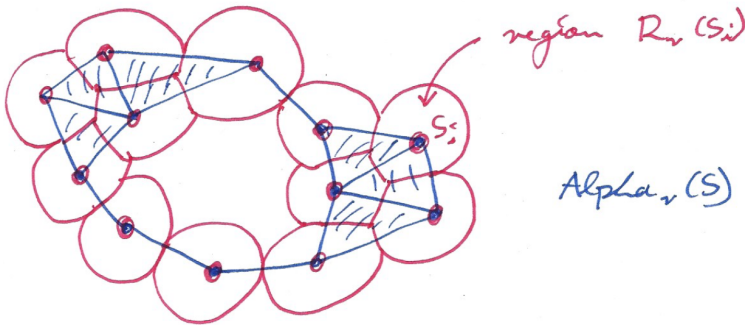


Fig. 106: An alpha complex and the balls $R_r(s)$.

Remark II.4.13 (Topological properties of alpha complexes) We have $R_r(s) \subset B_r(s)$. Hence $\mathrm{Alpha}_r(S) \leqslant \check{\mathrm{C}}\mathrm{ECH}_r(S)$. Furthermore, $\big(\mathring{R}_r(s)\big)_{s\in S}$ is a good open cover of its union. Furthermore, $\bigcup_{s\in S} B_r(s) \simeq |\mathrm{Alpha}_r(S)| \simeq |\check{\mathrm{C}}\mathrm{ECH}_r(S)|$ by the nerve theorem (and since the $\big(R_r(s)\big)_{s\in S}$ are closed and convex as intersection of closed convex sets and cover the same region as $(B_r(s))_{s\in S}$).

don't we have to take the interiors of both V_s and $B_r(s)$ such that the sets are open and we can apply the theorem to this good *open* cover?

Remark II.4.14 We can also set up weighted alpha complexes: as before for weighted VORONOI diagrams we use balls of radius $r_{s_i}^2 = w_{s_i}$. An application is the modelling of biomolecules, where the atoms are modelled as balls and their radius describes the range of the VAN DER WAALS interaction. ○

II.5 Simplicial homology

Recall from figure 2 that for a finite set $S \subset \mathbb{R}^d$ we wanted to compute invariants $I\big(K(S)\big)$ for associated simplicial complexes $K(S)$.

There are numerical invariants $I\colon \{\text{simplicial complexes}\} \to \mathbb{R}$ like the EULER characteristic and algebraic invariants $I\colon \{\text{simplicial complexes}\} \to \{\text{groups}\}$ like homology and the fundamental group.

Let K be a (finite, abstract) simplicial complex with ordered vertex set $V = \{v_1, \dots, v_n\}$, in particular we can without loss of generality choose $V = [n] := \{1, \dots, n\}$. Further let R be a commutative ring with 1.

DEFINITION II.5.1 (CHAIN MODULE)
The j-th chain module $C_j(K;R)$ of K is the set of formal linear combinations of j-faces of K with coefficients in R. chain module

The chain module is a free R-module. Here we need the 1 of R to define the linear combinations.

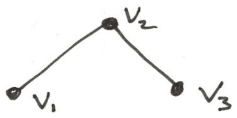


Fig. 107

Example II.5.2 (Chain modules) Consider the ASC K realised in figure 107. Its chain modules are

$$C_0(K;R) = \{\alpha \cdot v_1 + \beta v_2 + \gamma v_3 : \alpha, \beta, \gamma \in R\},$$

where e.g. v_1 means $\{v_1\}$, and

$$C_1(K;R) = \{\delta \cdot v_1 v_2 + \varepsilon \cdot v_2 v_3 : \delta, \varepsilon \in R\},$$

where e.g. $v_1 v_2$ means $\{v_1, v_2\}$. ◇

The j-faces of K form an R-basis of $C_j(K;R)$. Hence (for $j \geqslant 1$)

$$\partial_j(v_0 \dots v_j) := \sum_{k=0}^{j} (-1)^k v_0 \dots v_{k-1} v_{k+1} \dots v_j =: \sum_{k=0}^{j} (-1)^k v_0 \dots \widehat{v_k} \dots v_j, \tag{10}$$

where $v_0 \dots v_j$ (should rather be $v_{k_0} \dots v_{k_j}$ with $k_0 < \dots < k_j$) is an ordered j-simplex and the hat signifies omission, defines (by linear extension) an R-linear map from $C_j(K;R)$ to $C_{j-1}(K;R)$.

DEFINITION II.5.3 (BOUNDARY MAP)
The j-th boundary map of K is boundary map

$$\partial_j\colon C_j(K;R) \to C_{j-1}(K;R)$$

defined by (10).

Remark II.5.4 (Special choices of R) When R is a field, then $C_j(K;R)$ is a vector space (where one can use GAUSS elimination to simplify matrices). The choice $R = \mathbb{Z}_2$ is widely used in computational topology. The universal case is $R = \mathbb{Z}$, as the universal coefficient theorem from Topology tells us that results for other rings can be deduced from this case. ○

Example II.5.5 Let $\sigma := v_0v_1v_2 = \{v_0, v_1, v_2\}$ be an oriented / ordered triangle with $v_0 < v_1 < v_2$. Then

$$\partial_2(\sigma) = v_1v_2 - v_0v_2 + v_0v_1.$$

Further,

$$\begin{aligned}\partial_1(\partial_2(\sigma)) &= \partial_1(v_1v_2 - v_0v_2 + v_0v_1) = \partial_1(v_1v_2) - \partial_1(v_0v_2) + \partial_1(v_0v_1)\\ &= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0.\end{aligned}$$

◇

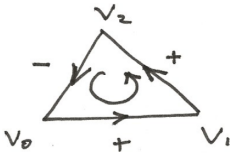


Fig. 108: An edge v_iv_j receives a plus if $v_i < v_j$ and a minus else.

Definition II.5.6 (∂_0)
Let $\partial_0 := 0$ by setting $C_{-1}(K;R) := R \cdot 0$ and $\partial_0(v) = 0$ for any vertex v of K.

Lemma II.5.7 (Most important lemma: $\partial^2 = 0$)
We have $\partial_q\partial_{q+1} = 0$ for all $q \geqslant 0$.

Proof. Consider $q+2$ vertices $v_0, \dots, v_{q+1}$ of a $(q+1)$-dimensional simplex. Then

$$\begin{aligned}\partial^2(v_0 \dots v_{q+1}) &= \partial\left(\sum_{k=0}^{q+1}(-1)^k v_0 \dots \widehat{v_k} \dots v_{q+1}\right) \overset{(\mathrm{L})}{=} \sum_{k=0}^{q+1}(-1)^k \partial\left(v_0 \dots \widehat{v_k} \dots v_{q+1}\right)\\ &= \sum_{k=0}^{q+1}(-1)^k \Bigg(\sum_{j=k+1}^{q+1}(-1)^{j-1} v_0 \dots \widehat{v_k} \dots \widehat{v_j} \dots\dots v_{q+1}\\ &\qquad + \sum_{k=0}^{q+1}(-1)^k \sum_{j=0}^{k-1}(-1)^j v_0 \dots \widehat{v_j} \dots \widehat{v_k} \dots\dots v_{q+1}\Bigg) = 0,\end{aligned}$$

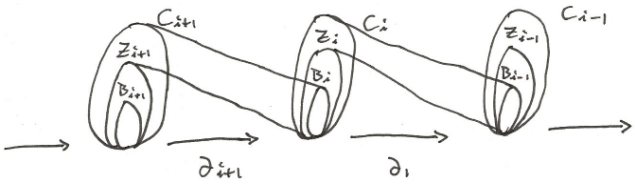
where we use that ∂ is linear in (L). Each ordered q-simplex occurs twice, but with opposite sign, hence the overall result is zero. □

Definition II.5.8 (Boundary module, module of cycles)
The j-th boundary module is $B_j(K;R) := \operatorname{im}(\partial_{j+1}) \subset C_j(K;R)$. — boundary module
The j-th cycle module is $Z_j(K;R) := \ker(\partial_j) \subset C_j(K;R)$. — cycle module

Since ∂_{j+1} and ∂_j are R-linear, $B_j(K;R)$ and $Z_j(K;R)$ are free R-submodules of the R-module $C_j(K;R)$. By lemma II.5.7,

$$B_j(K;R) \leqslant Z_j(K;R),$$

so we can take the quotient (do we need commutativity of R for this?).



Definition II.5.9 (Simplicial homology module)
The j-th homology module is — homology module

$$\boxed{H_j(K;R) := Z_j(K;R)/B_j(K;R).}$$

Remark II.5.10 (Special choices of R) If $R = \mathbb{Z}$, then H_j is an abelian group and if R is a field, then H_j is a vector space. ○

Remark II.5.11 Each finitely generated abelian group G has two uniquely determined subgroups F and T, where F is free and T (the torsion subgroup) is finite. ○

Definition II.5.12 (Betty number)
The q-th Betty number β_q of K is the free rank of $H_q(K;\mathbb{Z})$. Betty number

The Betty numbers are topological invariants.

Example II.5.13 For example, a homology group could be $H_j(K;\mathbb{Z}) \cong \mathbb{Z}^{104} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_5)$. In this case $\beta_j = 104$, $F = \mathbb{Z}^{104}$ and $T = \mathbb{Z}_2 \oplus \mathbb{Z}_5$. ◇

Theorem II.5.1: Euler characteristic and Betty numbers

For an n-dimensional simplicial complex K we have

$$\chi(K) = \sum_{k=0}^{n} (-1)^k f_k(K) = \sum_{k=0}^{n} (-1)^k \beta_k.$$

Definition II.5.14 (Homology class, homologous)
The homology class of $c \in Z_j(K;R)$ is $[c] := c + B_j(K;R) \in H_j(K;R)$. Two cycles $c, d \in Z_j(K;R)$ are homologous and we write $c \sim d$ if they belong to the same homology class: $c \in [d]$. homology class

Hence a homology class is represented by a cycle, which is not a boundary ("non-bounding cycle"), which one could see as a "hole" in the complex.

Example II.5.15 We have $\partial\Delta_3 \cong \mathbb{S}^2$, that is, the boundary of the 3-simplex (the tetrahedron with vertices $1, \ldots, 4$) is homeomorphic to $\mathbb{S}^2$.

Fig. 109: The tetrahedron with ordered triangles as faces.

We have

$$\delta_3(1234) = -123 + 124 - 134 + 234$$

To compute $H_2(\partial\Delta_3;\mathbb{Z})$ we write

∂_3	123	124	134	234
12	1	1	0	0
13	−1	0	1	0
14	0	−1	−1	0
23	1	0	0	1
24	0	1	0	−1
34	0	0	1	1

Adding the second and third row to the first one eliminates it. The first three rows can be eliminated, so (this is nonsense)

$$Z_3(\partial\Delta_3;\mathbb{Z}) = \ker(\delta_3) = \left\{ s \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} : s \in \mathbb{Z} \right\}$$

We interpret $(-1,1,-1,1)^\mathsf{T}$ as a linear combination of triangles $-123 + 124 - 134 + 234$.

We have $B_2(\partial\Delta_3;\mathbb{Z}) =$

To compute H_1 we observe that the k-th column in the boundary matrix δ_2 is a linear combination of the first three columns, that is

$$B_1 = \mathrm{im}(\delta_2) =)(($$

We have four vertices, six edges, four triangles and one 2-simplex, so $C_0(\partial\Delta_3;\mathbb{Z}) \cong \mathbb{Z}^4$, $C_1(\partial\Delta_3;\mathbb{Z}) \cong \mathbb{Z}^6$, $C_2(\partial\Delta_3;\mathbb{Z}) \cong \mathbb{Z}^4$ and $C_3(\partial\Delta_3;\mathbb{Z}) \cong \mathbb{Z}$.

To compute the 0-th homology we write

$$H_0(\partial\Delta_3;\mathbb{Z}) = Z_0(\partial\Delta_3;\mathbb{Z})/B_0(\partial\Delta_3;\mathbb{Z}) = \ker(\delta_0)/\operatorname{im}(\delta_1).$$

Since $\delta_0 \colon C_0 \to C_{-1} = R \cdot 0$, we have $\ker(\delta_0) = C_0(\partial\Delta_3;\mathbb{Z}) \cong \mathbb{Z}^4$.

We have $\delta_1(12) = 2-1$, $\delta_1(13) = 3-1$ etc. We have $(4-3)+(3-2) = 4-2$, $(3-1)-(2-1) = 3-2$ and $(4-3)+(3-1) = 4-1$, so $\operatorname{im}(\delta_1) \cong \mathbb{Z}^3$, implying $H_0(\partial\Delta_3;\mathbb{Z}) \cong \mathbb{Z}^4/\mathbb{Z}^3 \cong \mathbb{Z}$.

Alternatively, we can write

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and calculate that both its kernel and image have dimension 3 over $\mathbb{Z}$.

To compute the first homology we write

$$H_1(\partial\Delta_3;\mathbb{Z}) = Z_1(\partial\Delta_3;\mathbb{Z})/B_1(\partial\Delta_3;\mathbb{Z}) = \ker(\delta_1)/\operatorname{im}(\delta_2).$$

finish this!

Together we get the homology vector

$$H_*(\partial\Delta_3;\mathbb{Z}) := \big(H_0(\partial\Delta_3;\mathbb{Z}), H_1(\partial\Delta_3;\mathbb{Z}), H_2(\partial\Delta_3;\mathbb{Z})\big) = (\mathbb{Z}, 0, \mathbb{Z}).$$ ◇

another Homology calculation example for the ASC with facets 12, 14 and 234.

Example II.5.16 Let (G, E) be a graph with C connected components. Then

$$H_*(G;\mathbb{Z}) = \big(\mathbb{Z}; \mathbb{Z}^{|E|-(|V|-C)}\big),$$ ◇

because in each connected component $G_j = (E_j, V_j)$, $j \in \{1, \dots, C\}$ the number of edges which are in the graph, but not in a spanning tree of G_j is $z_j := |E_j| - (|V_j| - 1)$. Hence for the whole graph this number is $\sum_{j=1}^{C} z_j = |E| - (|V| - C)$, as $|E| = \sum_{j=1}^{C} |E_j|$ and similarly for $|V|$. The edges of the spanning tree can be contracted and this contraction gives a homotopy equivalence.

For example,

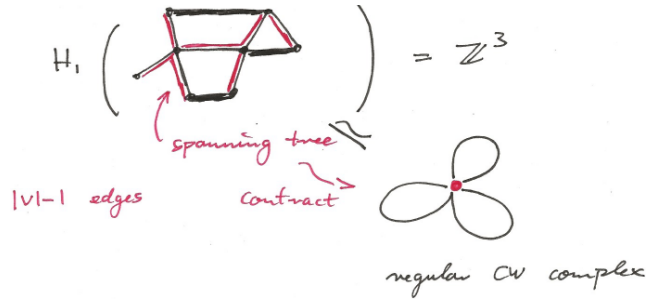


Fig. 110: Contracting the edges of a spanning tree gives a homotopy equivalence to a regular CW complex, from which one can determine the first homology of that graph.

Example II.5.17 (Homology of the sphere)
We have $H_*(\mathbb{S}^d;\mathbb{Z}) = (\mathbb{Z}, 0, \dots, 0, \mathbb{Z})$. ◇

II.6 SMITH Normal Form

12.07.2022

If R is a field, then $B_k = \operatorname{im}(\delta_{k+1})$ and $Z_k = \ker(\delta_k)$ and thus H_k can be computed via GAUSSian elimination. But GAUSSian elimination might fail (if the space has torsion) for $R = \mathbb{Z}$ in case multiplicative inverses are needed for the elimination.

Instead of multiplicative inverses (it suffices if we can cancel rows) we can instead utilise the

Lemma II.6.1 (Lemma of BÉZOUT)
For all $a, b \in \mathbb{Z}$ there exist $s, t \in \mathbb{Z}$ such that $\gcd(a, b) = s \cdot a + t \cdot b$.

Remark II.6.2 If d is a common divisor of a and b, that is, there exist $x, y \in R$ with $d = a \cdot x$ and $d = b \cdot y$ and every common divisor of a and b divides d, then d is called a greatest common divisor of a and b and we write $d = \gcd(a, b)$. In general rings, the gcd of two elements may not exist. ○

DEFINITION II.6.3 (PRINCIPAL IDEAL DOMAIN)
Let R be a commutative ring with $1 \neq 0$. Then R is a principal ideal domain (PID) if

- $a \cdot b \neq 0$ for $a, b \in R \backslash \{0\}$ (all elements of R are regular),
- every ideal $I \subset R$ is generated by some $x \in R$, written as $I = R \cdot x$.

principal ideal domain

Remark II.6.4 An ideal is a subset of a ring containing 0, which is closed with respect to addition and multiplication with elements from the ring. ○

Remark II.6.5 The ring of matrices is not a PID, because e.g. there exist nilpotent matrices and so the first condition in Definition II.6.3 is not fulfilled. ○

Example II.6.6 (PID)
The integers $\mathbb{Z}$ form a PID. ◇

Remark II.6.7 The Lemma of BÉZOUT holds in every PID. ○

II.6.1 Homology computation via the Smith Normal Form

Let R be a PID, e.g $R = \mathbb{Z}$ and $A \in R^{m \times n}$, e.g. $A = \delta_k$. Then there is (we will prove this by using an algorithm to construct them) a regular $m \times m$ matrix S and a regular $n \times n$ matrix T such that

$$SAT = \left(\begin{array}{ccc|c} \alpha_1 & 0 & 0 & \\ 0 & \ddots & 0 & \mathbf{0} \\ 0 & 0 & \alpha_r & \\ \hline & \mathbf{0} & & \mathbf{0} \end{array}\right) \in R^{m \times n} \tag{11}$$

with $\alpha_j \mid \alpha_{j+1}$ for all $j \in \{1, \ldots, r-1\}$ and $\alpha_k \neq 0$ for all $k \in \{1, \ldots, r\}$. This is the SMITH normal form (SNF) of A.

SMITH normal form

Remark II.6.8 (Invariant factors) The α_k are the invariant factors of A. We have $\alpha_k = \frac{d_k(A)}{d_{k-1}(A)}$, where $d_k(A)$ is the gcd of all $k \times k$-minors (subdeterminants) of A and $d_0(A) := 1$ (here we need that R has a unit elements $1 \neq 0$). For computation, this structural result is useless, because the computation of d_k has exponential running time in k. ○

Remark II.6.9 (How SNF determines the homology)
Let (11) be the SNF of $\delta_{k+1}\colon C_{k+1}(K;\mathbb{Z}) \to C_k(K;\mathbb{Z})$ and $s := \operatorname{rank}\big(C_k(K;\mathbb{Z})\big) - \operatorname{rank}(\delta_k)$, where $\operatorname{rank}\big(C_k(K;\mathbb{Z})\big)$ is the number of basis elements of $C_k(K;\mathbb{Z})$ and $\operatorname{rank}(\delta_k)$ can be read off from the SNF of δ_k, then

$$H_k(K;\mathbb{Z}) = \mathbb{Z}^{r-s} \oplus \bigoplus_{j=1}^{r} \mathbb{Z}_{\alpha_j}\,.$$

○

Remark II.6.10 The largest torsion coefficient of a simplicial complex grows as $\Theta(2^{n^2})$, even though the SNF can be computed in polynomial time (so it is not strongly polynomial). The proof uses randomised methods, but there is a construction using HADAMARD matrices (which have entries ± 1 and determinant $n^{\frac{n}{2}} \in \Theta(2^{n\log(n)})$) such that the greatest torsion coefficient is in $\Theta(2^{n\log(n)})$. ○

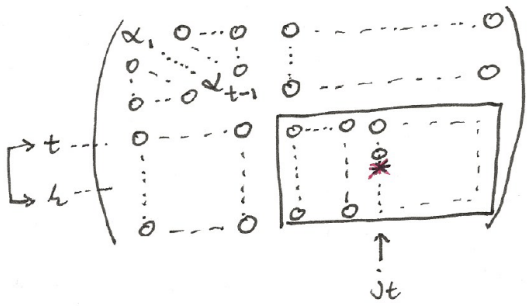
II.6.2 | Algorithm for SNF computation

Input: $A \in R^{m\times n}$.
Output: S, T regular matrices such that SAT is in SNF.

We modify A successively by row *and* column operations (in mixed order) implemented by regular square matrices $S_1, \dots, S_k$ and $T_1, \dots, T_\ell$ such that $S = S_k \dots S_1$ and $T = T_1 \dots T_\ell$.

Initialise: $S = \mathrm{id}_m$, $T = \mathrm{id}_n$, $SAT = A$.

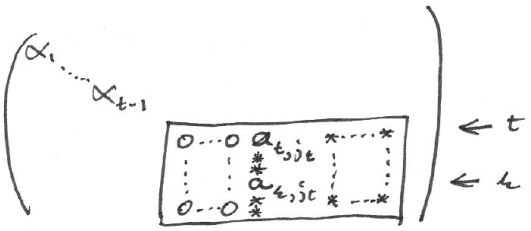
Step 1: Proceed recursively and choose pivot. Suppose



where $j_t \geqslant t$ is the smallest column index with a non-zero entry.

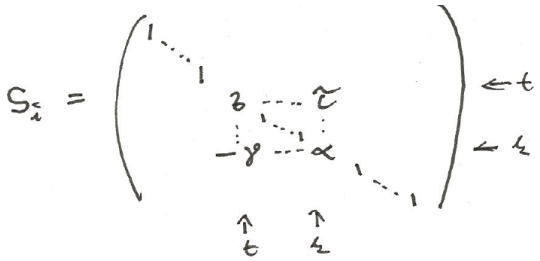
If the entry $a_{t,j_t} = 0$, then there is a smallest $k > t$ with $a_{k,j_t} \neq 0$ (the red entry above). We then exchange the t-th and k-th row.

Then

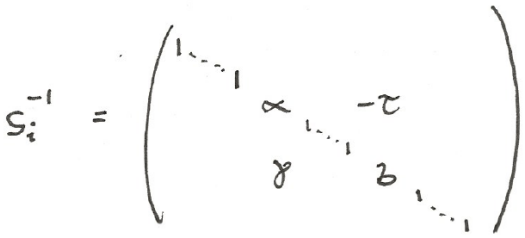


Step 2: Improve the pivot (should be the smallest possible). If for $a_{t,j_t} \neq 0$ there is an entry $a_{k,j_t} \neq 0$ with with (not $a_{t,j_t} \mid a_{k,j_t}$), then let $\beta := \gcd(a_{t,j_t}, a_{k,j_t})$. By lemma II.6.1 there exist $\sigma, \tau \in R$ such that $\beta = a_{t,j_t}\sigma + a_{k,j_t}\tau$. Via row operations (by left multiplication by a regular matrix S_i) we replace the t-th row by $\sigma \cdot (t\text{-th row}) + \tau \cdot (k\text{-th row})$.

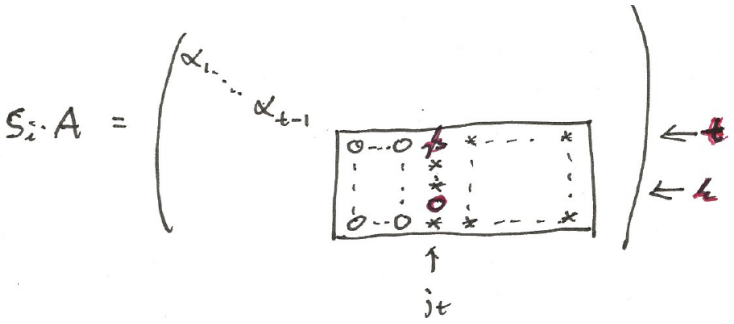
For $\alpha = \frac{a_{t,j_t}}{\beta}$ and $\gamma = \frac{a_{k,j_t}}{\beta}$ we have $\sigma \cdot \alpha + \tau \cdot \gamma = 1$. The matrix



is invertible with inverse

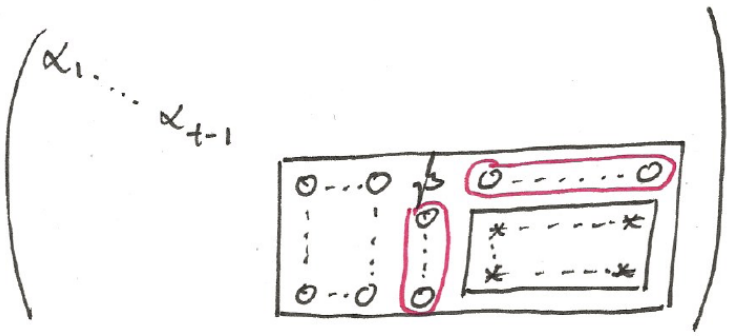


and



We repeat this step until no further improvement is possible. Since $\beta | a_{t,j_t}, a_{k,j_t}$, the procedure terminates.

Step 3: Eliminate further non-zero entries in row t and column j_t via row and column operations.



For row entries: add multiples of row t to remaining rows. For column entries: repeat step 2 for columns instead of rows.

Warning: Column operations might cause new row entries in column j_t and we possibly have to repeat row operations. However, since β has only finitely many prime factors, the process eventually becomes stationary (e.g β becomes 1).

Step 4: Move zero-columns to the right. We obtain the block matrix

$$\begin{pmatrix} \operatorname{diag}(\alpha_1, \dots, \alpha_t) & 0 \\ 0 & \star \end{pmatrix}$$

We still have to take care that $\alpha_{t-1} \mid \alpha_t$.

First of all, an example.

Example II.6.11 Consider $A = \left(\begin{smallmatrix} 6 & 1 \\ 4 & 3 \end{smallmatrix}\right)$. Let $\beta = \gcd(6,4) = 2 = 6 \cdot 1 + 4 \cdot (-1)$. With $\alpha = \frac{6}{2} = 3$ and $\gamma = \frac{4}{2} = 2$ we get

$$S_1 = \begin{pmatrix} \sigma & \tau \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

and thus

$$S_1 A = \begin{pmatrix} 2 & -2 \\ 0 & 7 \end{pmatrix}.$$

Adding the first to the second column via $T_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ we get $S_1 A T_1 = \operatorname{diag}(2,7)$. ◇

Step 5: ensure divisibility. If $\alpha_i \nmid \alpha_{i+1}$, we add the $(i+1)$-th column to the i-th column and apply row operations to "repair" α_i by replacing it with $\beta = \gcd(\alpha_i, \alpha_{i+1})$.

Example II.6.12 (Continued) Recall that $S_1 A T_1 = \operatorname{diag}(2,7)$. Adding the second column to the first via $T_2 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ yields $S_1 A T_1 T_2 = \left(\begin{smallmatrix} 2 & 0 \\ 7 & 7 \end{smallmatrix}\right)$. Let $\beta = \gcd(2,7) = 1 = 2 \cdot (-3) + 7 \cdot 1$. Hence $\alpha = \frac{2}{1} = 2$ and $\gamma = \frac{7}{1} = 7$. Hence $S_2 = \left(\begin{smallmatrix} -3 & -1 \\ -7 & 2 \end{smallmatrix}\right)$ and we get $S_2 S_1 A T_1 T_2 = \left(\begin{smallmatrix} 1 & 7 \\ 0 & 14 \end{smallmatrix}\right)$. Finally, with $T_3 = \left(\begin{smallmatrix} 1 & -7 \\ 0 & 1 \end{smallmatrix}\right)$ we get $S_2 S_1 A T_1 T_2 T_3 = \operatorname{diag}(1,14)$, which is the SNF of A. ◇

Example II.6.13 (Homology of $\mathbb{R}\mathrm{P}^2$ via SNF)
Consider $\mathbb{R}\mathrm{P}^2$ as a cell complex (CW complex) with one vertex (0-cell), one edge (1-cell) and one disk (2-cell). We have $\delta_2(\text{this disk}) = 2 \cdot (\text{this edge})$ and the (1-)boundary of C_1 is $0 \cdot C_0$.

Fig. 111: A CW-complex representation of $\mathbb{R}\mathrm{P}^2$.

Hence $Z_2(\mathbb{R}\mathrm{P}^2; R) = \ker(\delta_2) = \{0\}$ and $B_2(\mathbb{R}\mathrm{P}^2; R) = \operatorname{im}(\delta_3) = \{0\}$ (since there are no 3-faces) and thus $H_2(\mathbb{R}\mathrm{P}^2; R) = \{0\}$. Further $Z_0(\mathbb{R}\mathrm{P}^2; R) = \ker(\delta_0) = R$ and $B_0(\mathbb{R}\mathrm{P}^2; R) = \{0\}$ and thus $H_0 = R$. In the case of $R = \mathbb{Z}$ from the Smith Normal Form we get $H_1 = \mathbb{Z}^{(1-0)-1} \oplus \bigoplus_{j=1}^{1} \mathbb{Z}_2 = \mathbb{Z}_2$ since δ_2 is already in SNF. Alternatively, $H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) = \mathbb{Z} / \mathbb{Z}_2 = \mathbb{Z}_2$. ◇

II.7 Persistent homology

Definition II.7.1 (Filtration)
A filtration of a (e.g. simplicial) complex K is a nested sequence of subcomplexes $(K^j)_{j=0}^m$ with

$$\varnothing = K^0 \subset K^1 \subset K^2 \subset \ldots \subset K^m = K.$$

A complex K together with a filtration $(K^j)_{j=0}^m$ is a filtered complex.

filtration

Example II.7.2 (Filtration)
The Čech complex $\check{\text{C}}\text{ech}_r(S)$, the Vietoris-Rips complex $\text{VR}_r(S)$ and the alpha complex $\text{Alpha}_r(S)$ (for all r) are filtrations. Another example is filtration by dimension, that is $K^b := \text{Skel}_{b+1}(K)$ with $K^0 = \text{Skel}_{-1}(K) := \varnothing$. One can also consider randomised filtrations, where we uniformly randomly choose whether to add one simplex or the other. ◇

Definition II.7.3 (Homology of a filtration)
Let K^ℓ be the ℓ-th intermediate complex of a filtered complex.

Let $Z_k^\ell := Z_k(K^\ell)$ be the k-th cycles of K^ℓ, $B_k^\ell := B_k(K^\ell)$ be the k-th boundaries of K^ℓ, $H_k^\ell := Z_k(K^\ell)/B_k(K^\ell)$ be the k-th homology of K^ℓ and $\beta_k^\ell := \text{rank}(H_k^\ell)$ the k-th Betti number of K^ℓ.

Over time steps ℓ, β_1^ℓ records the one-dimensional holes. We call β_1 the signature function.

signature function

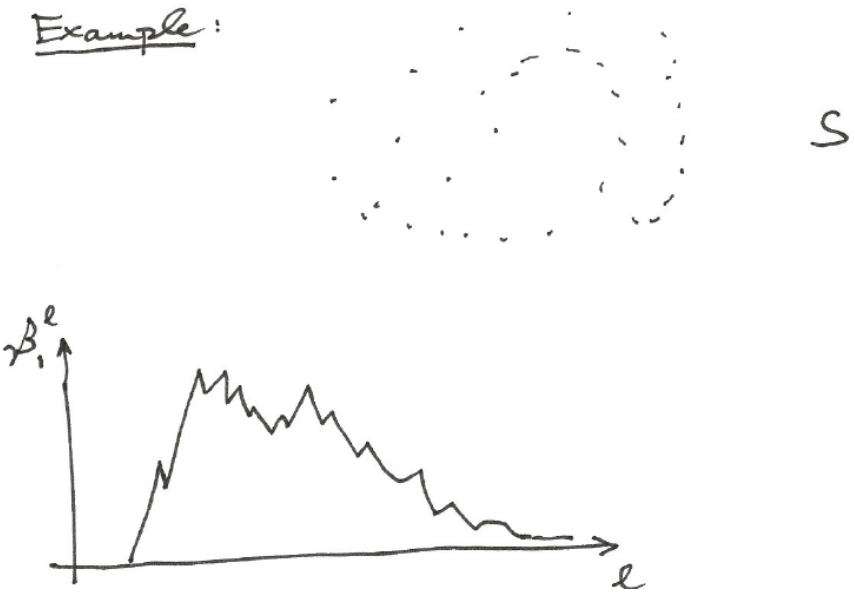


Fig. 112: An example for a data set S and for a plot of the corresponding signature function. If ℓ depends on e.g. a radius r, then $\beta_1^\ell \to 0$ for $\ell \to \infty$.

It is hard to distinguish between features and noise (e.g. spin glasses in material science). We want to identify features as substructures that persist over (a longer period of) time, e.g. find the non-bounding cycles that remain non-boundaries for at least the next p intermediate complexes of the filtration.

Definition II.7.4 (Persistent homology)
Let $(K^\ell)_{\ell=0}^m$ be a filtration, $k \in \mathbb{N}$, $\ell \in \{0, \dots, m\}$ and $p \in \{0, \dots, m-\ell\}$. The p-persistent k-th homology group of K^ℓ is

$$H_k^{\ell,p} := Z_k^\ell \backslash \left(B_k^{\ell+p} \cap Z_k^\ell\right)$$

and the p-persistent k-th Betti number of K^ℓ is

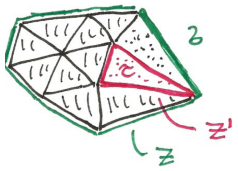
$$\beta_k^{\ell,p} := \operatorname{rank}\left(H_k^{\ell,p}\right).$$

Note that $H_k^{\ell,0} = H_k^\ell$ by lemma II.5.7.

Remark II.7.5 (Welldefinedness) The module $H_k^{\ell,p}$ is well-defined because $B_k^{\ell+p} \cap Z_k^\ell$ is the intersection of two submodules of $C_k^{\ell+p}$ (because $(K_\ell)_{\ell=0}^m$ is a filtration) and thus itself a module and because it is a submodule of Z_k^ℓ. ○

Remark II.7.6 (Functorial way to define persistent homology groups)
It two cycles are homologous in K^ℓ, then they also exist and are homologous in $K^{\ell+p}$ (because the inclusion $K^\ell \subset K^{\ell+p}$ induces an injective simplicial map $K^\ell \hookrightarrow K^{\ell+p}$). For the map $\eta_k^{\ell,p} \colon H_k^\ell \to H_k^{\ell+p}$, which maps a homology class in H_k^ℓ to one containing it in $H_k^{\ell+p}$, we set $H_k^{\ell,p} := \operatorname{im}(\eta_k^{\ell,p})$. ○

Example II.7.7 (Creating, destroying simplices)
If the dotted triangle σ (we could also instead choose σ to be the new outer edge of this triangle) is added first together with the two edges (τ is not yet added), then the outer boundary cycle Z of the whole complex is homologous to the boundary Z' of τ, so $Z \sim Z'$. We say that a cycle is created by adding σ and killed by adding τ. ◇

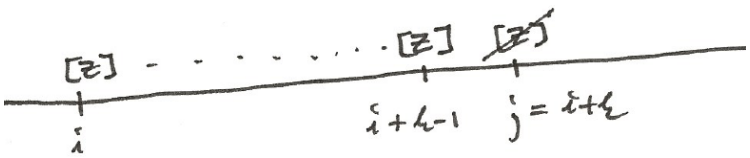


Definition II.7.8 (Persistence, creator, destroyer)
Let Z be a non-bounding k-cycle that is created at time i by the addition of the simplex σ, and let $Z' \sim Z$ be a homologous k-cycle that is turned into a boundary at time j by the simplex τ. Then the persistence of σ (and of its homology class $[Z]$) is $j - i$. persistence

We say that σ is the creator of $[Z]$ and τ is the destroyer of $[Z]$. In particular we say that σ is a positive simplex and τ is a negative simplex. destroyer creator

The persistence is the time σ "survives".

Example II.7.9
A creation at time i and death at time $i + k$ yields a persistence of k.



II.7.1 | Visualisation of persistence

19.07.2022

Suppose that the filtration $(K^\ell)_{\ell=0}^m$ of a simplicial complex K does not have torsion, e.g. by choosing $R = \mathbb{Z}_2$. Then for any bounding k-cycle $z \in B_k$ in the terminal complex K there is a representing pair of simplices (σ, τ) that create and destroy $[z]$ at times i and j, respectively.

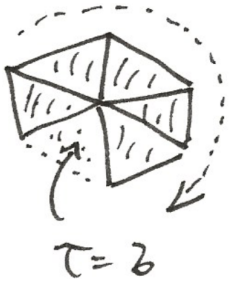


Fig. 113

Remark II.7.10 We can have $\tau = \sigma$, though. In figure 113 we have $\beta_1 = 0$ for all times. ◦

Example II.7.11 (Barcode diagram)
Every pair (σ, τ) with $\sigma \neq \tau$ is visualised by an half-open interval $[i, j)$, the k-interval of the k-cycle $z \in Z_k$. Non-bounding cycles (that is, $z \in Z_k \backslash B_k$) in K are represented by intervals $[i, \infty)$.

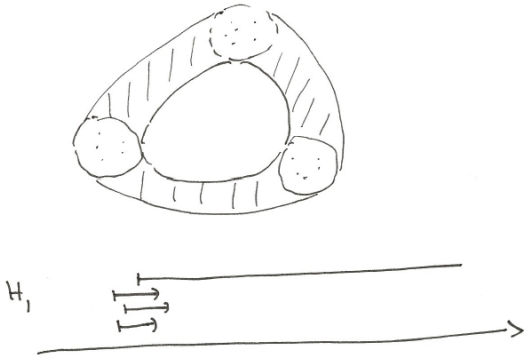


Fig. 114: This representation of persistence is called hierarchy of barcodes for H_*.

What is this shape above the barcode?

Note that the vertical axis doesn't carry any meaning in the bar code diagram. ◇

Example II.7.12 (Persistence diagram)
Every point in the persistence diagram below represents a pair (σ, τ), plotted at the coordinates (i, j), where i is the time of birth and j is the time of death of the pair.

persistence diagram

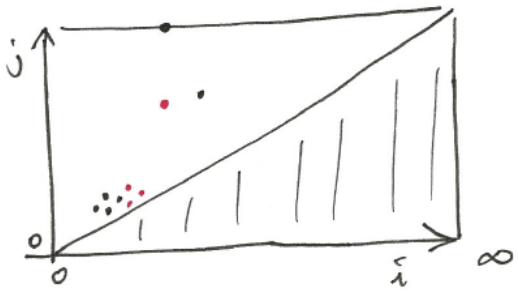
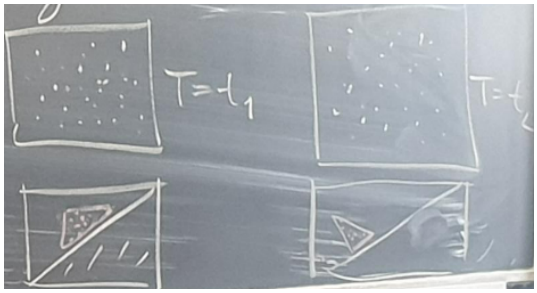


Fig. 115: The black dots represent 0-dimensional homology, the red ones 1-dimensional homology.

Important. All points close to the diagonal (most likely) represent noise, while points far away from the diagonal represent persistent information of the filtration.

An application is to spin glasses, which look very similar to the human at different points in time:



But, with some luck, the persistence diagrams differ visibly and one can conclude that something changed between those two points in time. ◇

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