Technische Universität Berlin

Lecture Notes

Complex Analysis I

read by Prof. Dr. Springborn in the summer semester 2021

Contents

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1 Holomorphic functions (in one variable)

1.1 The complex numbers

13.04.2021

Definition 1.1.1 (Complex numbers $\mathbb{C}$)
The complex plane $\mathbb{C}$ is the real vector space $\mathbb{R}^2$ with an additional operation. In $\mathbb{R}^2$, we can add vectors and multiply them with real scalars. For $x_1, x_2, y_1, y_2 \in \mathbb{R}$ we define the multiplication

complex plane $\mathbb{C}$

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1) \tag{1}$$

which turns $\mathbb{C}$ into a field containing the field $\{(x, 0) : x \in \mathbb{R}\}$ isomorphic the real numbers.

We write $x \in \mathbb{C}$ for $(x, 0)$ and i for $(0, 1)$ such that for $x, y \in \mathbb{R}$

i

$$(x, y) = x + iy.$$

Remark 1.1.2 (History) This is the end of the mathematisation of the concept of complex numbers. The history was different: complex numbers came up in algebra, when people figured out how to solve algebraic equations. For quadratic equations in one variable, there is a well-known formula, and one can live with the fact that sometimes, a quadratic equation doesn't have a solution, like $x^2 + 1 = 0$. Then people figured out how solve equations of degree 3 and something strange happened: you could solve equations of degree 3 by algebraic manipulation and get three real solutions, but in between you would have to calculate with numbers whose square is negative. It took a long time until complex numbers were established as something neither blasphemous nor mysterious. The whole discussion of "Yes, but does it exist?" ends with the above Definition: a field which contains the real numbers and a number whose square is -1. ∘

Definition 1.1.3 (Real and imaginary part)
The projections of $\mathbb{R}^2$ onto the entries are called real and imaginary parts, respectively:

$$\Re\colon\ \mathbb{C} \to \mathbb{R},\quad (x, y) \mapsto x, \qquad \Im\colon\ \mathbb{C} \to \mathbb{R},\quad (x, y) \mapsto y,$$

such that $z = \Re(z) + i\Im(z)$ for $z \in \mathbb{C}$.

Definition 1.1.4 (Absolute value)
The Euclidean norm $\|(x, y)\| := \sqrt{x^2 + y^2}$ defines a topology on $\mathbb{C}$, and thus also the notions of neighbourhood, convergence and continuity. We write $|z|$ instead of $\|z\|$ and call it the absolute value, that is

$$|z| := \sqrt{(\Re(z))^2 + (\Im(z))^2}.$$

Every vector $(x, y) \in \mathbb{R}^2$ of length 1 can be represented in angle notation as

angle notation

$$(x, y) = (\cos(\varphi), \sin(\varphi))$$

with a $\varphi \in \mathbb{R}$, determined uniquely up to a multiple of 2π. Hence every complex number $z \in \mathbb{C}$ can be represented as

$$z = |z|(\cos(\varphi) + i\sin(\varphi)). \tag{2}$$

The angle φ is the argument of z. For $z \neq 0$, the argument is only defined up to a multiple of 2π, for $z = 0$, the argument is either not defined or arbitrary.

argument

Euler's equation states

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

hence one can write the the polar representation (2) as

$$z = |z|e^{i\varphi}.$$

Definition 1.1.5 (Complex conjugation)
Complex conjugation is the map

$$\overline{\cdot}\colon \mathbb{C} \to \mathbb{C}, \qquad x + iy \mapsto \overline{x+iy} := x - iy.$$

and $\overline{x+iy}$ is the complex conjugate of $x + iy$.

We have $|z|^2 = z\bar{z}$ for all $z \in \mathbb{C}$ and hence $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$ for all $z \in \mathbb{C}\backslash\{0\}$. The latter calculation shows that $\mathbb{C}$ is a field and not only a ring, as one can also divide by complex numbers; the reciprocal of a complex number is again a complex number:

$$\frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

Remark 1.1.6 (Complex multiplication as an $\mathbb{R}$-linear map) Consider the complex number $w := a + ib = |w|e^{i\varphi}$ for $a, b \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$. What is the effect of multiplying with a complex number? For $x, y \in \mathbb{R}$ we have (cf. (1)!)

$$\begin{aligned}(a+ib)(x+iy) = ax - by + i(bx+ay) &= \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \underbrace{|w|}_{\in\mathbb{R}} \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}}_{\text{rotation matrix}\in\mathrm{SO}(2)} \begin{pmatrix} x \\ y \end{pmatrix},\end{aligned}$$

where $\mathrm{SO}(2) \subset \mathbb{R}^{2\times 2}$ is the special orthogonal group of two dimensions. Hence the map $z \mapsto w \cdot z$ is a scale rotation with center 0, angle φ (which is the argument of w) and scale factor $|w|$. scale rotation

The map

$$\mathbb{C} \to \mathbb{R}\cdot\mathrm{SO}(2) = \left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R}\right\} \subset \mathbb{R}^{2\times 2}, \qquad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is a field isomorphism (Exercise!). ○

1.2 Differentiability

Definition 1.2.1 (Complex differentiability, Holomorphy, Entire)
Let $U \subset \mathbb{C}$ be an open subset and $z_0 \in U$. A function $f\colon U \to \mathbb{C}$ is (complex) differentiable on U if the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \in \mathbb{C}.$$

exists. In that case, $f'(z_0)$ is the derivative of f at z_0. If f is differentiable for all $z_0 \in U$, then it is holomorphic or (complex) analytic. A holomorphic function on $\mathbb{C}$ is an entire function. holomorphic entire

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Remark 1.2.2 (Relation to real differentiability, computation rules)
The definition of the derivative is verbatim the same as in real analysis. Hence many of the same rules and theorems also hold in the complex case: the derivative is linear, the product

and the chain rule hold and the proofs are the same. Hence polynomials $p(z) := \sum_{k=0}^{n} a_k z^k$ are entire and rational functions $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials, are defined and holomorphic on the open set $U := \{z \in \mathbb{C} : q(z) \neq 0\}$.

Not all theorems of real analysis hold in the complex case. For example, $\mathbb{C}$ is not an ordered field, so there is no sensible way to define the relation $<$ on $\mathbb{C}$ such that it is compatible with addition and multiplication. Hence all theorems which rely on greater- or smaller-relations need not hold. For example, there is no such thing as the Mean Value Theorem (that is, $f(x_1) - f(x_0) = f'(\xi)(x_1 - x_0)$ for some ξ in between x_0 and x_1) in complex analysis. One can however compare absolute values:

$$|f(z_1) - f(z_0)| \leqslant M|z_1 - z_0|$$

holds for all $z_0, z_1 \in \mathbb{C}$ if M is an upper bound for $|f'(z)|$ and suitable conditions hold. ○

1.3 Power series

A large class of holomorphic functions are the power series. Power series are a more important tool in Complex Analysis than in Real Analysis, because, as we will see later, every holomorphic function is represented by a power series around every point of its domain of definition. In Real Analysis, the function $e^{-\frac{1}{x}}\, \mathbb{1}_{(0,\infty)}(x)$ is smooth but not analytic, that is, representable by a power series, in zero.

If a power series centered at zero, $z \mapsto \sum_{k=0}^{\infty} a_k z^k$, converges for $z_1 \in \mathbb{C}$, then it converges for all $z \in \mathbb{C}$ with $|z| < |z_1|$ because for all but finitely many $k \in \mathbb{N}$ we have

$$|a_k z^k| = \underbrace{|a_k z_1^k|}_{\leqslant 1} \left|\frac{z^k}{z_1^k}\right| \leqslant \left|\frac{z}{z_1}\right|^k,$$

so $\sum_{k=0}^{\infty} a_k z^k$ is majorised by the convergent geometric series $\sum_{k=0}^{\infty} q^k$ with $q := |\frac{z}{z_1}| < 1$.

Hence there is an $R \in [0, \infty] := \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ so that $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely if $|z| < R$ and diverges if $|z| > R$. In the complex case, we cannot make a statement for $|z| = R$, anything can happen on parts of that circle. If $r \in (0, R)$, then the power series converges uniformly on the disk

$$\overline{D}_r := \{z \in \mathbb{C} : |z| \leqslant r\}$$

In particular, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is continuous on $D_R := \{z \in \mathbb{C} : |z| < R\}$. All of the above works similarly for power series $\sum_{k=0}^{\infty} a_k (z - c)^k$ with centre $c \neq 0$.

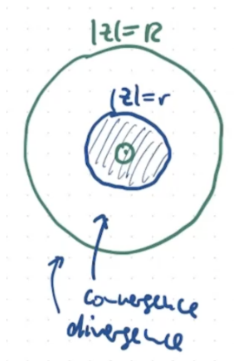

Fig. 1: Convergence radius of a power series.

Theorem 1.3.1: Power series may be differentiated termwise

Suppose the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ has radius of convergence $R \in (0, \infty]$ and let

$$f \colon \{z \in \mathbb{C} : |z - z_0| < R\} \to \mathbb{C}, \qquad z \mapsto \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

be the function defined by it.

1. The power series $\sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$ has the same radius of convergence R and therefore defines a function

$$g \colon \{z \in \mathbb{C} : |z - z_0| < R\} \to \mathbb{C}, \qquad z \mapsto \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}.$$

2. The function f is holomorphic.

Proof. ① Use the formula for the radius of convergence (like in the real case)

$$\frac{1}{R} := \limsup_{k\to\infty} \sqrt[k]{|a_k|}$$

(with obvious modifications for 0 and ∞).

② see Jänich, Ferus' notes, Ahlfors, or later in this course. □

Corollary 1.3.1 (Power series uniqueness theorem)
If f is given by a power series with positive radius of convergence, i.e.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

then

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

In particular, two power series with centre z_0 have the same coefficients, if they define the same function in the neighbourhood of z_0.

Proof. Repeated application of Theorem 1.3.1 yields

$$f^{(m)}(z) = \sum_{k=m}^{\infty} k \cdot (k-1) \cdot \ldots \cdot (k-m+1) a_k (z - z_0)^{k-m}$$

and hence

$$f^{(m)}(z_0) = m \cdot (m-1) \cdot \ldots \cdot 1 \cdot a_m = m! \cdot a_m.$$ □

Example 1.3.2 (Well known entire functions)
All functions known from Real Analysis which are defined by power series are also defined on disks in the complex plane. In particular, functions defined by power series that converge everywhere on the real line also converge everywhere in the complex plane, for example

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad \cos(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \qquad \sin(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}, \tag{3}$$

$$\cosh(z) := \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \qquad \text{and} \qquad \sinh(z) := \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

◇

Example 1.3.3 (Chebyshev polynomials (Tut I))
We show that there exists a sequence of polynomials $(T_n : \mathbb{C} \to \mathbb{C})_{n\in\mathbb{N}}$ such that $\cos(nx) = T_n(\cos(x))$ holds for all $x \in \mathbb{C}$.

We clearly have $T_1(z) = z$, $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ (by (3)) and thus

$$\cos(2x) = \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)^2 - 1 = 2\cos^2(x) - 1$$

and hence $T_2(z) = 2z^2 - 1$. We complete the proof by induction (details are left as an exercise):

$$\begin{aligned}
\cos(nx) &= \frac{1}{2}\left(e^{inx} - e^{-inx}\right) \\
&= \frac{1}{2}\left(e^{i(n-1)x} + e^{-i(n-1)x}\right)\left(e^{ix} + e^{-ix}\right) - \frac{1}{2}\left(e^{i(n-2)x} + e^{-i(n-2)x}\right) \\
&= 2\cos((n-1)x)\cos(x) - \cos((n-2)x).
\end{aligned}$$

◇

1.4 Complex and real differentiability

Definition 1.4.1 ((Total) differentiability in $\mathbb{R}^2$)
Let $U \subset \mathbb{C} = (\mathbb{R}^2, +, \cdot)$ be an open set and $f\colon U \to \mathbb{C}$. Then f is differentiable at $z_0 \in U$ in the real sense if there exists an $\mathbb{R}$-linear map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(z) = f(z_0) + F(z - z_0) + \alpha(z) \qquad \text{with } \lim_{z \to z_0} \frac{\alpha(z)}{|z - z_0|} = 0.$$

An $\mathbb{R}$-linear map on $\mathbb{R}^2$ is represented by a real (2×2)-matrix: for all $x, y \in \mathbb{R}$ we have

$$F \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

for some $a, b, c, d \in \mathbb{R}$. The function f is differentiable in the complex sense if F is also $\mathbb{C}$-linear on $\mathbb{C}$, that is $F(\lambda z) = \lambda F(z)$ for all $\lambda \in \mathbb{C}$ (the additivity is the same as in the real case). In particular $F(z) = F(z \cdot 1) = zF(1)$, so a $\mathbb{C}$-linear map F on $\mathbb{C}$ is just multiplication by the complex number $F(1)$. If F is $\mathbb{C}$-linear, it is in particular $\mathbb{R}$-linear, so by (4) we have

$$F(1) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = a + ib.$$

On the other hand, we have seen in Remark 1.1.6 that multiplication with a complex number has a particular matrix representation:

$$F \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (a + ib)(x + iy) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence f is complex differentiable if and only if the matrix representation of $\mathrm{d}_{z_0} f$ has the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. If $f\colon U \to \mathbb{R}^2$ is an function and $U \subset \mathbb{R}^2$ is open, for $z = x + iy$ we can separate f into its real and imaginary components:

$$f(z) = f(x, y) = (u(x, y), v(x, y)) = u(x, y) + iv(x, y).$$

Hence

$$F \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{d}_{z_0} f \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \Bigg|_{z = (x_0, y_0)} \begin{pmatrix} x \\ y \end{pmatrix} \overset{!}{=} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or equivalently, the equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \tag{5}$$

have to hold. We summarise our findings in a Theorem.

Theorem 1.4.1: Real and complex differentiability

A function $f\colon U \to \mathbb{C}$ is complex differentiable in $z_0 \in \mathbb{C}$ if it is differentiable in the real sense and one (and hence both) of the following two conditions hold:

- The derivative $\mathrm{d}_{z_0} f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{C}$-linear as a map on $\mathbb{C}$.
- The Cauchy-Riemann differential equations (5) hold in z_0.

In this case we have
$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0).$$

This theorem allows us to transfer theorems from two dimensional Real Analysis to Complex Analysis, in particular the Schrankensatz, as a complex differentiable function is a real differentiable function on $\mathbb{R}^2$ with the additional condition that the CAUCHY-RIEMANN equations hold.

Example 1.4.2 (Complex differentiability)

- The function $f(z) = \overline{z}$ is, in real terms,
$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ -y \end{pmatrix}.$$
The JACOBI matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, so the CAUCHY-RIEMANN equations are nowhere satisfied and hence f is nowhere differentiable in the complex sense.

  Intuitively, this makes sense as complex conjugation is reflection on the real axis and thus locally not well described by a scale rotation (multiplication with $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$).

  (V:) We can also directly check Definition 1.2.1: let $z_0 \in \mathbb{C}$. For $z \in \mathbb{C}\setminus\{z_0\}$ we can write $z - z_0 = r_z e^{i\varphi_z}$ for $r_z > 0$ and $\varphi_z \in \mathbb{R}$. Then
$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0} = \frac{r_z e^{-i\varphi_z}}{r_z e^{i\varphi_z}} = e^{-2i\varphi_z}.$$
The limit $\lim_{z\to z_0} e^{-2i\varphi_z}$ does not exist.

- Let $g(z) := \overline{z}^2 = (x - iy)^2 = x^2 - y^2 - 2ixy$ is, in real terms
$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 - y^2 \\ -2xy \end{pmatrix}.$$
The JACOBI matrix is $2\left(\begin{smallmatrix} x & -y \\ -y & -x \end{smallmatrix}\right)$. The CAUCHY-RIEMANN equations are only fulfilled in $(0, 0)$. It is not a holomorphic function, as $\{0\}$ is not an *open* subset of $\mathbb{C}$. ◇

We want to given an example for a case where we use real differentiability for a complex function. In Real Analysis one learns that if a function defined on an interval has zero derivative, the function is constant. This was proven with the Mean Value Theorem and that proof does not transfer the complex case, as there is no order relation in the complex plane. But we can use the Theorem of Multivariate Real Analysis stating that if function $\mathbb{R}^2 \supset U \to \mathbb{R}^2$, where U is connected, has zero derivative, it is constant.

THEOREM 1.4.2: CONSTANCY CRITERION

If f is holomorphic on the open and connected set $U \subset \mathbb{C}$ and $f'(z) = 0$ for all $z \in U$, then f is constant.

Proof. This follows from the corresponding theorem of Multivariate Real Analysis, as instead of the Mean Value Theorem we can use the Schrankensatz, with is still valid in $\mathbb{C}$. □

Open and connected subsets of the complex plane play a crucial role in Complex Analysis.

Definition 1.4.3 (Domain)
A domain (in $\mathbb{C}$) is an open and connected subset of $\mathbb{C}$.

domain

Corollary 1.4.4 (Real-valued Constancy criterion)
If f is holomorphic on a domain and real-valued, then f is constant.

Proof. If $f = u + iv$ is real-valued, then $v \equiv 0$. By the Cauchy-Riemann equations we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$, so u and therefore f are constant by Theorem 1.4.2. □

Hence there are no interesting real-valued holomorphic functions on a domain.

1.5 The argument function and the complex logarithm

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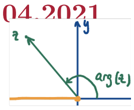


Fig. 2: The argument of a complex number non lying on the non-positive real axis is well defined as the angle it makes with the positive real axis.

We have defined the argument of a complex number $z \in \mathbb{C}$ as the angle that z, viewed as a position vector in $\mathbb{R}^2$, has with the positive real axis. This angle is only well defined up to a multiple of 2π and in particular, there is no way to define arg as a single-valued function on the whole plane without introducing discontinuities. The standard way to resolve this issue is to cut the plane along the non-positive real axis.

Definition 1.5.1 (Argument function)
On the complement of the non-positive real axis

$$U := \mathbb{C} \setminus \{x \in \mathbb{R} : x \leqslant 0\} = \{re^{i\varphi} : r > 0,\ \varphi \in (-\pi, \pi)\}$$

one has a well-defined argument function

argument function

$$\arg\colon U \to (-\pi, \pi), \qquad re^{i\varphi} \mapsto \varphi.$$

Remark 1.5.2 (Continuity of arg) One cannot define the argument as a continuous function on the whole complex plane as on the complement of U one can assign both π and $-\pi$ as arguments of the complex numbers, but both choices yield a discontinuous argument function. ○

One could choose a different slit, e.g. by cutting along the positive imaginary axis or along any curve starting in zero and going to infinity. The definition we have chosen is the "principal value" of the argument function.

It is not entirely obvious how to write an equation for $\arg(z)$ with $z \in U$. For example, this works:

$$\arg(x+iy) = \begin{cases} \arctan\left(\frac{y}{x}\right), & \text{for } x > 0, \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right), & \text{for } y > 0, \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right), & \text{for } y < 0. \end{cases} \tag{6}$$

Remark 1.5.3 In many computer languages, there is the function atan2(y, x), which is defined for all $(x, y) \neq 0$ and gives $\arg(x + iy)$ on U and has the value π on the nonpositive real axis, which is exactly its set of discontinuity points. ○

From (6), we can, for e.g. $x > 0$, compute the partial derivatives

$$\frac{\partial \arg(x+iy)}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial \arg(x+iy)}{\partial y} = \frac{x}{x^2+y^2}. \tag{7}$$

It is left as an exercise that the partial derivatives are the same for $y > 0$ and $y < 0$. The partial derivatives of arg are continuous (in fact, they can be continuously extended to

$\mathbb{C}\backslash\{0\}$). The function $\arg\colon U \to \mathbb{R}$ is therefore differentiable in the real sesnse. But as a real valued non-constant function, it is not holomorphic by corollary 1.4.4.

The complex exponential function (3) is not injective, as $e^{iy} = e^{(2k\pi+y)i}$ holds for all $y \in \mathbb{R}$ and all $k \in \mathbb{Z}$. But we can define the complex logarithm as the inverse function of the complex exponential restricted to the horizontal strip $\{\varrho + i\xi \in \mathbb{C} : \varrho \in \mathbb{R}, \xi \in (-\pi, \pi)\}$: if $z = re^{i\varphi}$ for $r = |z| > 0$ and $\varphi \in (-\pi, \pi)$, then

$$\log(z) := \log(|z|) + i\varphi,$$

where $\log(|z|)$ is the natural log of the positive real number $|z| = r$.

Definition 1.5.4 (Principal value logarithm)
The (principal value) logarithm function is

$$\log\colon \mathbb{C}\backslash\{x \in \mathbb{R} : x \leqslant 0\} \to \mathbb{C}, \qquad z \mapsto \log(|z|) + i \arg(z).$$

logarithm

Example 1.5.5 (The complex logarithm is holomorphic)
We check the Cauchy-Riemann equations for

$$u(x,y) := \Re\left(\log(x+iy)\right) = \log(\sqrt{x^2+y^2}) = \frac{1}{2}\log(x^2+y^2)$$

and

$$v(x,y) := \Im\left(\log(x+iy)\right) = \arg(x+iy).$$

We have

$$\frac{\partial u(x,y)}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2} \overset{(7)}{=} \frac{\partial \arg(x+iy)}{\partial y} = \frac{\partial v(x,y)}{\partial y},$$
$$\frac{\partial u(x,y)}{\partial y} = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2} \overset{(7)}{=} -\frac{\partial \arg(x+iy)}{\partial y} = -\frac{\partial v(x,y)}{\partial y}.$$

Hence by Theorem 1.4.1 the complex logarithm is holomorphic with derivative

$$\log'(z = x+iy) = \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

◇

1.6 Harmonic functions

If we talk about real and complex differentiability, there is one further connection that is important.

Suppose a holomorphic function $f = u + iv$ is two times differentiable in the real sense. Then the Cauchy-Riemann equations imply

We will later see that holomorphic functions are infinitely often differentiable in the complex sense.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \overset{(5)}{=} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \overset{(5)}{=} -\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 v}{\partial y \partial x}.$$

By Schwarz' theorem, we can interchange the order of differentiation to obtain

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where Δ is the Laplace operator. Analogously, we obtain $\Delta v = 0$.

Definition 1.6.1 (Harmonic function)
A function f defined on an *open* subset $U \subset \mathbb{C}$ that satisfies the Laplace equation $\Delta f = 0$ is a harmonic function. harmonic

Remark 1.6.2 (Harmonic functions in other dimensions)
In one (real) dimension, the harmonic functions are the affine linear functions, but in higher dimensions, they can be more complicated. Hence harmonic functions can be seen as a non-obvious generalisation of affine linear functions to higher dimensions. ○

By the above calculations, the real and imaginary parts of a holomorphic function are harmonic functions.

Suppose $u \colon U \to \mathbb{R}$ is a harmonic function on a domain $U \subset \mathbb{R}$. Does there exists a harmonic function $v \colon U \to \mathbb{R}$ such that $f = u + iv$ is holomorphic? For this to be the case, the partial derivatives of v would have to fulfil the Cauchy-Riemann equations. In Real Analysis, one learns that a necessary condition that for "given x- and y-derivatives, is there a function with this x- and y-derivative?", is that

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

holds, which is satisfied because u is harmonic.

This condition is also sufficient if U is simply connected (or: diffeomorphic to $\mathbb{R}^2$ or convex or star-shaped. In Analysis II one learns that a closed 1-form has an antiderivative (or: a rotation-free vector field is a gradient field of a function) if U is simply connected. We conclude:

Lemma 1.6.3 (Harmonic function is real part of holomorphic function)
On a simply connected domain $U \subset \mathbb{C}$, every harmonic function is the real part of a holomorphic function.

Example 1.6.4 The function $u(x, y) = \log(\sqrt{x^2 + y^2})$ is harmonic on the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ (which is not simply connected). So on the slit complex plane $\mathbb{C} \setminus \{x \in \mathbb{R} :$ punctured plane
$x \leqslant 0\}$, which is simply connected, there is a harmonic function v such that $f = u + iv$ is holomorphic. The function v is only determined unique up to an additive real constant. If we choose $v(1, 0) = 0$, then $v(x, y) = \arg(x + iy)$ (and hence $f = \log$).

On $\mathbb{C}^*$, there is no harmonic function yielding a well defined holomorphic function, because that function would discontinuous on the slit. ◇

We close this section with a simple consequence of the fact that the real and imaginary parts of holomorphic functions are harmonic.

Theorem 1.6.1: Composition of harmonic and holomorphic map

Let $f \colon U \to \mathbb{C}$ be holomorphic and $h \colon f(U) \to \mathbb{R}$ harmonic. Then $h \circ f$ is harmonic.

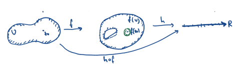


Fig. 3: Illustration of the setup of Theorem 1.6.1.

The idea of the proof is that we consider for h a second function H to make it the real part of holomorphic function. But this doesn't work globally. Being harmonic is a local property: if you can show that it is true on every neighbourhood of every point, it is true globally.

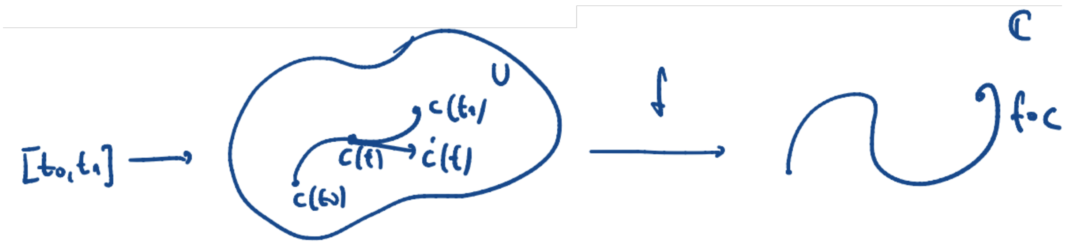
Proof. By lemma 1.6.3 for any point $f(z_0) \in f(U)$ (which is open), the harmonic function h is the real part of a holomorphic function H defined on a simply connected neighbourhood of $f(z_0)$, e.g. an sufficiently small (such that it is contained in $f(U)$) open disk around $f(z_0)$. By the chain rule for differentiation, the composition $H \circ f$ is holomorphic on an open neighbourhood of z_0. So the real part of $h \circ f$ is harmonic on a neighbourhood of z_0.

Since this is true for all $z_0 \in U$, $h \circ f$ is harmonic on U. $\square$

1.7 Conformal maps

We will discuss the geometric properties of holomorphic functions.

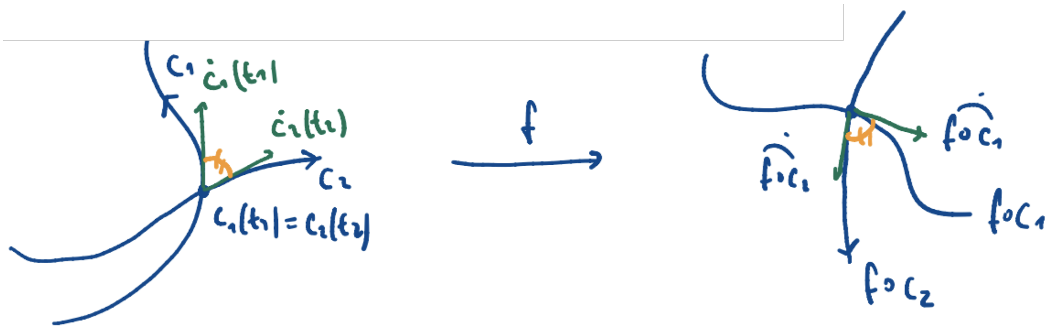
Suppose $f: U \to \mathbb{C}$ is holomorphic on a domain U and let $c: [t_0, t_1] \to U$ be a differentiable curve in U, whose velocity $\dot{c}$ vanishes nowhere.



The image curve under f is $f \circ c$ and its velocity vector is, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)(t) = f'(c(t)) \cdot c'(t).$$

If $f'(c(t)) = re^{i\varphi} \neq 0$, then the velocity vector of c is rotated by an angle of φ and scaled by a factor of $r > 0$. If we take two curves c_1 and c_2, which intersect in some point $c_1(t_1) = c_2(t_2)$, then f rotates their velocity vectors by the same angle. Hence the angle of intersection $\sphericalangle(\dot{c_1}(t_1), \dot{c_2}(t_2))$ remains the same after f is applied to both curves.



Definition 1.7.1 (Conformal map)
A map that preserves angles is conformal. conformal

We have just seen:

Theorem 1.7.1: Characterisation of conformal maps

Holomorphic functions with nonvanishing derivative are conformal.

Example 1.7.2 The function $f(z) = z^2$ is entire, as it is a polynomial. It maps the straight lines (that make an angle φ with the real axis) $c_\varphi(t) := te^{i\varphi}$ to $(f \circ c_\varphi)(t) = t^2 e^{2i\varphi}$. The function f doubles all angles at zero, in particular, it does not preserve angles.

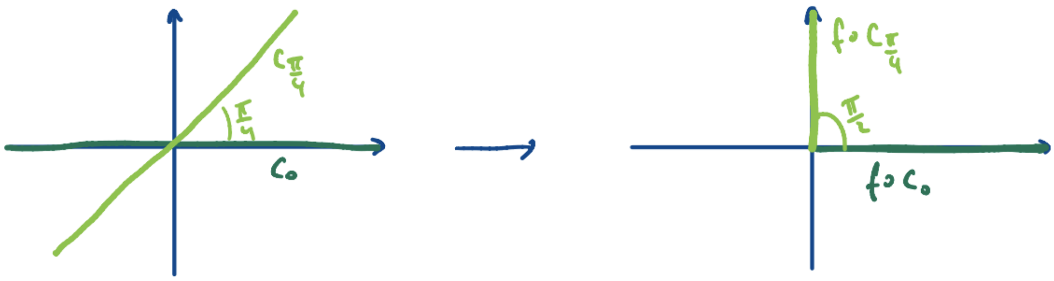


Fig. 4: Straight line curves and their image curves under $f(z) := z^2$.

This doesn't contradict Theorem 1.7.1, as $f'(0) = 0$ and 0 is the point where all curves c_φ intersect. As an exercise one checks that $f(z) := z^n$ for $n \geqslant 1$ multiplies all angles at zero with n. At all other points $f(z) = z^2$ preserves the angles, as $z = 0$ is the only zero of f'. ⋄

We will now look at a sort of converse of Theorem 1.7.1. Let's look at angle preserving linear maps because a nonlinear map is conformal if its real derivative - a linear map - is conformal. We only consider invertible maps, because this makes the discussion less complicated and prohibit that nonzero vectors are mapped to the zero vector, which is tricky since the angle between the zero vector and any other vector is not defined. **21.04.2021**

Lemma 1.7.3 (Characterisation of conformal maps)
For an invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ the following statements are equivalent:

1. *F preserves angles.*
2. *F preserves orthogonal angles: if z and w are orthogonal, then $F(z)$ and $F(w)$ are also orthogonal.*
3. *F is $\mathbb{C}$-linear (that is, $F(iz) = iF(z)$ for all $z \in \mathbb{C}$) or F is $\mathbb{C}$-antilinear (that is, $F(iz) = -iF(z)$).*

Proof. "1 $\implies$ 2": is obvious.

"3 $\implies$ 1": is almost already known: we know that F is $\mathbb{C}$ linear if and only if

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

i.e. if F is a scale rotation. If F is $\mathbb{C}$-antilinear, then F is a scale-reflection in a line through the origin.

"2 $\implies$ 3": We write $1 = \binom{1}{0}$ and $i = \binom{0}{1}$. So 1 and i are orthogonal and so are $1 + i = \binom{1}{1}$ and $1 - i = \binom{1}{-1}$. By 2 $F(1+i)$ and $F(1-i)$ are orthogonal and hence (by the additivity of F)

$$\begin{aligned} 0 &= \langle F(1+i), F(1-i) \rangle = \langle F(1) + F(i), F(1) - F(i) \rangle \\ &= \langle F(1), F(1) - F(i) \rangle + \langle F(i), F(1) - F(i) \rangle = \langle F(1), F(1) \rangle - \langle F(i), F(i) \rangle. \end{aligned}$$

Hence $F(1)$ and $F(i)$ have the same non-zero (as F is invertible) length and are orthogonal. Hence (as we are in the plane) $F(i) = iF(1)$ or $F(i) = -iF(1)$, as a 90 degree rotation is multiplication with $\pm i$ in $\mathbb{C}$. □

Addendum: How can we distinguish between the two cases in 3: we look at the determinant: we have

$$\det(F(1), F(i)) = \det(F(1), \pm iF(1)) = \pm\det(F(1), iF(1)) = |F(1)|^2 > 0$$

Hence if F also preserves orientation, F is $\mathbb{C}$-linear.

We have essentially proved the following

Theorem 1.7.2: Geometric characterisation of holom. maps

A real differentiable map $f : U \to \mathbb{C}$ on a domain $U \subset \mathbb{C}$ is holomorphic if its derivative in the real sense is everywhere angle and orientation preserving.

The zero function is holomorphic, but it doesn't preserve angles, so the theorem can't yield an only-if statement.

Stereographic projection

Definition 1.7.4 (Stereographic projection)
The stereographic projection from the north pole $e_3 := (0, 0, 1) \in \mathbb{R}^3$ is

stereographic projection

$$\mathbb{S}^2 \backslash \{e_3\} \to \mathbb{C}, \qquad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \mapsto \frac{1}{1-\zeta}(\xi + i\eta)$$

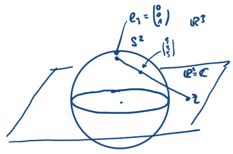


Fig. 5: The stereographic projection from the north pole e_3.

Its inverse is

$$x + iy =: z \mapsto \frac{1}{1+|z|^2}\begin{pmatrix} 2x \\ 2y \\ |z|^2 - 1 \end{pmatrix}.$$

It is a bijective and conformal map (Exercise!).

Definition 1.7.5 (Riemann sphere)
The Riemann sphere (or: extended complex plane)

$$\hat{\mathbb{C}} := \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$

is the complex plane $\mathbb{C}$ with the extra point ∞ added.

Here, $\mathbb{C}P^1 = (\mathbb{C}^2 \backslash \{0\})/_\sim$ with $x \sim y$ if there exists a $\lambda \in \mathbb{C} \backslash \{0\}$ such that $x = \lambda y$.

The point ∞ corresponds to the north pole of $\mathbb{S}^2$ under stereographic projection. The stereographic projection is a bijective map from $\mathbb{S}^2$ to $\hat{\mathbb{C}}$. Since $\mathbb{S}^2$ has a topology induced by the ambient $\mathbb{R}^3$, the stereographic projection induces a topology on $\hat{\mathbb{C}}$.

In the extended complex plane, it makes more concrete sense to say that a sequence of complex numbers converges to ∞; it really means the sequence of complex numbers converges to the point $\infty \in \hat{\mathbb{C}}$ in the topology on $\hat{\mathbb{C}}$, it is not only a short form for saying that this sequence diverges properly.

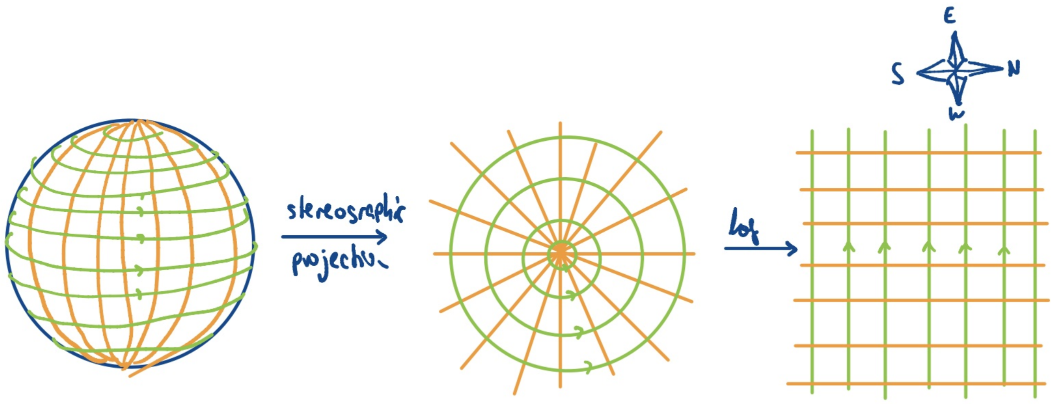


Fig. 6: The stereographic projection maps the meridians and circles of longitude (or latitude) to straight lines through the origin and concentric circles around the origin. The complex logarithm maps circles (all points with the same absolute value) to circles with the same real part because the real part of the logarithm is the logarithm of the absolute value. The orange rays in the middle picture are defined by all their points having the same argument. As the imaginary part of the logarithm is the argument, they get mapped to horizontal lines.

Since the stereographic projection and the logarithm preserve angles, their composition is an angle preserving map that maps circle of latitude and longitude to orthogonal families of straight lines. This is known as MERCATOR's projection. If one prints it correctly, one gets a map of the earth which is uniquely determined by the fact that the directions of the compass are exactly represented on the map: not only is north, south, east and west always up, down, left and right, but also all angles between these principal directions are represented correctly in the map. It turns out that this is the only map projection with this property.

The stereographic projection is a even conformal homeomorphism between the $\mathbb{S}^2$ and $\hat{\mathbb{C}}$.

1.8 Möbius transformations

A MÖBIUS transformation is an example of a holomorphic map, which is simple, but also complicated enough to be interesting. Let us consider functions of the form

$$f(z) = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{C}$, that is, rational functions with denominator and numerator being polynomials of degree at most 1 (hence the name *fractional linear transformations*).

We want to exclude the case that the numerator and denominator are linearly independent - that one is a multiple of the other - because then the function would be constant. If $ad = bc$ and $d \neq 0$, then

$$f(z) = \frac{adz+bd}{cdz+d^2} = \frac{bcz+bd}{cdz+d^2} = \frac{b\cancel{(cz+d)}}{d\cancel{(cz+d)}} = \frac{b}{d}.$$

Similarly, if $ad = bc$ and $c \neq 0$, then $f(z)$ is also constant.

DEFINITION 1.8.1 (MÖBIUS TRANSFORMATION (PRELIMINARY DEFINTION))
A MÖBIUS transformation is a function

$$f\colon \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$.

Note that the MÖBIUS transformations with the coefficients a, b, c, d and $\lambda a, \lambda b, \lambda c, \lambda d$, where $\lambda \in \mathbb{C}^*$, are the same function. Hence the coefficients determine the function, but the function only determines the coefficients only up to a non-zero scale factor $\lambda \in \mathbb{C}^*$.

Hence we can (but do not need to) require that $ad - bc = 1$. If we do, then the MÖBIUS transformation determines the coefficients up to a global sign change, i.e. a factor of ± 1.

A *transformation* is usually a bijective map. This is indeed the case for the MÖBIUS transformation: if $w = \frac{az+b}{cz+d}$ (which is not equal to $\frac{a}{c}$, as this would require $b = d = 0$, violating $ad - bc \neq 0$), then $cw - a \neq 0$ and thus

$$czw + dw = az + b \iff (cw - a)z = -dw + b \iff z = \frac{dw - b}{-cw + a}.$$

Hence the MÖBIUS transformation $f(z) = \frac{az+b}{cz+d}$ is injective (one-to-one) and its inverse is $f^{-1}(w) = \frac{dw-b}{-cw+a}$.

There is something that is not nice: $f(z)$ is not defined if the denominator vanishes, that is $z = -\frac{d}{c}$. Furthermore, it doesn't take the value $\frac{a}{c}$ (because there the inverse is not well defined). This is all presuming that $c \neq 0$. If $c = 0$, then f is a similarity transformation (translation + scale rotation), it is just $\frac{a}{d}z + \frac{b}{d}$, which is a polynomial - and hence boring.

Our way out of this is to consider the MÖBIUS transformations as functions from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ instead of from $\mathbb{C}$ to $\mathbb{C}$.

DEFINITION 1.8.2 (MÖBIUS TRANSFORMATION)

A MÖBIUS transformation is a function

MÖBIUS transformation

$$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \qquad z \mapsto \frac{az + b}{cz + d},$$

where

$$\begin{cases} f\left(-\frac{d}{c}\right) := \infty \quad \text{and} \quad f(\infty) = \frac{a}{c}, & \text{if } c \neq 0, \\ f(\infty) = \infty, & \text{if } c = 0. \end{cases}$$

and $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$.

THEOREM 1.8.1: MÖBIUS GROUP

The MÖBIUS transformations form a group of bijective functions under composition.

Proof. The only thing left to show is that the composition of two MÖBIUS transformation is again a MÖBIUS transformation. Let $f(z) := \frac{az+b}{cz+d}$ and $g(z) := \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$, where $ad - bc \neq 0 \neq \tilde{a}\tilde{d} - \tilde{b}\tilde{c}$. Then for $z \in \hat{\mathbb{C}}$ we have

$$(f \circ g)(z) = \frac{a \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + b}{c \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + d} = \frac{a\tilde{a}z + a\tilde{b} + b\tilde{c}z + b\tilde{d}}{c\tilde{a}z + c\tilde{b} + d\tilde{c}z + d\tilde{d}} = \frac{(a\tilde{a} + b\tilde{c})z + a\tilde{b} + b\tilde{d}}{(c\tilde{a} + d\tilde{c})z + c\tilde{b} + d\tilde{d}} =: \frac{\hat{a}z + \hat{b}}{\hat{c}z + \hat{d}}.$$

□

But this shows even more: consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a\tilde{a} + b\tilde{c} & a\tilde{b} + b\tilde{d} \\ c\tilde{a} + d\tilde{c} & c\tilde{b} + d\tilde{d} \end{pmatrix} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$$

Hence the map

$$\Phi\colon \mathrm{SL}(2,\mathbb{C}) \mapsto \text{Möb}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right),$$

where Möb is the group of MÖBIUS transformations, is group homomorphism with kernel $\{\pm \mathrm{id}_2\} \subset \mathbb{R}^{2\times 2}$ (so Φ is two-to-one). One can get an isomorphism between the projective special linear group $\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\ker(\Phi)$ and Möb.

Example 1.8.3 (Affine transformations on $\mathbb{C}$ (Tut II))
Consider the affine transformation $A\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto az + b$. It is bijective if and only if $a \neq 0$. The bijective affine transformations are a group with respect to composition. If $a = 1$, $b = 0$, A has infinitely many fixed points, if $a = 1$ and $b \neq 0$, it has none and if $a \neq 1$, it has exactly one. One can view these transformations as MÖBIUS transformations $A\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $A(\infty) = \infty$. ◇

27.04.2021

We will now see an example of a particularly simple MÖBIUS transformation, which is not as simple as a rotations, translations or scalings.

Example 1.8.4 (Inversion as MÖBIUS transform) Consider the MÖBIUS transformation

$$f(z) := \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \overline{\left(\frac{z}{|z|^2}\right)}.$$

It can be written as a composition:

$$z \overset{f_1}{\longmapsto} \frac{z}{|z|^2} \overset{f_2}{\longmapsto} \overline{\left(\frac{z}{|z|^2}\right)},$$

where f_1 is the inversion in the unit circle and f_2 is complex conjugation. The map f_1 is holomorphic on $\mathbb{C}^*$. ◇

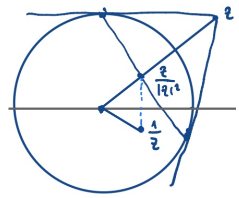


Fig. 7: The intersection of the ray connecting z to the origin and the segment connecting the tangency points on the unit sphere is $\frac{z}{|z|^2}$, so $\frac{1}{z}$ is its reflection at the real axis.

THEOREM 1.8.2: IMAGES OF CIRCLES AND STRAIGHT LINES UNDER INVERSION IN THE UNIT CIRCLE

Inversion in the unit circle $f(z) := \frac{1}{z}$ maps

1. circles that do not pass through 0 to circles,
2. circles that do pass through 0 to straight lines,
3. straight lines that do not pass through 0 to circles,
4. straight lines that do pass through zero to straight lines.

Proof. 1 Consider a circle with centre c and radius r, that is

$$\{z \in \mathbb{C} : |z-c|^2 - r^2 = 0\}. \tag{8}$$

The points $w = \frac{1}{z}$ of the image of the circle (8) satisfy

$$0 = \left|\frac{1}{w} - c\right|^2 - r^2 = \left(\frac{1}{w} - c\right)\left(\frac{1}{\overline{w}} - \bar{c}\right) - r^2 = \frac{1}{|w|^2} - c\frac{1}{\overline{w}} - \bar{c}\frac{1}{w} + |c|^2 - r^2.$$

If $w \neq 0$, then this is equivalent to (by multiplying through with $|w|^2 = w\overline{w}$)

$$0 = 1 - cw - \overline{cw} + (|c|^2 - r^2)|w|^2 \tag{9}$$

If $|c| \neq r$ (i.e. if the original circle does not pass through 0), then this is equivalent to (by completing the square)

$$\begin{aligned} 0 &= \frac{1}{|c|^2 - r^2} - \frac{cw + \overline{cw}}{|c|^2 - r^2} + |w|^2 \\ &= \left(w - \frac{\bar{c}}{|c|^2 - r^2}\right)\left(\overline{w} - \frac{c}{|c|^2 - r^2}\right) - \frac{|c|^2}{(|c|^2 - r^2)^2} + \frac{1}{|c|^2 - r^2} \\ &= \left|w - \frac{\bar{c}}{|c|^2 - r^2}\right|^2 - \frac{r^2}{(|c|^2 - r^2)^2}, \end{aligned}$$

which is the equation of a circle with centre $\frac{\bar{c}}{|c|^2 - r^2}$ and radius $\frac{r}{|c|^2 - r^2}$.

2 If the original circle passes through 0, then $|c|^2 = r^2$, so (9) becomes

$$1 = cw + \overline{cw},$$

which is the equation for a line. Let $c = c_1 + ic_2$ and $w = w_1 + iw_2$, then

$$cw + \overline{cw} = 2(c_1 w_1 - c_2 w_2)$$

That is, a linear expression in w and $\overline{w}$ is a real linear expression in $\Re(w)$ and $\Im(w)$.

3 and 4 follow from 1 and 2 as $f^{-1} = f$. □

Definition 1.8.5 (Möbius circle)
A Möbius circle is either circle in $\mathbb{C}$ or the union of a straight line in $\mathbb{C}$ and $\{\infty\}$. Möbius circle

This definition makes sense as all points of a circle passing through zero are mapped to a straight line, except 0, which is mapped to ∞ by f.

Corollary 1.8.6 (Inverted Möbius circles in the unit circle are Möbius circles)
The map f maps Möbius *circles to* Möbius *circles.*

Warning: this maps does *not* map the centre of a circle to the centre of the image circle (except when the centre is 0).

Lemma 1.8.7 (Unique Möbius transform with $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$)
For any three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, there is a unique $f \in$ Möb with

$$f(z_1) = 0, \qquad f(z_2) = 1, \qquad f(z_3) = \infty. \tag{10}$$

Proof. Existence. The map

$$f(z) = \frac{z_2 - z_3}{z_2 - z_1}\frac{z - z_1}{z - z_3}$$

fulfills the condition: we have $f(z_1) = \frac{z_2-z_3}{z_2-z_1}\frac{0}{z_1-z_3} = 0$, $f(z_2) = \frac{\cancel{z_2-z_3}}{\cancel{z_2-z_1}}\frac{\cancel{z_2-z_1}}{\cancel{z_2-z_3}} = 1$ and $f(z_3) = “\frac{z_2-z_3}{z_2-z_1}\frac{z_3-z_1}{0}” = \infty$ as well as $a := z_2 - z_3$, $b := z_1(z_3 - z_2)$, $c := z_2 - z_1$, $d := z_3(z_1 - z_2)$ and thus

$$\begin{aligned} ad - bc &= (z_2 - z_3)z_3(z_1 - z_2) - z_1(z_3 - z_2)(z_2 - z_1) \\ &= (z_2 - z_3)z_3(z_1 - z_2) - z_1(z_2 - z_3)(z_1 - z_2) \\ &= \underbrace{(z_3 - z_1)}_{\neq 0}\underbrace{(z_1 - z_2)}_{\neq 0}\underbrace{(z_2 - z_3)}_{\neq 0} \end{aligned}$$

as the z_k are pairwisely distinct. Alternatively: if $ad = bc$, then f were constant, but the constructed f takes the values 0 and 1 so $ad \neq bc$.

Uniqueness.

1. Suppose $f(z) := \frac{az+b}{cz+d}$ satisfies $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Then $f(0) = 0$ implies that $b = 0$, $f(\infty) = \infty$ implies that $c = 0$ and $f(1) = \frac{a}{d} = 1$ implies $a = d$.

   So the only MÖBIUS transformation that fixes $0, 1$ and ∞ is the identity (if $a = 0$, then $f(1) \neq 1$).

2. Suppose f_1, f_2 are MÖBIUS transformations satisfying (10). Then $g = f_2 \circ f_1^{-1}$ fixes $0, 1$ and ∞, so $g = \mathrm{id}$ by the previous step, so $f_1 = f_2$. □

Now it is easy to prove the interesting theorem.

THEOREM 1.8.3: 3 POINTS + THEIR IMAGES DETERMINE MÖB UNIQUELY

If $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ are each three points, then there is a unique MÖBIUS transformation f satisfying $f(z_i) = w_i$ for $i \in \{1, 2, 3\}$.

Proof. Existence. Let g and h be the MÖBIUS transformations sending z_1, z_2, z_3 and w_1, w_2, w_3 to $0, 1$ and ∞ respectively, which exist by lemma 1.8.7. Then $f := h^{-1} \circ g$ satisfies $f(z_i) = w_i$ for $i \in \{1, 2, 3\}$.

Uniqueness.

1. Suppose $f \in$ Möb and $f(z_i) = z_i$ for $i \in \{1, 2, 3\}$. Then $f = \mathrm{id}$. Indeed let $g \in$ Möb be the map with $g(z_1) = 0$, $g(z_2) = 1$ and $g(z_3) = \infty$. Then $h := g \circ f \circ g^{-1} \in$ Möb satisfies $h(0) = 0$, $h(1) = 1$, $h(\infty) = \infty$. By lemma 1.8.7, $h = \mathrm{id}$ and thus $f = g^{-1} \circ h \circ g = \mathrm{id}$.

2. Suppose f_1 and f_2 are MÖBIUS transformations with $f_j(z_i) = w_i$, $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$. Then $f_2^{-1} \circ f_1 \in$ Möb fixes z_1, z_2, z_3, so by the previous step, $f_2^{-1} \circ f_1 = \mathrm{id}$, hence $f_2 = f_1$. □

Example 1.8.8 (The MÖBIUS transformation $f(z) := \frac{z-i}{z+i}$)
The MÖBIUS transformation $f(z) := \frac{z-i}{z+i}$ satisfies $f(i) = 0$, $f(-i) = \infty$ and $f(\infty) = 1$. If $z \in \mathbb{R}$, then $|f(z)| = \frac{|z-i|}{|z+i|} = 1$, so f maps the $\mathbb{R} \cup \{\infty\}$ to the unit circle. We have $f(1) = \frac{1-i}{1+i} = \frac{-2i}{2} = -i$ and $f(-1) = \frac{-1-i}{-1-i} = i$.

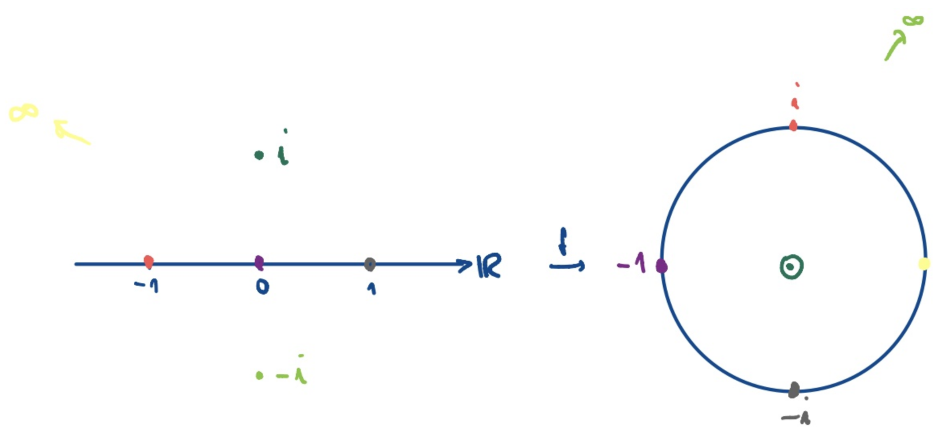


Fig. 8: The (pre)images of ± 1, $\pm i$ and ∞ of $f(z) := \frac{z-i}{z+i}$.

Before we show that *all* MÖBIUS transformations map MÖBIUS circles to MÖBIUS circles, we show a Lemma which makes the proof of that theorem very easy.

Lemma 1.8.9 (Decomposition of Möbius transformations)
Every Möbius *transformation* $f(z) := \frac{az+b}{cz+d}$ *is a composition of* Möbius *transformations of the following form:*

$$z \mapsto z + b, \qquad b \in \mathbb{C} \qquad \text{(translation)}$$
$$z \mapsto az, \qquad a \in \mathbb{C}^* \qquad \text{(scale-rotation)}$$
$$z \mapsto \frac{1}{z}. \qquad \text{(inversion)}$$

Clearly the first two Möbius transformations map Möbius circles to Möbius circles and for the third one, we have shown it in Theorem 1.8.2.

Theorem 1.8.4: $f \in$ Möb maps M-circles to M-circles

Every Möbius transformation maps Möbius circles to Möbius circles.

Proof. **(Of lemma 1.8.9)** If $c \neq 0$, then

$$\frac{az+b}{cz+d} = -\frac{ad-bc}{c}\frac{1}{cz+d} + \frac{a}{c}$$

by polynomial division. So f is the composition of the following maps

$$z \longmapsto cz =: z_1, \qquad z_1 \longmapsto z_1 + d =: z_2, \qquad z_2 \longmapsto \frac{1}{z_2} =: z_3,$$
$$z_3 \longmapsto -\underbrace{\frac{ad-bc}{c}}_{\neq 0} z_3 =: z_4, \qquad z_4 \longmapsto z_4 + \frac{a}{c}.$$

The case $c = 0$ is clear. □

We have seen that Möbius transformations have simple forms and are flexible; they map any three points to any other three points, and they map Möbius circles to Möbius circles, but they do not preserve lengths, they are not isometries. But is there any other quantity that is preserved? The answer is yes:

Definition 1.8.10 (Cross ratio)
The cross-ratio of four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is cross-ratio

$$\operatorname{cr}(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_2 - z_3}\frac{z_3 - z_4}{z_4 - z_1}.$$

If one of the points is ∞, this is supposed to be evaluated by cancelling infinities.

For example,

$$\operatorname{cr}(z_1, z_2, \infty, z_4) = \text{``}\frac{z_1 - z_2}{\cancel{z_2 - \infty}}\frac{\cancel{\infty - z_4}}{z_4 - z_1}\text{''} := -\frac{z_1 - z_3}{z_4 - z_1} = \frac{z_3 - z_1}{z_4 - z_1}$$

Theorem 1.8.5: When is $\operatorname{cr}(z_1, z_2, z_3, z_4) \in \mathbb{R}$?

The cross-ratio of four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is real if and only if the four points lie on a Möbius circle.

Theorem 1.8.6: Möbius transformations preserve cross ratio

① For any $f \in$ Möb and any four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ we have

$$\mathrm{cr}(z_1, z_2, z_3, z_4) = \mathrm{cr}(f(z_1), f(z_2), f(z_3), f(z_4)).$$

② Conversely, Möbius are the only transformation that preserves the cross ratio: if $\mathrm{cr}(z_1, z_2, z_3, z_4) = \mathrm{cr}(w_1, w_2, w_3, w_4)$, there there exists a $f \in$ Möb with $f(z_j) = w_j$ for $j \in \{1, \ldots, 4\}$.

So in summary: for two points, their distance is not changed under isometries (so, Euclidean motions), so this is the only invariant for Möbius transformations. For three points, there is no invariant, as one can map any three points to any three other points and for four points, there is only one invariant, which is the cross-ratio.

One can prove both theorems by a straightforward and lengthy calculation, but one can also prove both theorems without preforming hardly any calculations by using the following observation:

Lemma 1.8.11 (Characterisation of the cross-ratio in terms of Möb)
The number $\mathrm{cr}(z_1, z_2, z_3, z_4)$ *is the value* $h(z_1)$ *of the* Möbius *transformation* h *with* $h(z_2) = 0$, $h(z_3) = 1$, $h(z_4) = \infty$.

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Proof. Consider the map

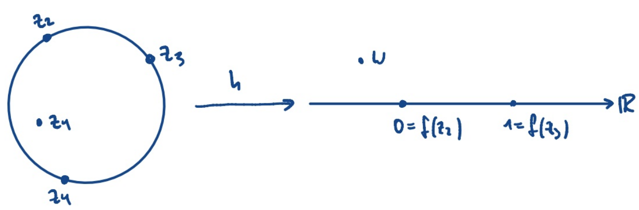
$$h(z) = \frac{z_3 - z_4}{z_3 - z_2} \frac{z - z_2}{z - z_4}$$

from lemma 1.8.7. Then

$$h(z_1) = \frac{z_3 - z_4}{z_3 - z_2} \frac{z_1 - z_2}{z_1 - z_4} = \frac{z_1 - z_2}{z_2 - z_3} \frac{z_3 - z_4}{z_4 - z_1} = \mathrm{cr}(z_1, z_2, z_3, z_4).$$

□

Proof. (of Theorem 1.8.5) Let $h \in$ Möb be the map satisfying $h(z_2) = 0$, $h(z_3) = 1$ and $h(z_4) = \infty$ and let $w := h(z_1) = \mathrm{cr}(z_1, z_2, z_3, z_4)$, where the last equality is by lemma 1.8.11. Consider the Möbius circle through z_2, z_3, z_4 (three points in the complex plane always determine a circle). If either point is equal to ∞, the Möbius circle is a line, otherwise it is a circle. As Möbius transformations map Möbius circles to circles, the Möbius circle is mapped to the extended real line (the line through the images of z_2, z_3, z_4, which are 0, 1 and ∞ respectively).



Clearly, the point z_1 is on the Möbius circle through z_2, z_3, z_4 if and only if its image w is contained in the image of that Möbius circle, that is, if and only if it is real. □

Corollary 1.8.12
The points z_1, z_2, z_3, z_4 *are on a* Möbius *circle in clockwise or anticlockwise order if and only if* $\mathrm{cr}(z_1, z_2, z_3, z_4) < 0$.

Fig. 9: Four points z_1, z_2, z_3, z_4 on a MÖBIUS circle in clockwise or anticlockwise order.

Proof. Left to the reader. □

Proof. (of Theorem 1.8.6) (1) Let $h \in$ Möb be the map with $h(z_2) = 0$, $h(z_3) = 1$ and $h(z_4) = \infty$, which exists by lemma 1.8.7. By lemma 1.8.11 we have $h(z_1) = \operatorname{cr}(z_1, z_2, z_3, z_4)$. The map $\tilde{h} := h \circ f^{-1}$ is a MÖBIUS transformation by Theorem 1.8.1 and satisfies $\tilde{h}(f(z_2)) = 0$, $\tilde{h}(f(z_3)) = 1$ and $\tilde{h}(f(z_4)) = \infty$. By lemma 1.8.11 we have

$$\operatorname{cr}(f(z_1), f(z_2), f(z_3), f(z_4)) = \tilde{h}(f(z_1)) = (h \circ f^{-1})(f(z_1)) = h(z_1) = \operatorname{cr}(z_1, z_2, z_3, z_4).$$

(2) Let $h, \tilde{h} \in$ Möb be maps with $h(z_2) = \tilde{h}(w_2) = 0$, $h(z_3) = \tilde{h}(w_3) = 1$ and $h(z_4) = \tilde{h}(w_4) = \infty$. By assumption $(\star)$ we have

$$h(z_1) \overset{1.8.11}{=} \operatorname{cr}(z_1, z_2, z_3, z_4) \overset{(\star)}{=} \operatorname{cr}(w_1, w_2, w_3, w_4) \overset{1.8.11}{=} \tilde{h}(w_1).$$

Hence $f := \tilde{h}^{-1} \circ h \in$ Möb satisfies $f(z_j) = w_j$ for $j \in \{1, \dots, 4\}$. □

The following theorem will be important later and we will not prove it completely, for now.

THEOREM 1.8.7: MÖBIUS TRANSFORMATIONS PRESERVING D

The MÖBIUS transformations that map the unit disk

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

onto itself are precisely the MÖBIUS transformations of the form

$$f(z) = e^{i\varphi} \frac{z - z_0}{1 - \overline{z_0} z},$$

where $z_0 \in D$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.

Proof. " $\Longleftarrow$ ": We first show f maps D to D.

(1) We show that $f(\mathbb{S}^1) = \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$. For $z \in \mathbb{S}^1$ we have

$$\begin{aligned} |f(z)|^2 &= \underbrace{|e^{i\varphi}|}_{=1} \frac{(z - z_0)(\bar{z} - \overline{z_0})}{(1 - \overline{z_0} z)(1 - \bar{z} z_0)} = \frac{|z|^2 - \overline{z_0} z - z_0 \bar{z} + |z_0|^2}{1 - \overline{z_0} z - z_0 \bar{z} - |z_0|^2 |z|^2} \\ &= \frac{1 - \overline{z_0} z - z_0 \bar{z} + |z_0|^2}{1 - \overline{z_0} z - z_0 \bar{z} - |z_0|^2} = 1. \end{aligned}$$

(2) Since f is continuous on $\hat{\mathbb{C}}$ and $f(\mathbb{S}^1) = \mathbb{S}^1$ and $f(z_0 \in D) = 0 \in D$, f maps the connected component of $\hat{\mathbb{C}} \backslash \mathbb{S}^1$ containing z_0 to the connected component containing 0, which is, $f(D) = D$.

" $\Longrightarrow$ ": See Dirk Ferus' script or wait until later in this course. □

THEOREM 1.8.8: MÖBIUS TRANSFORMATIONS PRESERVING H (TUT)

The MÖBIUS transformations $f(z) = \frac{az+b}{cz+d}$ with $f(H) = H$ are characterised by $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

Proof. " $\Longrightarrow$ ": One solution is a computational one: we know that $\tilde{f}(z) := e^{i\varphi}\frac{z-z_0}{1-\overline{z_0}z}$ obeys $\tilde{f}(D) = D$ and we know a map g with $g(H) = D$ from example 1.8.14. We can thus obtain the maps $H \to H$ as $f = g^{-1} \circ \tilde{f} \circ g$.

Another approach is as follows. As $ad - bc \neq 0$, we can assume that at least one of the coefficients a, b, c, d is nonzero. As the coefficients $\lambda a, \lambda b, \lambda c, \lambda d$ for any $\lambda \in \mathbb{C}^*$ give the same MÖBIUS transform, we can assume that nonzero coefficient to be real. If $f(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$, then all the numbers $f(0) = \frac{b}{d}$, $f(\infty) = \frac{a}{c}$, $f^{-1}(0) = -\frac{b}{a}$ and $f^{-1}(\infty) = -\frac{d}{c}$ must be in $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Since the ratios must be real and one of the coefficients is real, we can assume $a, b, c, d \in \mathbb{R}$. (One might have to be careful if ∞ is a fixed point of f.)

As $f(H) = H$, we must have $\Im(f(i)) > 0$. We have

$$\Im\left(\frac{ai+b}{ci+d}\right) = \Im\left(\frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)}\right) = \frac{\Im(ac+bd+i(ad-bc))}{c^2+d^2} = \frac{ad-bc}{c^2+d^2},$$

and as $c^2 + d^2 > 0$, we must have $ad - bc > 0$.

" $\Longleftarrow$ ": If $a, b, c, d \in \mathbb{R}$, then $f(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$. With the above calculation we have $f(i) \in H$ as $ad - bc > 0$ and thus $f(H) = H$ by connectedness. □

Example 1.8.13 (MÖBIUS transformation f with $f(0) = i$, $f(i) = \infty$, $f(\infty) = 1$)
We know that the MÖBIUS transformation f_1 with $f_1(0) = 0$, $f_1(i) = 1$, $f_1(\infty) = \infty$ is $f_1(z) = \mathrm{cr}(z, 0, i, \infty) = -iz$. Furthermore, the MÖBIUS transformation f_2^{-1} with $f_2^{-1}(i) = 0$, $f_2^{-1}(\infty) = 1$, $f_2^{-1}(1) = \infty$ is $f_2^{-1}(z) = \mathrm{cr}(z, i, \infty, 1) = \frac{z-i}{z-1}$. Then $f = f_1 \circ f_2 = \frac{z+1}{z-i}$. ◇

Example 1.8.14 (Find MÖBIUS transformation with $f(\mathbb{H}) = \mathbb{D}$ (Tut II))
We find the MÖBIUS transformation f with $f(0) = -1$, $f(i) = 0$ and $f(\infty) = 1$ and show that $f(\mathbb{H}) = \mathbb{D}$, where $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ is the upper half plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.
As before $f_1(z) = -iz$ and $f_2^{-1}(z) = \frac{(z+1)(0-1)}{(-1-0)(1-z)} = \frac{z+1}{1-z}$ and thus $f_2(z) = \frac{z-1}{z+1}$ and hence $f(z) = \frac{z-i}{z+i}$. MÖBIUS transformations map generalised circles to generalised circles. The "circle" through $0, 1, \infty$ (that is $\hat{\mathbb{R}} := \{x \in \mathbb{C} : x \in \mathbb{R}\} \cup \{\infty\}$) is mapped to the unit circle, as $f(0) = -1$, $f(1) = -i$ and $f(\infty) = 1$ (three points uniquely determine a circle). The extended real line $\hat{\mathbb{R}}$ separates $\hat{\mathbb{C}}$ into two connected components, $\mathbb{H}$ and $\hat{\mathbb{C}}\backslash\overline{\mathbb{H}}$. Similarly, the unit circle separates $\hat{\mathbb{C}}$ into two connected components, $\mathbb{D}$ and $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$. As f is a homeomorphism, we either have $f(\mathbb{H}) = \mathbb{D}$ or $f(\mathbb{H}) = \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ by connectedness. As $f(i) = 0 \in \mathbb{D}$, we conclude $f(\mathbb{H}) = \mathbb{D}$. ◇

Remark 1.8.15 (Connection to hyperbolic geometry) H is one of the models of the hyperbolic plane with the metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y}$. Geodesics are circle perpendicular to the real axis or lines parallel to the imaginary axis. As geodesics are mapped to geodesics, the isometries in this model are exactly the MÖBIUS transforms. ○

We conclude this chapter of MÖBIUS applications with one mathematical application.

Example 1.8.16
We first show: For any pair of non intersecting circles, there is a MÖBIUS transformation that maps these circles to concentric circles (circles with the same centre).

Step 1. We consider two cases.

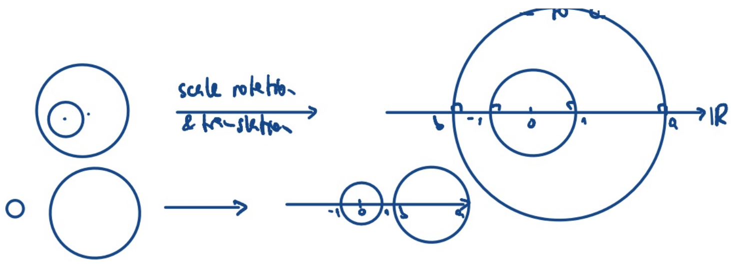


Fig. 10: One can make it such that the small circle has the centre 0 and has radius 1. We can make the centre of the larger circle to be on the real axis. Both circles are perpendicular to the real axis.

Step 2. We want to find a number $q \in \mathbb{R}\backslash\{\pm 1, 0\}$ (or $|q| > 1$?) such that

$$c := \operatorname{cr}(a, b, -1, 1) \stackrel{!}{=} \operatorname{cr}(q, -q, -1, 1) = \frac{2q(-2)}{(-q-1)(1-q)} = \frac{4q}{1-q^2}.$$

Solving for q we obtain

$$1 - q^2 = \frac{4q}{c} \iff q^2 + \frac{4q}{c} - 1 = 0 \iff q_\pm = -\frac{2}{c} \pm \sqrt{\frac{4}{c^2} + 1}.$$

Let $q := q_+$ be the positive solution.

Step 3. Let f be the MÖBIUS transformation with

$$f(a) = q, \qquad f(b) = -q, \qquad f(-1) = -1, \qquad f(1) = 1. \qquad \diamond$$

Then f maps the circles to concentric circles.

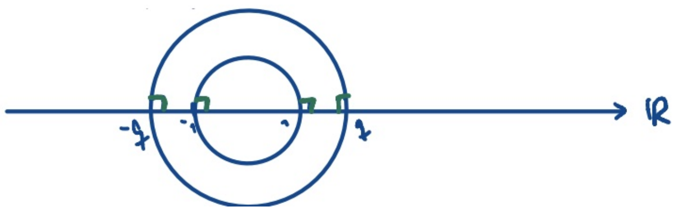
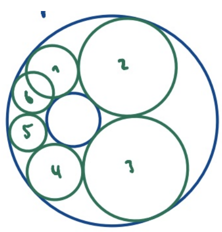
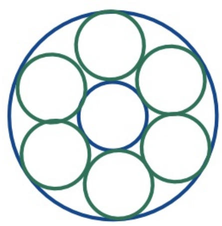


Fig. 11: As MÖBIUS transformations are holomorphic, they preserve angles, so the centres are real, and therefore 0.

This can be use to prove a weird theorem of STEINER: Start with two circles with one being contained in the other. One can start to draw circles between them that touch both circles like in the figure on the right and then continue drawing such circles that also touch the previous circle. Either one new circle overlaps the others, or the circle again touches the first circle, in which case the procedure repeats.



Theorem. (Steiner) Given the blue circles, whether or not the sequence of green circles "closes up", that is the n-th green circle touches the first, depends only on the blue circles and not on the choice of the first circle.



Proof. Apply a suitable MÖBIUS transformation: if we make the two blue circles concentric, it is obvious. $\square$

2 Complex integration and CAUCHY's integral theorem

2.1 Contour integrals / Integrals along curves

We now switch over to JÄHNICH's textbook.

We will consider complex functions f defined on open subsets $U \subset \mathbb{C}$ and curves γ in the domain of definition U, which are represented as maps from a closed interval to the space. We don't have to worry about the parametrisation of the curve as we will see that the integral of a function over a curve is independent of the parametrisation of the curve.

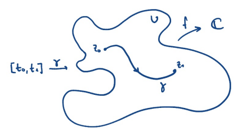


Fig. 12: A curve in a domain.

DEFINITION 2.1.1 (COMPLEX INTEGRAL)
Let $U \subset \mathbb{C}$ be any subset, $f: U \to \mathbb{C}$ be continuous and $\gamma: [t_0, t_1] \to U$ be a continuously differentiable curve. Then

$$\boxed{\int_\gamma f(z)\,\mathrm{d}z := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,\mathrm{d}t}$$

is the integral of f along γ.

integral

We can consider complex-valued functions as $\mathbb{R}^2$-valued functions, which we integrate componentwise, as we know how to integrate real valued functions of a real variable. Note that $\gamma'(t) \in \mathbb{C}$.

The substitution rule implies that the integral does not depend on the parametrisation of γ (only on the orientation, i.e. on the direction that it is traced in); if $\tau: [s_0, s_1] \to [t_0, t_1]$ is a continuously differentiable reparametrisation of γ with $\tau(s_k) = t_k$ for $k \in \{0, 1\}$, then $\tilde{\gamma} := \gamma \circ \tau: [s_0, s_1] \to U$ is the reparametrised curve and

Note that τ doesn't need to be bijective (that is, monotonic).

$$\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z = \int_{s_0}^{s_1} f((\gamma \circ \tau)(s))\gamma'(\tau(s))\tau'(s)\,\mathrm{d}s = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_\gamma f(z)\,\mathrm{d}z$$

by the substitution rule (aka the change of variables formula).

But if τ reverses the orientation, that is, $\tau(s_0) = t_1$ and $\tau(s_1) = t_0$, we get (by an analogous calculation)

$$\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

DEFINITION 2.1.2 (CONTOUR INTEGRAL (EXTENDED DEFINITION))
If $\gamma: [t_0, t_1] \to U$ is only piecewise continuously differentiable, i.e. if there is a subdivision

$$t_0 = \tau_0 < \tau_1 < \ldots < \tau_n = t_1$$

such that $\gamma \in \mathcal{C}([t_0, t_1])$ is continuously differentiable on $[\tau_j, \tau_{j+1}]$ for $j \in \{0, \ldots, n-1\}$, then

$$\int_\gamma f(z)\,\mathrm{d}z := \sum_{j=0}^{n-1} \int_{\gamma|_{[\tau_j, \tau_{j+1}]}} f(z)\,\mathrm{d}z.$$

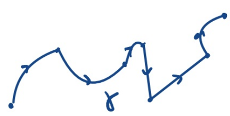


Fig. 13: A piecewise continuously differentiable curve.

Theorem 2.1.1: Triangle inequality for complex integrals (Tut IV)

Let $f\colon \mathbb{C} \supset U \to \mathbb{C}$ be continuous and $\gamma\colon [t_0, t_1] \to U$ be a $\mathcal{C}^1$ curve such that $|(f \circ \gamma)(t)| \leqslant M$ for all $t \in [t_0, t_1]$. Then

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant M \int_{t_0}^{t_1} |\gamma'(t)|\,\mathrm{d}t =: M\,\mathrm{len}(\gamma).$$

Proof. (from Ferus' lecture notes) Let $J := \int_\gamma f(z)\,\mathrm{d}z$. If $J = 0$, then the statement is clear. Let $J \neq 0$. Then

$$\begin{aligned} 1 &= \frac{\int_\gamma f(z)\,\mathrm{d}z}{J} = \Re\left(\frac{\int_\gamma f(z)\,\mathrm{d}z}{J}\right) = \Re\left(\int_\gamma \frac{f(z)}{J}\,\mathrm{d}z\right) = \Re\left(\int_{t_0}^{t_1} \frac{f(\gamma(t))\gamma'(t)}{J}\,\mathrm{d}t\right) \\ &= \int_{t_0}^{t_1} \Re\left(\frac{f(\gamma(t))\gamma'(t)}{J}\right)\mathrm{d}t \leqslant \int_{t_0}^{t_1} \frac{|f(\gamma(t))||\gamma'(t)|}{|J|}\,\mathrm{d}t \leqslant \frac{M}{|J|}\int_{t_0}^{t_1} |\gamma'(t)|\,\mathrm{d}t. \end{aligned}$$

□

For anyone with a background in vector calculus (either over divgrad or differential forms), there is a little excursion.

Excursion. Consider $f(x+iy) = u(x,y) + iv(x,y)$ and $\gamma(t) = \xi(t) + i\eta(t)$. Then (ignoring the argument t)

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_{t_0}^{t_1} (u(\xi,\eta) + iv(\xi,\eta)) \cdot (\xi' + i\eta')\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \left((u(\xi,\eta)\cdot\xi' - v(\xi,\eta)\cdot\eta') + i\left(v(\xi,\eta) + u(\xi,\eta)\eta'\right)\right)\mathrm{d}t \\ &= \int_\gamma \omega + i\int_\gamma *\omega \\ &= \underbrace{\int_{t_0}^{t_1}\left\langle \begin{pmatrix} u \\ -v \end{pmatrix}, \gamma' \right\rangle \mathrm{d}t}_{\text{curve integral of vector field } \binom{u}{-v} = \overline{f}} + i\underbrace{\int_{t_0}^{t_1}\left\langle \begin{pmatrix} v \\ u \end{pmatrix}, \gamma' \right\rangle \mathrm{d}t}_{\text{curve integral of vector field } \binom{v}{u}}, \end{aligned}$$

where $\omega := u\,\mathrm{d}x - v\,\mathrm{d}y$ and $*\omega := v\,\mathrm{d}x + u\,\mathrm{d}y$ are differential forms. We have

$$\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ -v \end{pmatrix}$$

and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is 90 degree rotation.

04.05.2021

Example 2.1.3 (Integrals of $z \mapsto z^k$ for $k \in \mathbb{Z}$) Consider the inversion

$$f\colon \mathbb{C}^* \to \mathbb{C}, \qquad z \mapsto \frac{1}{z}$$

and the curve

$$\gamma\colon [0, 2\pi] \to \mathbb{C}^*, \qquad t \mapsto Re^{it}$$

which traces a circle of radius $R > 0$ centred at the origin. Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{\cancel{Re^{it}}} i\cancel{Re^{it}}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

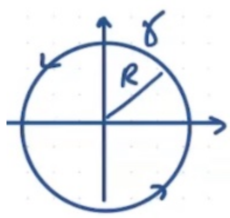


Fig. 14: The curve γ.

For $k \in \mathbb{Z}\backslash\{-1\}$ and $R = 1$ we have

$$\int_{|z|=1} z^k \,\mathrm{d}z = \int_0^1 e^{ikt} i e^{it} \,\mathrm{d}t = i \int_0^1 e^{i(k+1)t} \,\mathrm{d}t = \cancel{i} \left[\frac{e^{i(k+1)t}}{(k+1)\cancel{i}} \right]_{t=0}^{2\pi}$$
$$= \frac{e^{2\pi i(k+1)} - 1}{k+1} = \frac{1-1}{k+1} = 0.$$

◇

Example 2.1.4 (Tut II) The integral of the identity over the path γ going from the origin to i and then to $1+i$ via straight lines can be split into the straight line paths

$$\gamma_1\colon [0,1] \to \mathbb{C}, \qquad t \mapsto ti, \qquad \text{and} \qquad \gamma_2\colon [0,1] \to \mathbb{C}, \qquad t \mapsto t+i,$$

so

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^1 ti \cdot i \,\mathrm{d}t + \int_0^1 (t+i)\cdot 1 \,\mathrm{d}t = \int_0^1 -t\,\mathrm{d}t + \int_0^1 t\,\mathrm{d}t + i = i.$$

◇

Excursion into vector analysis (cont.) If $f(x+iy) = u(x,y) + iv(x,y)$, then we have seen that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i \int_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y)\,.$$

You may know the general STOKES theorem for differential forms or the special case called GREEN's theorem.

THEOREM 2.1.2: GREEN'S THEOREM

Let $B \subset \mathbb{R}^2$ be a compact set with a piecewise $\mathcal{C}^1$ boundary ∂B and let P, Q be functions of class $\mathcal{C}^1$ on a domain containing B. Then

$$\int_{\partial B} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_B -\frac{\partial P}{\partial y} + \frac{\partial Q}{\mathrm{d}x}\,\mathrm{d}x\,\mathrm{d}y.$$

Applying GREEN's Theorem (or any other formulation) to $\int_\gamma f(z)\,\mathrm{d}z$ we obtain

THEOREM 2.1.3: CAUCHY'S INTEGRAL THEOREM (VECTOR ANALYSIS VERSION)

If f is holomorphic on U with continuous derivative, and if γ is a piecewise $\mathcal{C}^1$ curve bounding a compact set $B \subset U$, then

$$\int_\gamma f(z)\,\mathrm{d}z = 0. \tag{11}$$

Proof. With GREEN's Theorem we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i \int_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y)$$
$$\overset{2.1.2}{=} \int_B \underbrace{\left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)}_{\overset{(5)}{=}0} \mathrm{d}x\,\mathrm{d}y + i \int_B \underbrace{\left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right)}_{\overset{(5)}{=}0} \mathrm{d}x\,\mathrm{d}y = 0$$

by the CAUCHY-RIEMANN equations. □

In this lecture we will see a completely different proof a stronger version. It can be stronger because we don't really need GREEN's theorem in full generality, we only need the case where integrand is zero. In particular we will see that it is enough that f is differentiable and we do not require continuity of its derivatives.

2.2 Cauchy's Integral Theorem of a Rectangle

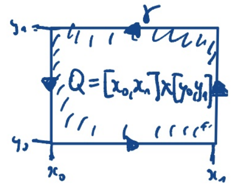


Fig. 15: A rectangular region as in Theorem 2.2.1

In this lecture we will state more and more general forms of CAUCHY's integral theorem. We start with a toy version.

Theorem 2.2.1: Cauchy's Integral Theorem of a Rectangle

Let $Q \subset \mathbb{C}$ be a closed rectangular region with sides parallel to the real and imaginary axes and let γ be a piecewise $\mathcal{C}^1$ parametrisation of the boundary of Q with orientation as shown in figure 15. If f is holomorphic on a domain containing Q, then (11).

Before we prove this, we consider complex functions possessing antiderivatives.

Lemma 2.2.1 (Cauchy's Theorem for functions with holomorphic antiderivative) *Let $f\colon U \to \mathbb{C}$ be continuous and have a holomorphic antiderivative F on U, that is, $F' = f$. Then, for any $\mathcal{C}^1$-curve $\gamma\colon [t_0, t_1] \to U$ we have*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(t_1)) - F(\gamma(t_0)).$$

If in particular γ is a closed curve, that is, $\gamma(t_0) = \gamma(t_1)$, we have (11).

Proof. This is just the Fundamental Theorem Of Calculus since

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t)) = \mathrm{d}F_{\gamma(t)}(\gamma'(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t).$$

□

Proof. (of Theorem 2.2.1) 1 We will prove that, for any $\varepsilon > 0$, $\left|\int_\gamma f(z)\,\mathrm{d}z\right| < \varepsilon$. Since f is holomorphic on U, for any $z \in U$ we have

$$f(z) = f(z_0) + f'(z_0)\cdot(z - z_0) + R_{z_0}(z),$$

where the error function $R_{z_0}\colon U \to \mathbb{C}$ satisfies

$$\lim_{z\to z_0} \frac{R_{z_0}(z)}{|z - z_0|} = 0. \tag{12}$$

Since $z \mapsto f(z_0) + f'(z_0)\cdot(z - z_0)$ is a polynomial of degree one - hence entire - and thus has a global antiderivative, its integral along the closed curve γ is zero by lemma 2.2.1. Therefore

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma R_{z_0}(z)\,\mathrm{d}z. \tag{13}$$

2 We will choose the point z_0 later. In the vicinity of z_0, R_{z_0} vanishes fast. The problem is that the rectangle is not in the vicinity of z_0 so the idea is to divide up the rectangle.

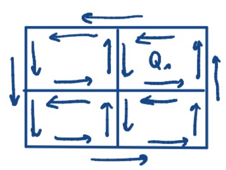


Fig. 16: Division of the rectangular region Q.

Let $\varepsilon > 0$. Divide Q into four equal subrectangles Q_1, Q_2, Q_3, Q_4 and let Q_1 be that subrectangle for which the integral along the boundary, γ_1, is largest in absolute value.

When we integrate over all curves, on the interior edges of the rectangle we go once in one direction and once in the other direction, so those integrals cancel each other out and we are left with the integrals over the boundary of Q. Then

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant 4\left|\int_{\gamma_1} f(z)\,\mathrm{d}z\right|.$$

Now subdivide the rectangle Q_1 into four equal subrectangles and let Q_2 be the rectangle for which the integral along its boundary curve, γ_2, is the largest. Continuing this process we obtain a infinite sequence of rectangles Q_k and boundary curves γ_k such that

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant 4^k \left|\int_{\gamma_k} f(z)\,\mathrm{d}z\right| \overset{(13)}{=} 4^k \left|\int_{\gamma_k} R_{z_0}(z)\,\mathrm{d}z\right|$$

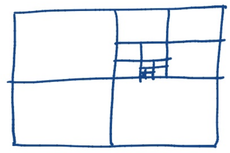


Fig. 17: The nested rectangles $(Q_k)_{k\in\mathbb{N}}$.

The intersection of all rectangles Q_k contains a single point z_0, that is, $\bigcap_{k=1}^\infty Q_k = \{z_0\}$. The x- and y-intervals of the rectangles form two sequences of nested intervals, whose lengths tend to zero.

3 We have by Theorem 2.1.1

$$\left|\int_{\gamma_k} R_{z_0}(z)\,\mathrm{d}z\right| \leqslant \operatorname{len}(\gamma_k) \cdot \sup_{z\in Q_k} |R_{z_0}(z)|.$$

We have $\operatorname{len}(\gamma_k) = 2^{-k}\operatorname{len}(\gamma)$. By (12) there is a $\delta > 0$ such that $|R_{z_0}(z)| < \tilde{\varepsilon}|z - z_0|$ for all z with $|z - z_0| < \delta$, where

$$\tilde{\varepsilon} := \frac{\varepsilon}{\operatorname{len}(\gamma)\operatorname{diam}(Q)}.$$

Choose $k \in \mathbb{N}$ so large that $\operatorname{diam}(Q_k) = 2^{-k}\operatorname{diam}(Q) < \delta$, then

$$\sup_{z\in Q_k} |R_{z_0}(z)| \leqslant \varepsilon \sup_{z\in Q_k} |z - z_0| \leqslant \varepsilon \operatorname{diam}(Q_k) = \varepsilon \cdot 2^{-k}\operatorname{diam}(Q).$$

Altogether, we have

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant \cancel{4^k 2^{-k}} \cdot \operatorname{len}(\gamma) \cdot \tilde{\varepsilon} \cdot \cancel{2^{-k}} \cdot \operatorname{diam}(Q) = \operatorname{len}(\gamma) \cdot \tilde{\varepsilon} \cdot \operatorname{diam}(Q) = \varepsilon.$$

□

Remark 2.2.2 We did not need to assume that f' is continuous (as in Theorem 2.1.3). ○

This version of Cauchy's integral theorem is fairly useless as in most cases we are not interested in integrating only over rectangular curves. But all analytical ideas are already in the proof.

2.3 Cauchy's Theorem of $\mathcal{C}^1$ images of rectangles

Now let us consider a version of Cauchy's Theorem that is actually useful.

Theorem 2.3.1: Cauchy's integral theorem for $\mathcal{C}^1$ images of rectangles

Let f be a holomorphic function on $U \subset \mathbb{C}$, let $Q \subset \mathbb{C}$ be a closed rectangular region, let γ be a $\mathcal{C}^1$ parametrisation of its boundary and let $\Phi\colon W \to \mathbb{C}$ be a continuously differentiable (in the real sense) map on some set $W \supset Q$ with $\Phi(Q) \subset U$. Then

$$\int_{\Phi\circ\gamma} f(z)\,\mathrm{d}z = 0.$$

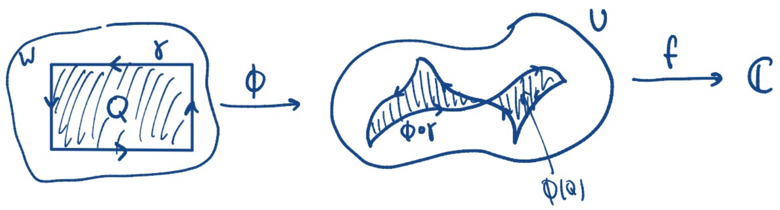


Fig. 18: The setup of Theorem 2.3.1.

Proof. We construct a sequence of rectangles $Q \supset Q_1 \supset Q_2 \supset \ldots$ as before with

$$\left|\int_{\Phi\circ\gamma} f(z)\,\mathrm{d}z\right| \leqslant 4^k \left|\int_{\Phi\circ\gamma_k} f(z)\,\mathrm{d}z\right| \tag{14}$$

with $\gamma_k := \partial Q_k$ and $\gamma := \partial Q$. But now we need to estimate $\operatorname{diam}(\Phi(Q_k))$ and $\operatorname{len}(\Phi\circ\gamma_k)$. To this end, we observe that since Φ is a $\mathcal{C}^1$ function, $\mathrm{d}\Phi$ is continuous on the compact set Q, so there exists a $C > 0$ such that $\|\mathrm{d}\Phi_z\| \leqslant C$ for all $z \in Q$. Hence

$$\operatorname{diam}(\Phi(Q_k)) \leqslant C\operatorname{diam}(Q_k) = C2^{-k}\operatorname{diam}(Q) \quad\text{and}\quad \operatorname{len}(\Phi\circ\gamma_k) \leqslant C\operatorname{len}(\gamma_k) = C2^{-k}.$$

Let $\varepsilon > 0$ and let $z_0 := \Phi\left(\bigcap_{k\in\mathbb{N}} Q_k\right)$. Choose $\delta > 0$ so small that $|R_{z_0}(z)| < \varepsilon|z - z_0|$ holds for all z with $|z - z_0| < \delta$. Now choose $k \in \mathbb{N}$ large enough that $C2^{-k}\operatorname{diam}(Q) \leqslant \delta$ holds. Then we have

$$\left|\int_{\Phi\circ\gamma} f(z)\,\mathrm{d}z\right| \overset{(14)}{\leqslant} 4^k \left|\int_{\Phi\circ\gamma_k} f(z)\,\mathrm{d}z\right| \leqslant \cancel{4^k 2^{-k}\cdot 2^{-k}} C^2 \operatorname{len}(\gamma)\operatorname{diam}(Q)\cdot\varepsilon.$$

□

This was the worst analysis we will do in this course. From now on, the proofs will only be simple applications.

Differentiable images of rectangles are also not really what one needs, but it is easy to adapt Theorem 2.3.1 to different situations.

Theorem 2.3.2: Cauchy's theorem for triangles

If f is holomorphic on U and γ is the boundary curve of a triangular region that is contained in U, then (11).

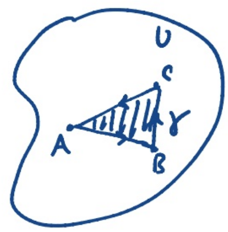


Fig. 19: A triangular region bounded by γ with vertices A, B and C contained in a domain U.

Proof. Apply Theorem 2.3.1 to the function

$$\Phi\colon [0,1]^2 \to U, \qquad (s,t) \mapsto (1-t)\underbrace{((1-s)A + sB)}_{\substack{\text{straight line segment}\\ \text{connecting } A \text{ and } B}} + t\underbrace{((1-s)A + sC)}_{\substack{\text{straight line segment}\\ \text{connecting } A \text{ and } C}}.$$

□

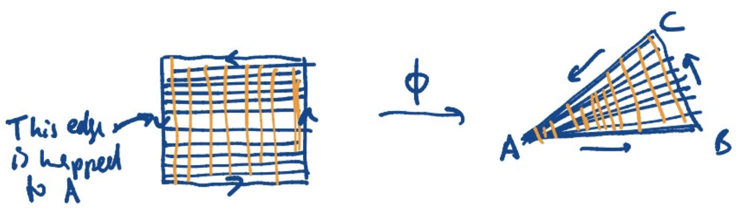


Fig. 20: The action of Φ, which maps the unit square to the triangle ABC.

Theorem 2.3.3: Cauchy's theorem for disk

If f is holomorphic on U and γ is the boundary circle of a closed disk that is contained in U, then (11).

Proof. Let $z_0 \in U$ be the centre and $r > 0$ the radius of the closed disk. Apply Theorem 2.3.1 to the function

$$\Phi\colon [0, 2\pi] \times [0, r] \to U, \qquad (s, t) \mapsto z_0 + te^{is}$$

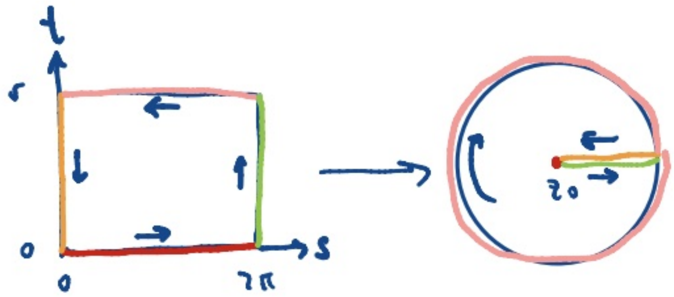


Fig. 21: The action of the map Φ. The bottom edge of the rectangle is mapped to the point z_0. The left edge is mapped to a radius of the circle. The remaining two segments are mapped to the boundary circle and the inverse of the first path.

Only the integral along the pink curve matters, as a single point does not contribute to the integral and the other two paths cancel each other out. The integral along the pink line is zero. $\square$

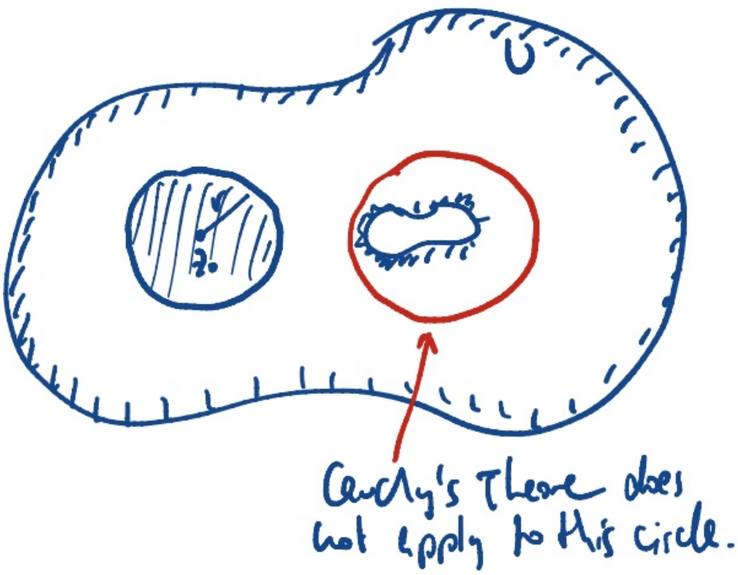


Fig. 22: For this theorem it is important that the disk is contained in U, for the red circle in the above drawing Cauchy's theorem does not hold.

Notation. Integrals along circles are very common, thus there is a special notation for this:

$$\int_{|z-z_0|=r} f(z)\,\mathrm{d}z := \int_\gamma f(z)\,\mathrm{d}z,$$

where $\gamma\colon [0,2\pi] \to \mathbb{C}$, $t \mapsto z_0 + re^{it}$. By convention, the circle $|z-z_0| = r$ is traversed in the counterclockwise direction.

Cauchy's integral theorem for $\mathcal{C}^1$-homotopic curves

Definition 2.3.1 ($\mathcal{C}^1$-homotopic)
Two curves $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are $\mathcal{C}^1$-homotopic in $U \subset \mathbb{C}$ if there exists a $\mathcal{C}^1$-function $H\colon [0,1]^2 \to U$, called homotopy, such that

- $H(0,\cdot) = \alpha$ and $H(1,\cdot) = \beta$,
- $H(\cdot,0) = \alpha(0) = \beta(0)$ and $H(\cdot,1) = \alpha(1) = \beta(1)$.

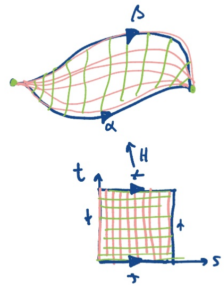


Fig. 23: Two $\mathcal{C}^1$-homotopic curves α and β.

The parameter domain $[t_0, t_1]$ of both curves can be chosen to be $[0,1]$ without loss of generality as we can always reparametrise accordingly.

Theorem 2.3.4: Cauchy's theorem for $\mathcal{C}^1$-homotopic curves

If $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are $\mathcal{C}^1$-homotopic curves in U and f is holomorphic on U, then

$$\int_\alpha f(z)\,\mathrm{d}z = \int_\beta f(z)\,\mathrm{d}z. \tag{15}$$

Proof. Choosing $\Phi = H$, Theorem 2.3.1 implies

$$\int_\alpha f(z)\,\mathrm{d}z - \int_\beta f(z)\,\mathrm{d}z = 0.$$

□

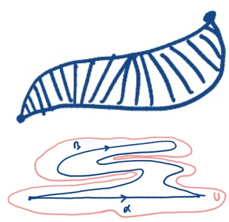


Fig. 24: Our case (below) is more general, as we can see in the both pictures above

Remark 2.3.2 Jähnich, at this point, only considers the case where all straight segments connecting $\alpha(t)$ and $\beta(t)$ are in U. ○

$\mathcal{C}^1$ homotopy is more general, but it is also not the most general case possible. We will see later that it suffices if H is continuous, and that is why $\mathcal{C}^1$ homotopy is not a well known concept, as it can be replaced by something even more general.

Cauchy's theorem for freely $\mathcal{C}^1$ homotopic closed curves

Definition 2.3.3 (freely $\mathcal{C}^1$-homotopic)
Two closed curves $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are freely $\mathcal{C}^1$-homotopic in $U \subset \mathbb{C}$ if there is a $\mathcal{C}^1$-function $H\colon [0,1] \times [0,1] \to U$ such that

- $H(0,\cdot) = \alpha$ and $H(1,\cdot) = \beta$,
- $H(\cdot,0) = H(\cdot,1)$.

U only needs to be a

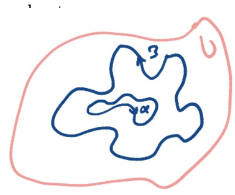
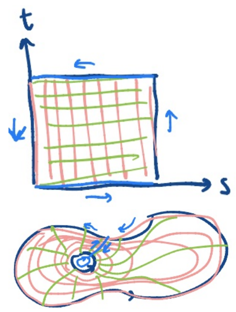


Fig. 25: Two closed freely $\mathcal{C}^1$ homotopic curves α and β.

Theorem 2.3.5: Cauchy's theorem for closed freely $\mathcal{C}^1$ homotopic curves

If $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are closed freely $\mathcal{C}^1$-homotopic curves in U and f is holomorphic on U, then (15).



Proof. We apply Theorem 2.3.1. The image of the boundary of $[0,1]^2$ under H is the curve α traced in the opposite direction, a segment connecting it to β, the curve β and the segment traced in the other direction. $\square$

An important special case is the

Theorem 2.3.6: Cauchy's integral theorem for annuli

If two nested (that is, one is contained in the other and they don't intersect) circles with centres z_0 and z_1 and radii r_0 and r_1 are contained in U together with the region between them, then for all holomorphic functions f on U we have

$$\int_{|z-z_0|=r_0} f(z)\,\mathrm{d}z = \int_{|z-z_1|=r_1} f(z)\,\mathrm{d}z.$$

A special case occurs if $z_0 = z_1$, and then the concentric circles in U bound an annulus in U.

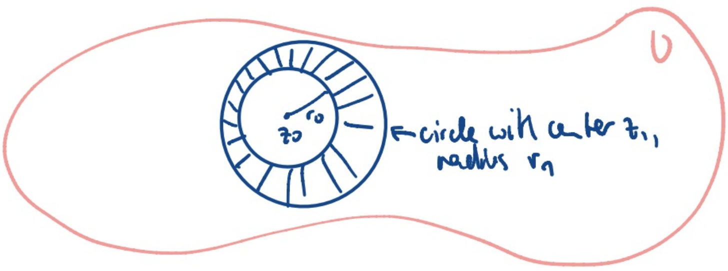


Fig. 27: The setup of Theorem 2.3.6

Let us now consider two example applications of Cauchy's theorem.

Example 2.3.4 ($\int_{\mathbb{R}} e^{-(x-ia)^2}\,\mathrm{d}x = \sqrt{\pi}$ **for all** $a \in \mathbb{R}$) Define

$$I\colon \mathbb{R} \to \mathbb{C}, \qquad a \mapsto \int_{\mathbb{R}} e^{-(x-ia)^2}\,\mathrm{d}x.$$

Then we have $I(a) = I(0)$ for all $a \in \mathbb{R}$.

For $R > 0$ consider the contour given by the $\mathcal{C}^1$ curves

$$\begin{aligned} &\gamma_1\colon [-R,R] \to \mathbb{R} \subset \mathbb{C}, && t \mapsto t, && \gamma_2\colon [0,a] \to \mathbb{C}, && t \mapsto R + it, \\ &\gamma_3\colon [-R,R] \to \mathbb{R} \subset \mathbb{C}, && t \mapsto ia - t, && \gamma_4\colon [0,a] \to \mathbb{C}, && t \mapsto R + ia - it. \end{aligned}$$

We then have

$$\int_{\gamma_1} e^{-z^2}\,\mathrm{d}z = \int_{-R}^{R} e^{-t^2} \xrightarrow{R\to\infty} I(0).$$

and

$$\left|\int_{\gamma_2} e^{-z^2}\,\mathrm{d}z\right| \leqslant \int_0^a e^{t^2-R^2}\,\mathrm{d}t \leqslant a e^{a^2-R^2} \xrightarrow{R\to\infty} 0$$

and analogously for γ_3. By Cauchy's Theorem

$$0 = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} e^{-z^2}\,\mathrm{d}z \xrightarrow{R\to\infty} I(0) - \int_{\mathbb{R}} e^{-(t-ia)^2}\,\mathrm{d}z.$$

$\diamond$

Example 2.3.5 (Todo (Tut IV)) For $a, b > 0$ we have

$$\int_0^{2\pi} \frac{1}{a^2\cos^2(t) + b^2\sin^2(t)}\,\mathrm{d}t = \frac{2\pi}{ab}.$$

Take $\gamma(t)\colon [0, 2\pi] \to \mathbb{C}$, $t \mapsto a\cos(t) + ib\sin(t)$, which is homotopic to $t \mapsto e^{it}$ in $\mathbb{C}^*$. Hence

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_{|z|=1} \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

Hence

$$\begin{aligned}
2\pi = \Im\left(\int_\gamma \frac{1}{z}\,\mathrm{d}z\right) &= \Im\left(\int_0^{2\pi} \frac{-a\sin(t) + ib\cos(t)}{a\cos(t) + ib\sin(t)}\,\mathrm{d}t\right) = \int_0^{2\pi} \Im\left(\frac{-a\sin(t) + ib\cos(t)}{a\cos(t) + ib\sin(t)}\right)\mathrm{d}t \\
&= \int_0^{2\pi} \Im\left(\frac{-(a^2 + b^2)\cos(t)\sin(t) + iab(\sin^2(t) + \cos^2(t))}{a^2\cos^2(t) + b^2\sin^2(t)}\right)\mathrm{d}t \\
&= \int_0^{2\pi} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\,\mathrm{d}t.
\end{aligned}$$

◇

3 First consequences of Cauchy's theorem

After we adapted the Cauchy integral theorem for images of rectangles to particular useful cases, we can now reap the rewards in this section. This section is not called "Consequences of Cauchy's theorem" because ultimately, everything in Complex Analysis is a consequence of Cauchy's theorem.

3.1 Cauchy's Integral Formula

Theorem 3.1.1: Cauchy's integral formula for disks

Let f be holomorphic on the (open) set $U \subset \mathbb{C}$, with $\overline{B_r(z_0)} \subset U$ for $z_0 \in \mathbb{C}$. Then for every $a \in B_r(z_0)$ we have

$$f(a) = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

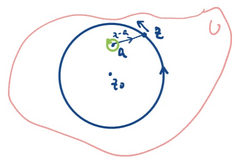


Fig. 28: By Theorem 3.1.1, the values of f in the interior of the disk are determined by the values on its boundary.

Proof. Choose $\varepsilon > 0$ so small that $\overline{B_\varepsilon(a)} \subset B_r(z_0)$. By Theorem 2.3.6,

$$\int_{|z-z_0|=r} \frac{f(z)}{z-a}\,\mathrm{d}z = \int_{|z-a|=\varepsilon} \frac{f(z)}{z-a}\,\mathrm{d}z, \tag{16}$$

because the integrand (which is not defined at a) is nevertheless holomorphic on the annulus (not containing a) bounded by the circles $|z-z_0| = r$ and $|z-a| = \varepsilon$ as it is the quotient of two holomorphic functions. We have

$$\begin{aligned}\int_{|z-a|=\varepsilon} \frac{f(z)}{z-a}\,\mathrm{d}z &= \int_{|z-a|=\varepsilon} \frac{f(a)+f(z)-f(a)}{z-a}\,\mathrm{d}z \\ &= \underbrace{\int_{|z-a|=\varepsilon} \frac{f(a)}{z-a}\,\mathrm{d}z}_{=:A} + \underbrace{\int_{|z-a|=\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z}_{=:B}.\end{aligned}$$

We have

$$A = f(a)\int_{|z-a|=\varepsilon} \frac{1}{z-a}\,\mathrm{d}z = f(a)\int_0^{2\pi} \frac{1}{\cancel{a}+\cancel{\varepsilon e^{it}}\cancel{-a}}\, i\cancel{\varepsilon e^{it}}\,\mathrm{d}t = f(a)\int_0^{2\pi} i\,\mathrm{d}t = 2\pi i f(a).$$

using the parametrisation $\gamma(t) = a + \varepsilon e^{it}$. It remains to show that $B = 0$. Note that B does not depend on ε as long as $\varepsilon > 0$ is small enough: one can immediately see this from Cauchy's theorem for annuli with concentric circles because if we change ε then we get the same result. Hence it is enough to show that

$$\lim_{\varepsilon\searrow 0} \int_{|z-a|=\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = 0.$$

We have

$$\int_{|z-a|=\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{f(a+\varepsilon e^{it})-f(a)}{\cancel{a}+\cancel{\varepsilon e^{it}}\cancel{-a}}\, i\cancel{\varepsilon e^{it}}\,\mathrm{d}t = i\int_0^{2\pi} \underbrace{f(a+\varepsilon e^{it})-f(a)}_{=:h_\varepsilon(t)}\,\mathrm{d}t.$$

Since f is continuous at a, $\lim_{\varepsilon\searrow 0} h_\varepsilon(t) = 0$ uniformly in $t \in [0, 2\pi]$, because continuous functions on compact sets are uniformly continuous. Hence

$$\lim_{\varepsilon\searrow 0} \int_0^{2\pi} h_\varepsilon(t)\,\mathrm{d}t = 0.$$

□

Example 3.1.1 (Tut IV) As sin and cos are entire we have

$$\int_{|z|=1} \frac{\sin(z)}{z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{|z-0|=1} \frac{2\pi i \sin(z)}{z-0}\,\mathrm{d}z = 2\pi i \sin(0) = 0$$

by Theorem 3.1.1 and analogously

$$\int_{|z|=1} \frac{\cos(z)}{z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{|z-0|=1} \frac{2\pi i \cos(z)}{z-0}\,\mathrm{d}z = 2\pi i \cos(0) = 2\pi i.$$

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Example 3.1.2 (Calculating $\int \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z$ (Tut IV)) How can we calculate

$$\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z,$$

where $z_1 \neq z_2$ are complex numbers with $\max(|z_1|, |z_2|) < R$?

We want to find constants $A, B \in \mathbb{C}$ such that

$$\frac{1}{(z-z_1)(z-z_2)} = \frac{A}{z-z_1} + \frac{B}{z-z_2},$$

which can be rewritten as

$$1 = z \cdot \big(A+B\big) - z_2 A - z_1 B.$$

Assuming $A = -B$ (to eliminate the z term), we get

$$1 = -z_2 A - z_1 B = -z_2 A + z_1 A$$

and thus

$$A = \frac{1}{z_1 - z_2} \qquad \text{and} \qquad B = \frac{1}{z_2 - z_1}$$

We found the partial fraction decomposition:

$$\frac{1}{(z-z_1)(z-z_2)} = \frac{A}{z-z_1} + \frac{B}{z-z_2} = \frac{1}{(z-z_1)(z_1-z_2)} + \frac{1}{(z-z_2)(z_2-z_1)}$$

and thus, by the Cauchy integral formula we have

$$\begin{aligned}\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z &= \int_{|z|=R} \frac{f(z)}{(z-z_1)(z_1-z_2)} - \frac{f(z)}{(z-z_2)(z_1-z_2)}\,\mathrm{d}z \\ &= 2\pi i \left(\frac{f(z_1)}{z_1-z_2} - \frac{f(z_2)}{z_1-z_2}\right) = 2\pi i \cdot \frac{f(z_1)-f(z_2)}{z_1-z_2}.\end{aligned}$$

Alternatively we can consider the path tracing a circle around z_1 of a sufficiently small $\varepsilon > 0$, the straight line segment connecting this circle to a small circle around z_2 with radius $\varepsilon > 0$ and the closing the curve up with a segment back to the first circle. The contributions of the connecting segments in the curve cancel each other out. Hence

$$\begin{aligned}\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z &= \int_{|z-z_1|=\varepsilon} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z + \int_{|z-z_2|=\varepsilon} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z \\ &\overset{3.1.1}{=} 2\pi i \left(\frac{f(z_1)}{z_1-z_2} + \frac{f(z_2)}{z_2-z_1}\right).\end{aligned}$$

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In particular we get the following corollary by choosing $a = z_0$.

Corollary 3.1.3 (Mean value property of holomorphic functions)
If f is holomorphic on a domain containing $\overline{B}_r(z_0)$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

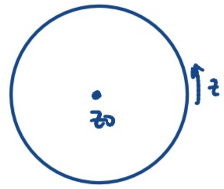


Fig. 29: A circle with centre z_0 and the orientation (counterclockwise) of its boundary curve.

Proof. With the parametrisation $z = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and using Theorem 3.1.1 for $a = z_0$ we obtain

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{\cancel{z_0} + \cancel{re^{it}} - \cancel{z_0}} \cdot \cancel{i} \cdot \cancel{re^{it}}\,\mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}).$$

$\square$

The Real Analysis version of Cauchy's Integral Formula would be: the values of a differentiable function $f\colon [a, b] \to \mathbb{R}$ at the endpoints determine all values in between, which is certainly not true.

3.2 The power series expansion theorem

11.05.2021

Digression (from Jähnich's book)

We consider another consequence of Cauchy's theorem that is not strictly necessary, but it is a nice example of ways to think and also a way to show that power series can be differentiated term by term - by showing that they can be integrated term by term.

Theorem 3.2.1: Complex Version of the Fundamental Theorem of Calculus

Let f be holomorphic on a convex domain U and $z_0 \in U$. Define

$$F\colon U \to \mathbb{C}, \qquad z \mapsto \int_{z_0}^{z} f(u)\,\mathrm{d}u, \tag{17}$$

where we write $\int_a^b$ for the integral along the straight line segment from a to b parametrised by $\gamma(t) = a + t(b - a)$ for $t \in [0, 1]$. Then F is an antiderivative of f, that is, F is holomorphic and $F' = f$.

The convexity is not necessary, it suffices that U is star-shaped with respect to z_0.

In the real case we only require continuity of f. Here we need holomorphicity, because then Cauchy's integral theorem holds and the integral in the definition of F should not depend on the path from z_0 to z (we take a straight line segment anyway, but it should also work for arbitrary paths).

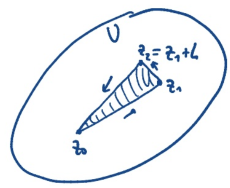


Fig. 30: The closed triangular region with vertices z_0, z_1 and $z_2 := z_1 + h$.

Proof. For $z_1 \in U$, we have to show that F is differentiable at z_1 and $F'(z_1) = f(z_1)$, that is

$$\lim_{h \to 0} \frac{F(z_1 + h) - F(z_1)}{h} = f(z_1).$$

Because U is convex, the closed triangular region with vertices z_0, z_1 and $z_2 := z_1 + h$ is contained in U as long as h is small enough. By Cauchy's Integral Theorem,

$$\int_{z_0}^{z_1} f(z)\,\mathrm{d}z + \int_{z_1}^{z_2} f(z)\,\mathrm{d}z + \int_{z_2}^{z_0} f(z)\,\mathrm{d}z = 0. \tag{18}$$

Hence,

$$F(z_2) - F(z_1) \overset{(17)}{=} \int_{z_0}^{z_2} f(z)\,\mathrm{d}z - \int_{z_0}^{z_1} f(z)\,\mathrm{d}z \overset{(18)}{=} \int_{z_1}^{z_2} f(z)\,\mathrm{d}z = \int_0^1 f(z_1 + th)h\,\mathrm{d}t \tag{19}$$

by using the parametrisation $\gamma\colon [0,1] \to \mathbb{C}$, $t \mapsto z_1 + th$ of the line segment $[z_1, z_2]$. By the Mean Value Theorem of Real Analysis, there exists $\tau, \tau' \in [0,1]$ such that

$$\frac{1}{h}\left(F(z_2) - F(z_1)\right) \overset{(19)}{=} \int_0^1 f(z_1 + th)\,\mathrm{d}t \overset{(\star)}{=} \Re(f(z_1 + \tau h)) + i\Im(f(z_1 + \tau' h)) \xrightarrow{h \to 0} f(z_1)$$

where the limit is due to the continuity of f. $\square$

It is easier to prove that power series can be integrated term-by-term than that they can be differentiated term-by-term, because integration makes function "nicer" and differentiation makes them "worse". Using Theorem 3.2.1, we can prove the second statement using the first. Why is this not usually taught like this? Because power series and differentiation is treated very early on and integration is treated much later.

Theorem 3.2.2: Power series can be integrated term-by-term

If the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$, then the power series $F(z) := \sum_{k=0}^{\infty} \frac{1}{k+1} a_k z^k$ also has radius of convergence R and $F' = f$.

The idea of this proof is that we can interchange limit and integral if the convergence is uniform.

Proof. On compact subsets of the disk $\{z \in \mathbb{C} : |z| < R\}$, the power series $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly, so we may interchange integration and this limit. If we set $f_n(z) := \sum_{k=0}^{n} a_k z^k$, then $F_n(z) := \sum_{k=0}^{n} \frac{1}{k+1} a_k z^{k+1}$ is the uniquely determined antiderivative of f_n with $F_n(0) = 0$ (the last condition is needed for the uniqueness). By Theorem 3.2.1 for z in the disk we have

$$F_n(z) = \int_0^z f_n(u)\,\mathrm{d}u,$$

so by the uniform convergence (UC) of $(f_n)_{n\in\mathbb{N}}$ on the closed line segment from 0 to z,

$$F(z) = \lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \int_0^z f_n(u)\,\mathrm{d}u \overset{\text{UC}}{=} \int_0^z \lim_{n\to\infty} f_n(u)\,\mathrm{d}u = \int_0^z f(u)\,\mathrm{d}u.$$

By Theorem 3.2.1, F is an antiderivative of f. $\square$

We can now prove the second part of Theorem 1.3.1.

Theorem 3.2.3: Differentiating power series term-by-term

If the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$, then the power series $g(z) := \sum_{k=1}^{\infty} k a_k z^{k-1}$ also has radius of convergence R and $g' = f$.

Proof. Integrate g term-by-term and use Theorem 3.2.2 to see that $f(z) - f(z_0)$ is an antiderivative of g, so $f' = g$. $\square$

End of digression.

Theorem 3.2.4: Power series expansion

Let f be a holomorphic function on U. For $z_0 \in U$ there exists a unique power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

with positive convergence radius representing f in some neighbourhood of z_0. The coefficients c_k are determined by CAUCHY's coefficient formula

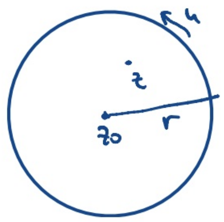
$$c_k = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z,$$

where the only condition on r is to be small enough such that $\overline{B}_r(z_0) \subset U$.
The radius of convergence is not smaller than the radius of the largest open disk around z_0 contained in U.

In other words: "Holomorphic functions can be represented by power series". Since power series are differentiable and their derivatives are again power series, we get the Theorem of GOURSAT.

Corollary 3.2.1 (GOURSAT)
Every holomorphic function is arbitrarily often complex differentiable. In particular it is $\mathcal{C}^\infty$ in the real sense.

Proof. (of Theorem 3.2.4) Uniqueness. Since power series are differentiable by Theorem 3.2.3, $f(z) = \sum_{k=0}^\infty c_k(z-z_0)^k$ implies $f^{(k)}(z_0) = k! c_k$, so the coefficients are $c_k = \frac{f^{(k)}(z_0)}{k!}$ are determined by f in any neighbourhood of z_0 and there can be at most one power series representing f.



Existence. Let $r > 0$ be small enough such that $\overline{B}_r(z_0) \subset U$. By Theorem 3.1.1 for $z \in B_r(z_0)$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|u-z_0|=r} \frac{f(u)}{u-z}\,\mathrm{d}u = \frac{1}{2\pi i} \int_{|u-z_0|=r} \frac{f(u)}{(u-z_0)-(z-z_0)}\,\mathrm{d}u \\ &= \frac{1}{2\pi i} \int_{|u-z_0|=r} \frac{f(u)}{u-z_0} \frac{1}{1-\frac{z-z_0}{u-z_0}}\,\mathrm{d}u =: (\star). \end{aligned}$$

Note that $\frac{z-z_0}{u-z_0} = \frac{z-z_0}{r} < 1$, so we can apply the formula for the geometric series $\frac{1}{1-q} = \sum_{k=0}^\infty q^k$ to $q := \frac{z-z_0}{u-z_0}$:

$$\frac{1}{1-\frac{z-z_0}{u-z_0}} = \sum_{k=0}^\infty \frac{(z-z_0)^k}{(u-z_0)^k}.$$

Hence

$$(\star) = \frac{1}{2\pi i} \int_{|u-z_0|=r} \frac{f(u)}{u-z_0} \sum_{k=0}^\infty \frac{(z-z_0)^k}{(u-z_0)^k}\,\mathrm{d}u = \frac{1}{2\pi i} \int_{|u-z_0|=r} \sum_{k=0}^\infty \frac{f(u)}{(u-z_0)^{k+1}} (z-z_0)^k\,\mathrm{d}u.$$

As the series converges uniformly (as the geometric series converges uniformly) in u with $|u-z_0| = r$, so the above term is equal to

$$\begin{aligned} \frac{1}{2\pi i} \sum_{k=0}^\infty \int_{|u-z_0|=r} \frac{f(u)}{(u-z_0)^{k+1}} (z-z_0)^k\,\mathrm{d}u &= \frac{1}{2\pi i} \sum_{k=0}^\infty \underbrace{\int_{|u-z_0|=r} \frac{f(u)}{(u-z_0)^{k+1}}\,\mathrm{d}u}_{=:c_k} (z-z_0)^k \\ &= \frac{1}{2\pi i} \sum_{k=0}^\infty c_k (z-z_0)^k. \end{aligned}$$

□

Corollary 3.2.2 (CAUCHY estimate for TAYLOR coefficients)
Let f be holomorphic on U and suppose $r > 0$ such that $\overline{B}_r(z_0) \subset U$. Assume that $|f(z)| \leqslant M$ for all z with $|z-z_0| = r$ for some $M > 0$ and let

$$f(z) = \sum_{k=0}^\infty c_k(z-z_0)^k$$

be the power series expansion of f around z_0. Then

$$|c_k| \leqslant M \cdot r^{-k} \qquad \forall k \in \mathbb{N}\,.$$

Proof. By theorem 3.2.4 we have

$$|c_k| = \left| \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{|f(z)|}{|z-z_0|^{k+1}} \,\mathrm{d}z \right| \leqslant \frac{1}{2\pi} \cdot (2\pi r) \frac{M}{r^{k+1}} = \frac{M}{r^k}$$

because $\frac{|f(z)|}{|z-z_0|^{k+1}} \leqslant \frac{M}{r^{k+1}}$ and the length of the curve is $2\pi r$. □

Theorem 3.2.5: Liouville (Cauchy, 1844)

A bounded entire function (that is, f is holomorphic on $\mathbb{C}$ and $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$) is constant.

Proof. The function f is represented by a power series by Theorem 3.2.4 and we can choose 0 as its centre: for all $z \in \mathbb{C}$ we have

$$f(z) = \sum_{k=0}^{\infty} c_k z^k.$$

By corollary 3.2.2 we have

$$|c_k| \leqslant \frac{M}{r^k},$$

for all $r > 0$, so $c_k = 0$ unless $k = 0$. □

Corollary 3.2.3 (Entire functions bounded away from zero (Tut V))
Let f be entire such that there exists a $c > 0$ with $|f(z)| \geqslant c$ for all $z \in \mathbb{C}$. Then f is constant.

Proof. As $f(z)$ is never zero, $\frac{1}{f}$ is well defined and holomorphic as composition of the holomorphic maps $f\colon \mathbb{C} \to \mathbb{C}^*$ and $\frac{1}{\cdot}\colon \mathbb{C}^* \to \mathbb{C}$. The inequality implies $\frac{1}{|f(z)|} \leqslant \frac{1}{c}$, so $\frac{1}{f}$ is constant by Theorem 3.2.5 and thus so is f. □

Corollary 3.2.4
Let f be entire and let $A, R > 0$ and $m \in \mathbb{N}$ be constants such that $f(z) \leqslant A|z|^m$ for all $z \in \mathbb{C}$ with $|z| > R$. Then f is a polynomial of degree at most m.

Proof. Homework 5.2. □

Corollary 3.2.5 (Fundamental Theorem of Algebra)
A polynomial $p \in \mathbb{C}_n[z]$ of degree $n \geqslant 1$ has at least one zero in $\mathbb{C}$.

Proof. (By contradiction) Suppose the polynomial $p(z) := \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$ and $n \geqslant 1$ has no zeros. As p is also holomorphic, so is $f := \frac{1}{p}$. Also

$$|f(z)| = \frac{1}{|a_0 + a_1 z + \ldots + a_n z^n|} = \frac{1}{|z^n| \left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + a_n\right|} \xrightarrow{z\to\infty} 0.$$

So there is an $R > 0$ such that $|f(z)| < 1$ for all $z \in \mathbb{C}$ with $|z| > R$. As f is continuous on the compact disk with radius R, it is bounded there, so f is bounded on $\mathbb{C}$. By Theorem 3.2.5, so f is constant and so is p, which is a contradiction to $n \geqslant 1$ and $a_n \neq 0$. □

12.05.2021

Theorem 3.2.6: Cauchy's Integral Formula for Derivatives

Under the same conditions as in Cauchy's Integral Formula for $f(a)$, we have

$$f^{(k)}(a) = \frac{k!}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{(z-a)^{k+1}}\,\mathrm{d}z.$$

Proof. By Theorem 3.2.4,

$$f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$$

in some open disk around z_0 and we have two equations for the coefficients:

$$c_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int\limits_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z.$$

□

This is another explanation for the fact that complex differentiable functions are so much nicer behaved than real differentiable functions. In Real Analysis, integrating makes functions smoother, while differentiating makes them rougher. In Complex Analysis, however, derivatives are also obtained by an integration process.

Example 3.2.6 (Bounding $f^{(n)}(0)$ when $f(z) \leqslant \frac{1}{1-|z|}$ on the open unit disk)
Let $D := \{z \in \mathbb{C} : |z| < 1\}$ and $f : D \to \mathbb{C}$ holomorphic such that $|f(z)| \leqslant \frac{1}{1-|z|}$ for all $z \in D$. How can we upper bound $|f^{(n)}(0)|$ in a sensible way?

For $r \in (0,1)$ we have by Theorem 3.2.6 and Theorem 2.1.1

$$|f^{(n)}(0)| = \left|\frac{n!}{2\pi i}\int_{|z|=r}\frac{f(z)}{(z-0)^{n+1}}\,\mathrm{d}z\right| \leqslant \frac{n!}{\cancel{2\pi}}\cdot\cancel{2\pi}r\max_{|z|=r}\frac{|f(z)|}{|z|^{n+1}} \leqslant n!\frac{r}{1-r}\frac{1}{r^{n+1}} = n!\frac{1}{(1-r)r^n}.$$

Standard real analysis shows that, given $n \in \mathbb{N}$, $(1-r)r^n$ is minimised for $r = \frac{n}{n+1}$, so we have

$$|f^{(n)}(0)| \leqslant n!\frac{1}{1-\frac{n}{n+1}}\frac{1}{\left(\frac{n}{n+1}\right)^n} = n!\frac{(n+1)^{n+1}}{n^n} = (n+1)!\left(1+\frac{1}{n}\right)^n \leqslant e(n+1)!\,.$$

◇

3.3 Morera's Theorem and Schwarz's reflection principle

Theorem 3.3.1: Morera (Holomorphicity criterion / Converse of Cauchy's Theorem)

Let $U \subset \mathbb{C}$ be an open subset, $f : U \to \mathbb{C}$ be a continuous function and suppose that for each curve γ that bounds a closed triangular region contained in U we have (11). Then f is holomorphic.

Proof. Remember the digression, where we showed that a holomorphic function on a convex domain has a holomorphic antiderivative. The proof works also if we only assume that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for boundary curve of triangles contained in the domain (instead of f being holomorphic).

So for any $z_0 \in U$ let U_0 be an open disk around z_0 that is contained in U. Then $f|_{U_0}$ has an holomorphic antiderivative, which is infinitely often differentiable by corollary 3.2.1, so $f|_{U_0}$ is holomorphic. Hence f is holomorphic. $\square$

Remark 3.3.1 (Motivation of the Schwarz reflection principle)
Suppose f is a holomorphic function defined on a domain U that intersects the real axis and suppose that $f(\mathbb{R}) \subset \mathbb{R}$. This is not an unusual set up: the standard functions from Real Analysis such as polynomials or the exponential functions can also take complex arguments but take real values on the real axis.

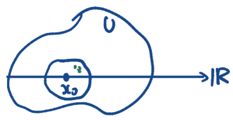


Fig. 31: TODO

If we represent f as a power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - x_0)^k$$

around a point $x_0 \in U \cap \mathbb{R}$, then the coefficients

$$c_k = \frac{f^{(k)}(x_0)}{k!}$$

are real. Hence, if z is contained in a disk around z_0 that is contained in U, then

$$\overline{f(z)} = \overline{\sum_{k=0}^{\infty} c_k (z - x_0)^k} = \sum_{k=0}^{\infty} c_k (\bar{z} - x_0)^k = f(\bar{z}).$$

$\circ$

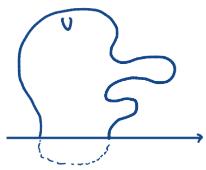


Fig. 32: U is open in the subspace topology of the closed upper half plane.

Theorem 3.3.2: Schwarz reflection principle

Let

$$H := \{z \in \mathbb{C} : \operatorname{Im}(z) \geqslant 0\}$$

be the closed upper half plane and let $U \subset \mathbb{C}$ be open in the subspace topology of H. Suppose $f\colon U \to \mathbb{C}$ is continuous and holomorphic on $U \backslash \mathbb{R}$ and $f(\mathbb{R}) \subset \mathbb{R}$. Then the function

$$\tilde{f}\colon U \cup \overline{U} \to \mathbb{C}, \qquad z \mapsto \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\bar{z})}, & \text{if } z \in \overline{U}, \end{cases}$$

where $\overline{U} := \{z \in \mathbb{C} : \bar{z} \in U\}$ (the bar denotes complex conjugation, NOT closure) is holomorphic.

This means there is an open set $\hat{U} \subset \mathbb{C}$ such that $U = \hat{U} \cap H$.

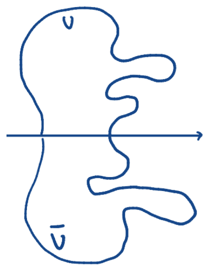


Fig. 33: A set U and its corresponding reflection $\overline{U}$.

Proof. (1) The function $\tilde{f}$ is holomorphic on $\overline{U}\backslash\mathbb{R}$ because complex conjugation is the $\mathbb{R}$ linear map

$$\tau\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

it is differentiable in the real sense with derivative $\mathrm{d}\tau_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Now on $\overline{U}\backslash\mathbb{R}$, $\tilde{f} = \tau \circ f \circ \tau$ is $\mathbb{R}$-differentiable and for any point $z \in \overline{U}\backslash\mathbb{R}$,

$$\mathrm{d}\tilde{f}_z = \mathrm{d}\tau_{f(\bar{z})} \circ \mathrm{d}f_{\bar{z}} \circ \mathrm{d}\tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Hence $\tilde{f}$ is $\mathbb{C}$-differentiable at z and $\tilde{f}'(z) = \overline{f'(\bar{z})}$.

(2) To prove that f is holomorphic on $U \cap \overline{U}$, we apply Theorem 3.3.1. For a boundary curve γ of a closed triangular region contained completely in $U\backslash\mathbb{R}$ or $\overline{U}\backslash\mathbb{R}$ we know that

$$\int_\gamma \tilde{f}(z)\,\mathrm{d}z = 0,$$

because $\tilde{f}$ is holomorphic on these domains by step (1).

(3) Consider a closed triangular region T that intersects $\mathbb{R}$. Let $T^+ := \{z \in T : \Im(z) \geqslant 0\}$ and T^- analogously. Let γ, γ^+ and γ^- be the boundary curve (oriented counterclockwise) of T, T^+ and T^-, respectively. Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma^+} f(z)\,\mathrm{d}z + \int_{\gamma^-} f(z)\,\mathrm{d}z$$

because the contributions of the segments of γ^+ and γ^- on the real axis cancel. It remains to show that $\int_{\gamma^+} f(z)\,\mathrm{d}z = \int_{\gamma^-} f(z)\,\mathrm{d}z = 0$.

(4) Let $T^+_\varepsilon := \{z \in T : \Im(z) \geqslant \varepsilon\}$ for $\varepsilon > 0$ and let γ^+_ε be its boundary curve. Then by the continuity of f

$$\int_{\gamma^+} f(z)\,\mathrm{d}z = \lim_{\varepsilon \searrow 0} \int_{\gamma^+_\varepsilon} f(z)\,\mathrm{d}z = 0,$$

where the second equality is by Cauchy's integral theorem of $\mathcal{C}^1$-images of rectangles. Analogously $\int_{\gamma^-} f(z)\,\mathrm{d}z = 0$. □

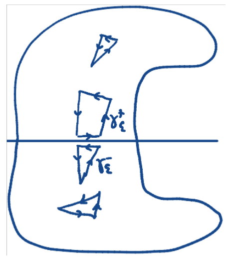


Fig. 34: Different closed triangular regions in the union of U with its reflection $\overline{U}$.

3.4 Zeros of holomorphic functions

The term *holomorphic* comes from the Greek "holo" (meaning "whole") and "morphic" (meaning "shape of" or "similar to"). Hence holomorphic functions are, in a sense, like polynomials. We know that holomorphic functions can be represented as a power series, which are a sort of like polynomials of infinite degree.

By corollary 3.2.5, any polynomial can be decomposed in a product of linear factors, which enables us to define the multiplicity of its zeros. A very similar concept can be defined for holomorphic functions using that they can be represented as a power series.

In the following let $U \subset \mathbb{C}$ be an open subset and $f\colon U \to \mathbb{C}$ be a holomorphic function.

Definition 3.4.1 (Order / Multiplicity of a zero)
The order or multiplicity of a zero $z_0 \in U$ of f is order

$$\operatorname{ord}(f, z_0) := \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$$

or $\operatorname{ord}(f, z_0) := \infty$ if $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$.

Example 3.4.2 (Order of zeros of entire functions) We will find the zeros and their multiplicities of the entire functions

$$f(z) := \cos(z), \qquad g(z) := \cos(z) - 1, \qquad h(z) := e^{z^2} - 1.$$

(1) We have

$$f(z) = 0 \iff 0 = e^{iz} - e^{-iz} = \underbrace{e^{iz}}_{\neq 0}(e^{2iz} + 1) \iff e^{2iz} = -1$$
$$\iff 2z = \pi + 2\pi k,\ k \in \mathbb{Z}$$

Hence the zeros of f are $\{z_k := \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. We have $f'(z_k) = -\sin(z_k) = \pm 1 \neq 0$, so $\operatorname{ord}(f, z_k) = 1$.

2. Similarly, $\cos(z) = 1$, is equivalent to $e^{iz} + e^{-iz} = 2$, that is $(e^{-i\frac{z}{2}} - e^{-i\frac{z}{2}})^2 = 0$, so $e^{-i\frac{z}{2}} - e^{-i\frac{z}{2}} = 1$, that is $e^{iz} = 1$, so $z \in 2\pi\mathbb{Z}$. We have $f'(2\pi\mathbb{Z}) = -\sin(2\pi\mathbb{Z}) = 0$ but $f''(2\pi\mathbb{Z}) = -\cos(2\pi\mathbb{Z}) \neq 0$, so $\operatorname{ord}(g, 2\pi k) = 2$.

3. We have $h(0) = 0$, $h'(z) = 2ze^{z^2}$ and $h''(z) = 2(2z^2+1)e^{z^2}$. Hence $\operatorname{ord}(h, 0) = 2$. The other zeros $z_0^{(k)}, z_1^{(k)}$ of h are implicitly defined by $(z_{0,1}^{(k)})^2 = 2\pi i k$ for $k \in \mathbb{Z}$. We have

$$h'(z) = 2z_{0,1}^{(k)} e^{(z_{0,1}^{(k)})^2} = 2z_{0,1}^{(k)} e^{2\pi i k} = 2z_{0,1}^{(k)} \neq 0$$

so $\operatorname{ord}(h, z_{0,1}^{(k)}) = 1$ for all $k \in \mathbb{Z}$. ◇

Theorem 3.4.1: Isolated singularities

Let U be a domain and let $z_0 \in U$ be a zero of order $k \in \mathbb{N} \cup \{\infty\}$. Then either

1. $k = \infty$ and $f = 0$

or

2. there is a holomorphic function $g \colon U \to \mathbb{C}$ such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^k g(z).$$

In particular, zeros of finite order are isolated.

$x \in S$ is isolated in $S \subset \mathbb{C}$ if there exists a neighbourhood of x in $\mathbb{C}$ that doesn't contain any other points of S.

Proof. In a disk around z_0, f is represented by a power series due to Theorem 3.2.4:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in B_R(z_0)$.

1. If $k = \infty$, then $a_n = \frac{f^{(n)}(z_0)}{n!} = 0$ by Definition 3.4.1, so $f(z) = 0$ for all $z \in B_R(z_0)$. All these z are zeros of infinite order. Hence the set of zeros of infinite order is open. Because f is continuous, the set of *all* zeros is closed in U. We will see in 2 that the set of zeros of finite order is discrete. Hence the set of zeros of infinite order is closed, as the set of finite order zeros is discrete, so the singletons of that set are open, so taking them away from the set of zeros doesn't change its closedness. Since the set of zeros of infinite order is nonempty, open and closed in U and U since is connected, it is equal to U.

2. If $k < \infty$, then $f(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^n = (z - z_0)^k \sum_{m=0}^{\infty} a_{k+m} (z - z_0)^m$ and $a_k \neq 0$. Now we can define

$$g(z) := \begin{cases} \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k, & \text{if } |z - z_0| < R, \\ \frac{f(z)}{(z - z_0)^k}, & \text{if } z \neq z_0. \end{cases}$$

(The definitions agree on the overlap $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ of both cases.) The function g is holomorphic because it is either a power series or a quotient of two holomorphic functions with nonvanishing denominator. □

Theorem 3.4.2: Identity Theorem for Holomorphic Functions

Let U be a domain and f_1 and f_2 be holomorphic on U. If the set

$$M := \{z \in U : f_1(z) = f_2(z)\}$$

has an accumulation point in U, then $f_1 = f_2$.

Proof. The set M is the set of zeros of the holomorphic function $f_1 - f_2$. If it has an accumulation point $z \in U$, that means that if there is a sequence $(z_j)_{j\in\mathbb{N}} \subset M$ with limit $z \in U$, then z is a zero of infinite order, as the set of finite order zeros is isolated by Theorem 3.4.1. Hence $f_1 - f_2 = 0$ by Theorem 3.4.1 1. □

Counterexample 3.4.3 The function $f(z) := \sin\left(\frac{1}{z}\right)$ is holomorphic on $U := \mathbb{C}^*$. It, like the zero function, has zeros at the points $\frac{1}{j\pi}$ for $j \in \mathbb{Z}\setminus\{0\}$. This set has 0 as a accumulation point, but f is not equal to the zero function. This is not a contradiction to Theorem 3.4.2 because the limit point is not in U. ◇

Local behaviour of holomorphic functions near zeros

We will now try to understand the local behaviour of a holomorphic function near one of its zeros. For a motivation, we investigate the simplest function with a zero of order n. **18.05.2021**

Example 3.4.4 For $n \in \mathbb{N}$, the function $f(z) := z^n$ has only one zero, $z = 0$, which has order n (as $f^{(n)} \equiv n! \neq 0$). This function is globally as easy as it is locally. Consider the disk in $B_r(0) \subset \mathbb{C}$. Then $f(B_r(0)) = B_{r^n}(0)$. Writing $z = \varrho e^{i\varphi}$ in polar form, we have $f(z) = \varrho^n e^{in\varphi}$. Hence the argument of all points is multiplied by n when applying f. So as we wrap around zero one time in the domain of f, the image point walks around zero n times. Or if we cut the disk $|z| = r^n$ along the negative real axis, then the preimage is the disk $|z| = r$ with n cuts.

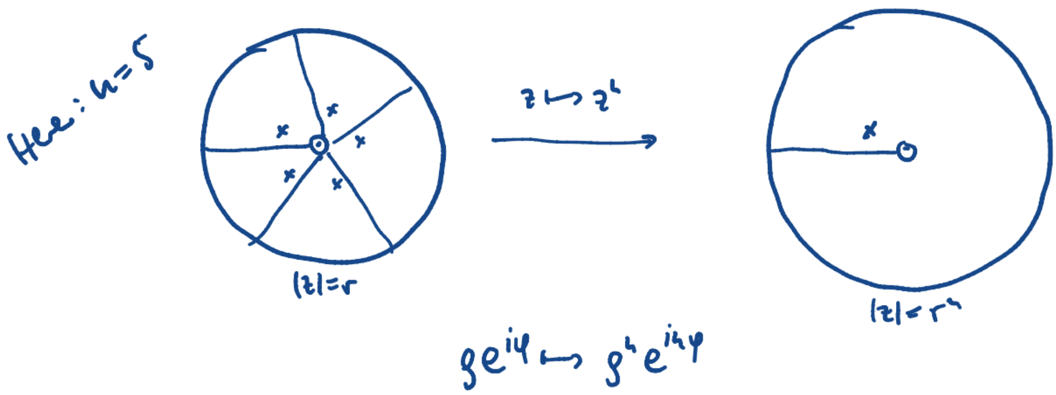


Fig. 35: Caption

In particular, every point except zero in the image of f has exactly n preimages. ◇

We will see that all holomorphic functions show distorted versions of this behaviour.

Let us back up a little and get to gather some more basic theorems.

Theorem 3.4.3: Inverse Function Theorem (Real Version)

If $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^n$ is $\mathcal{C}^1$, $x_0 \in U$ and $\mathrm{d}f_{x_0}\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, then there is an open neighbourhood U_0 of x_0 such that $f(U_0)$ is also open and $f|_{U_0}$ is invertible with continuously differentiable inverse $\mathrm{d}f^{-1}|_{U_0\,f(x_0)} = (\mathrm{d}f_{x_0})^{-1}$.

This implies the complex version:

Theorem 3.4.4: Inverse Function Theorem

If f is holomorphic on U and $f'(z_0) \neq 0$ for some $z_0 \in U$, then there is an open neighbourhood W of $f(z_0)$ on which an holomorphic inverse function $g\colon W \to \mathbb{C}$ exists. We have $g' = \frac{1}{f' \circ g}$.

Proof. The function f is differentiable in the real sense and $\mathrm{d}f_{z_0}$ is the $\mathbb{R}$-linear map $v \mapsto f'(z_0)v$ (multiplication with a complex number is a $\mathbb{R}$-linear map), which is non-singular (if $f'(z_0)v = 0$, then $v = 0$, so $\ker(f'(z_0)) = \{0\}$). The inverse $\mathbb{R}$-linear map $v \mapsto f'(z)v$ is $v \mapsto \frac{1}{f'(z)}v$. So the complex version follows from the real one. □

Let us apply the previous theorem to a particular function. Consider again example 3.4.4. For the function $f(z) = z^n$ there is no well-defined inverse function locally around 0, because the function is not injective. Start with a point w in the image and pick on of its preimages. If we move around zero in the clockwise direction, we move from one preimage to another, so there has to be some discontinuity.

Theorem 3.4.5: Locally defined n-root function

For $w_0 \in \mathbb{C}^*$ and $n \in \mathbb{N}_0$ there exists an open neighbourhood W_0 of w_0 and a holomorphic function R on W_0 such that

$$(R(w))^n = w$$

for all $w \in W_0$.

Check that this is also true for real exponents $n > 0$

Proof. Let $z_0 \neq 0$ be one of the n-th roots of w_0 apply Theorem 3.4.4 to the function $f(z) = z^n$ around z_0. □

We will need this in a more specialised context.

Lemma 3.4.5

Let f be a holomorphic function on U, let $z_0 \in U$ and assume $f(z_0) \neq 0$. For $n \in \mathbb{N}_{>0}$, there exists an open neighbourhood U_0 of z_0 and a holomorphic function g (the locally defined n-th root of f) such that

$$g^n = f|_{U_0}.$$

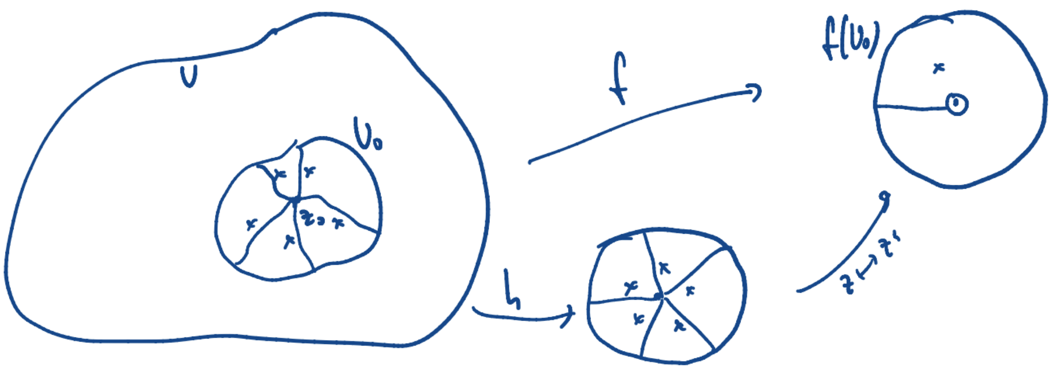
Proof. Let R be the n-th root function defined in a neighbourhood W of $f(z_0)$, which exists by Theorem 3.4.5. Let $U_0 := f^{-1}(W)$, which is open as f is continuous and we are done. □

Definition 3.4.6 (Biholomorphic)
A holomorphic function $f\colon U \to \mathbb{C}$ that has a holomorphic inverse $f^{-1}\colon f(U) \to \mathbb{C}$ is biholomorphic. biholomorphic

The following theorem states in the neighbourhood of a n-th order zero, f behaves like the n-th power function up to a biholomorphic deformation.

Theorem 3.4.6: Local behaviour of a holo. function near a 0

Let f be a holomorphic function on U, let $f(z_0) = 0$ and $n := \operatorname{ord}(f, z_0) < \infty$. Then there is an open neighbourhood U_0 of z_0 and an biholomorphic function h on U_0 such that $h(z_0) = 0$ and $f|_{U_0} = h^n$.
In particular, f takes any non-zero value $w \in f(U_0)$ exactly n times in U_0.



Proof. Since z_0 is an n-th order zero of f,

$$f(z) = (z - z_0)^n g(z)$$

for some holomorphic function g on U with $g(z_0) \neq 0$ by Theorem 3.4.1 (2). By lemma 3.4.5 there is an open neighbourhood U_0 of z_0 and a holomorphic function H on $\tilde{U}_0$ such that $H^n = g|_{\tilde{U}_0}$. Let $h(z) := (z - z_0)H(z)$ on $\tilde{U}_0$. Then $h^n = f|_{\tilde{U}_0}$.

Since $h'(z_0) = H(z_0) \neq 0$ (by the chain rule), the function h is invertible with holomorphic inverse in a neighbourhood U_0 by Theorem 3.4.4. $\square$

Corollary 3.4.7 (Biholomorphy)
A injective holomorphic function is biholomorphic.

Counterexample 3.4.8 (Real Analysis) The function $x \mapsto x^3$ is injective on $\mathbb{R}$ with inverse $x \mapsto \sqrt[3]{x}$ which is differentiable at 0 (the graph has a vertical tangent). $\diamond$

Proof. Suppose f is holomorphic on U. We first show that if f' has no zeros, then f is injective. If $f'(z_0) = 0$ for some $z_0 \in U$, then the function

$$g(z) := f(z) - f(z_0)$$

has a zero of order at least two at z_0, as $g(z_0) = 0$ and $g'(z_0) = f'(z_0) \neq 0$. So g takes any nonzero value in neighbourhood of zero at least twice by Theorem 3.4.6. So f takes any value in a neighbourhood of $f(z_0)$ except for $f(z_0)$ at least twice. So f is not injective.

The inverse of f, which exists as f is injective, is differentiable with derivative $(f^{-1})' = \frac{1}{f' \circ f^{-1}}$. $\square$

3.5 Preservation of Domain, Maximum Principle, Schwarz's Lemma

Theorem 3.5.1: Preservation of Domain

If f is holomorphic and not constant on a domain U, then $f(U)$ is also a domain.

Proof. The image $f(U)$ is connected because it is the image of the connected set U under the continuous function f.

Suppose $w_0 = f(z_0) \in f(U)$. We have to show that $f(U)$ contains an open neighbourhood of w_0. Since f is not constant, the function $g(z) := f(z) - f(z_0)$ has a zero of finite order at z_0 by Theorem 3.4.1. Hence there is an open neighbourhood W of 0 such that g takes any nonzero value in W at least once by Theorem 3.4.6. So f takes any value in the open neighbourhood $f(z_0) + W$ at least once. □

If we forget about connectedness, we get the following corollary:

Theorem 3.5.2: Open mapping theorem

The image $f(U)$ of a holomorphic function on U (this implies that U is open in $\mathbb{C}$) is open.

Counterexample 3.5.1 (Real Analysis) Theorem 3.5.2 is not true for real differentiable functions: the function $x \mapsto x^2$ is differentiable on $\mathbb{R}$, but the image $[0, \infty)$ is not open. ◇

Theorem 3.5.3: Maximum Principle

If f is holomorphic and nonconstant on a domain U, then $|f|$ does not attain a supremum on U.

Proof. Let $z_0 \in U$ and $w_0 := f(z_0)$. As $f(U)$ is open by Theorem 3.5.2, it contains an open disk of radius $\varepsilon > 0$ around w_0 which is not contained in the closed disk $\overline{B}_{|w_0|}(0)$. Hence the ε-disk contains the point $w_1 = f(z_1)$ with $|f(z_1)| = |w_1| > |w_0|$. □

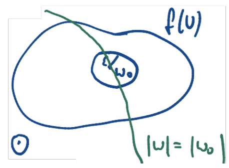


Fig. 36: Illustration of the proof of Theorem 3.5.3, the point in the lower left corner representing the origin.

Similarly, one can prove

Theorem 3.5.4: Maximum principle for $\Re(f)$, $\Im(f)$

For a non-constant holomorphic function f on a domain both $\Re(f)$ and $\Im(f)$ do neither attain a infimum nor a maximum on U.

Proof. We only show that if f a holomorphic function on a domain $U \subset \mathbb{C}$ that is non vanishing and not constant and if $|f|$ attains its infimum in U, then f is constant.

Consider the function $g(z) := \frac{1}{f}$, which is holomorphic on U and well-defined as f does not vanish. If $|f|$ attains its infimum, then $|g|$ attains it supremum, so g has to be constant, so f has to be constant, which is a contradiction. □

19.05.2021

An equivalent formulation of Theorem 3.5.3, which better illustrates its name is

Theorem 3.5.5: Maximum Principle (Version 2)

If f is holomorphic and not constant on U and $K \subset U$ is compact, then $|f|_K|$ attains its maximum on the boundary of K, ∂K.

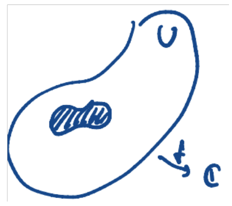


Fig. 37: A compact connected subset of a domain $U \subset \mathbb{C}$ and a function $f\colon U \to \mathbb{C}$.

Proof. Since $|f|_K|$ is continuous, it attains its supremum on the compact set K by a Theorem of Weierstrass. Suppose $|f|_K|$ attains its maximum at a point z_0 in the interior $\mathring{K}$ of K, which is an open set. By Theorem 3.5.3, this implies f is constant on the connected component of $\mathring{K}$ containing z_0. Hence $|f|_K|$ attains its supremum on ∂K. □

One can also deduce Theorem 3.5.3 from Theorem 3.5.5 (Exercise!).

Theorem 3.5.6: Schwarz's Lemma

Let $f\colon D \to D$ be holomorphic with $f(0) = 0$. Then
1. $|f'(0)| \leqslant 1$,
2. $|f(z)| \leqslant |z|$.

If we have $|f'(0)| = 1$ or there is a point $z_0 \in D$ where $|f(z_0)| = |z_0|$, then f is a rotation, that is $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

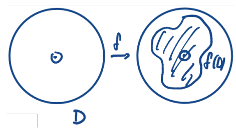


Fig. 38: A holomorphic map $f\colon D \to D$.

Remark 3.5.2 (Rigidity of holomorphic functions) In Real Analysis, the corresponding statement would be: for a differentiable function $f\colon [0,1) \to [0,1)$ we have $|f'(0)| \leqslant 1$ and $|f(x)| \leqslant |x|$ and if $|f'(0)| = 1$, then either $f(x) = x$ or $f(x) = -x$. ○

Proof. 2 Since f has a zero of order at least one at zero, there exists a holomorphic function g on D such that $f(z) = zg(z)$ by Theorem 3.4.1. For all $z \in D$ we have $|f(z)| = |z||g(z)| < 1$ because $f(D) \subset D$. For $r \in (0,1)$ and $z \in \mathbb{C}$ with $|z| = r$ we have

$$|f(z)| = r|g(z)| < 1$$

and so

$$|g(z)| < \frac{1}{r}.$$

By Theorem 3.5.5 the function $|g|$ attains its supremum in the compact set $K := \overline{B}_r(0)$ on the boundary $\partial K = \{z \in \mathbb{C} : |z| = r\}$. So $|g(z)| < \frac{1}{r}$ if $|z| \leqslant r$. This implies $|g(z)| \leqslant 1$ for all $z \in D$: given $z \in D$, we find a $r > |z| \geqslant 0$ such that $|g(z)| \leqslant \frac{1}{r} \xrightarrow{r \nearrow 1} 1$. Therefore, $|f(z)| \leqslant |z||g(z)| \leqslant |z|$.

1 We have

$$|f'(0)| = \left|\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}\right| = \lim_{z \to 0} \underbrace{\left|\frac{f(z)}{z}\right|}_{\leqslant 1} \leqslant 1.$$

Now suppose there exists a $z_0 \in D$ with $|f(z_0)| = |z_0||g(z_0)| = |z_0|$. Then we have $|g(z_0)| = 1$. Since $|g(z)| \leqslant 1$ for all $z \in D$, $|g|$ attains its supremum in D. By Theorem 3.5.3, g is constant, that is, there exists a $a \in \mathbb{C}$ with $|a| = 1$ such that $g(z) = a$. Hence $f(z) = az$.

Finally, suppose $|f'(0)| = 1$. Note that $f'(z) = g(z) + zg'(z)$ and hence $f'(0) = g(0)$. So $|g|$ attains its supremum in zero and is thus constant as before. □

Corollary 3.5.3 ($\mathrm{ord}(0, f) = n \implies |f(z)| \leqslant |z|^n$ (Tut VI))
Let $f\colon D \to D$ be holomorphic and let 0 be a zero of n-th order, where $n \geqslant 1$. Then $|f(z)| \leqslant |z|^n$ for all $z \in D$.

Proof. Consider the holomorphic function $g(z) := \frac{f(z)}{z^{n-1}}$ on D. Then g has a zero of order 1 at zero. We show that $g(D) \subset D$. Consider $g_k(z) := z^{-k} f(z)$ for $k \in \mathbb{N}$. By SCHWARZ's lemma, we have $|f(z)| \leqslant |z|$ and thus $\left|\frac{f(z)}{z}\right| \leqslant 1$ and thus $g_1(D) \subset D$ and g_1 is holomorphic. Applying this iteratively we get $|g_{n-2}| \leqslant |z|$ and thus $|g_{n-1}| \leqslant 1$ and thus $g_{n-1}(D) = g(D) \subset D$. Applying SCHWARZ's lemma to g_{n-1} yields the claim. □

In the last theorem we dealt with functions mapping the unit disk into the unit. Now let us consider a stronger assumption.

THEOREM 3.5.7: HOLOMORPHIC MAPPINGS $D \to D$

Let $f\colon D \to D$ be holomorphic and injective Then there is a $\varphi \in \mathbb{R}$ and a $z_0 \in D$ such that
$$f(z) = e^{i\varphi}\frac{z - z_0}{1 - \overline{z_0}z}.$$

In particular, f then is the restriction to D of a rotated MÖBIUS transformation.

Proof. By corollary 3.4.7, f is biholomorphic.

Case 1. Assume that $f(0) = 0$. Then Theorem 3.5.6 1 implies that $|f'(0)| \leqslant 1$. As $f^{-1}(0) = 0$, we also have
$$\frac{1}{|f'(0)|} = |(f^{-1})'(0)| \leqslant 1$$
by Theorem 3.5.6 and Theorem 3.4.4 and hence $|f'(0)| = 1$. By Theorem 3.5.6, there exists a $a = e^{i\varphi} \in \mathbb{C}$ with $|a| = 1$ for some $\varphi \in \mathbb{R}$ such that $f(z) = az = e^{i\varphi}z$. This proves the theorem with $z_0 = 0$.

Case 2. Assume that $z_0 := f^{-1}(0) \neq 0$. We will show that the MÖBIUS transformation
$$g(z) := \frac{z - z_0}{1 - \overline{z_0}z}$$
maps D bijectively onto D.

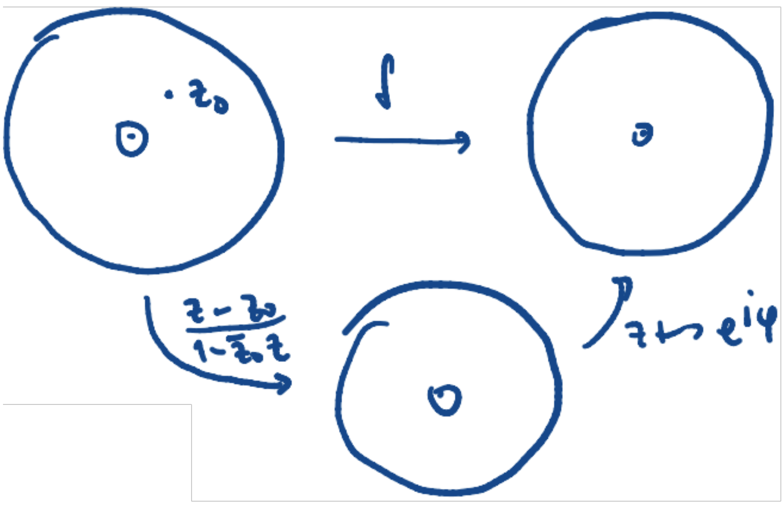


Fig. 39: TODO

Since $f \circ g^{-1}$ is a biholomorphic map $D \to D$ preserving the origin, we have $f \circ g^{-1} = e^{i\varphi} w$ by Case 1. Then $f(z) = e^{i\varphi} g(z)$, which proves the theorem.

It suffices to show $g(D) = D$, as MÖBIUS transformations are bijective maps from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. First we show that $g(\mathbb{S}^1) = \mathbb{S}^1$. If $|z| = 1$, then $\frac{1}{z} = \overline{z}$ and thus

$$\overline{g(z)} = \frac{\overline{z} - \overline{z_0}}{1 - z_0 \overline{z}} = \frac{\frac{1}{z} - \overline{z_0}}{1 - z_0 \frac{1}{z}} = \frac{1 - \overline{z_0} z}{z - z_0} = \frac{1}{g(z)}$$

and so $|g(z)| = \overline{g(z)} g(z) = 1$.

So g^{-1} also maps $\mathbb{S}^1$ to $\mathbb{S}^1$. As a MÖBIUS transformation, g^{-1} either maps the connected components of $\mathbb{C} \setminus \mathbb{S}^1$ to themselves or to each other. Since $g^{-1}(z_0) = 0 \in D$, g^{-1} maps D to D and hence so does g. □

Remark 3.5.4 The injective holomorphic maps mapping D onto D are the MÖBIUS transformations mapping D onto D (cf. Theorem 1.8.7). ○

Remark 3.5.5 Using a MÖBIUS transformation mapping D to the upper half plane H, we see that the MÖBIUS transformations $z \mapsto \frac{az+b}{cz+d}$ with real coefficients fulfilling $ad - bc > 0$ are not only all MÖBIUS transformations mapping H to itself (cf. Theorem 1.8.8), but they are all injective holomorphic maps from H to H. ○

The set $\{z \mapsto e^{i\varphi} \frac{z - z_0}{1 - \overline{z_0} z}\}$ forms a three-dimensional manifold of transformations, as there are three real parameters φ, $\Re(z_0)$ and $\Im(z_0)$, while in general, spaces of holomorphic functions are infinite-dimensional.

There is a very close connecting between Complex Analysis and 2D Hyperbolic Geometry.

Remark 3.5.6 (Geometric interpretation of SCHWARZ's Lemma) If we consider D as the hyperbolic plane in the POINCARÉ disk model. Then SCHWARZ's Lemma says that any holomorphic function f on D with $f(D) \subset D$ is a contraction mapping in the hyperbolic metric:

$$d_{\text{hyp}}(f(z_1), f(z_2)) \leqslant d_{\text{hyp}}(z_1, z_2).$$

The maps $z \mapsto e^{i\varphi} \frac{z - z_0}{1 - \overline{z_0} z}$ are exactly the orientation preserving isometries of the hyperbolic plane in the POINCARÉ disk model. ○

4 Isolated singularities

4.1 Three types of isolated singularities

Definition 4.1.1 (Isolated singularity)
Let f be holomorphic on U. A point $z_0 \in \mathbb{C}\backslash U$ is a isolated singularity of f if there is an open neighbourhood U_0 of z_0 such that $U_0 \cap U = U_0\backslash\{z_0\}$, that is, there is an $\varepsilon > 0$ such that isolated singularity

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\} \subset U.$$

An isolated singularity is "point-shaped hole" in the domain of definition. As with real numbers, where we have continuous completion, there are isolated singularities that are not really singularities and are thus called removable.

Definition 4.1.2 (Removable isolated singularity)
An isolated singularity z_0 of $f: U \to \mathbb{C}$ is removable if there is a holomorphic function $\tilde{f}$ on $U \cup \{z_0\}$ (still open!) such that $f = \tilde{f}|_U$. removable

Hence removable singularities are that isolated singularities z_0 of f that can easily be "fixed" by assigning the "appropriate value" to f at z_0.

Theorem 4.1.1: Riemann'scher Hebbarkeitssatz

If $z_0 \in \mathbb{C}\backslash U$ is an isolated singularity of a holomorphic function $f: U \to \mathbb{C}$, then the following statements are equivalent.

1. The singularity z_0 is removable.
2. f is bounded in a neighbourhood of z_0: there is a $\varepsilon > 0$ and a $M \geqslant 0$ such that
$$|f(z)| \leqslant M \qquad \text{for all } z \in U \cap B_\varepsilon(z_0).$$
3. We have $\lim\limits_{z \to z_0} (z - z_0)f(z) = 0$.

Proof. "1 $\implies$ 2": If z_0 is removable, then by Definition 4.1.2 there exists a holomorphic continuation $\tilde{f}$, which is bounded in a neighbourhood of z_0 because it is continuous. As $f = \tilde{f}|_U$, the statement follows.

"2 $\implies$ 3": is clear by the normal rules of doing limits.

"3 $\implies$ 1": Suppose $\lim_{z \to z_0}(z - z_0)f(z) = 0$. Consider the function

$$g(z) := \begin{cases} (z - z_0)^2 f(z), & \text{if } z \neq z_0, \\ 0, & \text{if } z = z_0. \end{cases}$$

Then g is holomorphic on U. But g is also differentiable in z_0: for $z \neq z_0$ we have

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{g(z)}{z - z_0} = \frac{(z - z_0)^2 f(z)}{z - z_0} = (z - z_0)f(z) \xrightarrow[3]{z \to z_0} 0.$$

So $g'(z_0) = 0$ and so g has a zero of order of at least two at z_0. Around z_0, g is represented by a power series of the form

$$g(z) = \sum_{k=2}^{\infty} a_k (z - z_0)^k$$

for all z in a neighbourhood of z_0 by Theorem 3.2.4. Hence the holomorphic function

$$\frac{g(z)}{(z-z_0)^2} = \sum_{k=0}^{\infty} a_{k+2}(z-z_0)^k$$

has a removable singularity at z_0. Defining $\tilde{f}(z) = \frac{g(z)}{(z-z_0)^2}$ concludes the proof. □

Corollary 4.1.3 (Dominating function (Tut VI))
Let f and g be entire functions such that $|f(z)| \leqslant |g(z)|$ for all $z \in \mathbb{C}$. Then there exists a $a \in \mathbb{C}$ such that $f = ag$.

Proof. Case 1. If g is bounded, then it is constant by Theorem 3.2.5 and hence so is f. So we can choose $a = \frac{g(z_0)}{f(z_0)}$ for any $z_0 \in \mathbb{C}$.

Case 2. If g is not bounded, consider the holomorphic function $h := \frac{f}{g} \colon \mathbb{C} \backslash M \to \overline{D}$, where $M := \{z \in \mathbb{C} : g(z) = 0\}$ is the zero set of g (all zeros (of finite order) are isolated by Theorem 3.4.1). If $z \in M$, then h has a removable singularity at z, so h can be holomorphically continued onto $\mathbb{C}$ and is bounded by 1, as h is bounded by 1 on $\mathbb{C} \backslash M$. By Theorem 3.2.5, h is constant and thus the statement follows. □

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Theorem 4.1.2: 3 types of singularities

Let z_0 be an isolated singularity of a holomorphic function f. There are three possibilities:

1. f is bounded in a neighbourhood of z_0 and hence z_0 is removable.
2. $\lim_{z \to z_0} |f(z)| = \infty$. Then z_0 is a pole of f and there exist a number $m \in \mathbb{N}$ such that $z \mapsto (z-z_0)^m f(z)$ has a removable singularity at z_0. The smallest such exponent m is the order of the pole. pole
3. If none of the above holds, z_0 is an essential singularity. essential singularity

Proof. We only have to prove that only at most one of the possibilities can hold, since by construction of 3, every isolated singularity must fall in one of the three categories.

1 holds by Theorem 4.1.1.

2 Suppose $\lim_{z \to z_0} |f(z)| = \infty$. By Theorem 4.1.1, $\frac{1}{f}$ is bounded in a neighbourhood of z_0, as $\lim_{z \to z_0} \frac{1}{|f(z)|} = 0$. Hence z_0 is a removable singularity of $\frac{1}{f}$. After removing the singularity, one obtains a holomorphic function $g := \frac{1}{f}$ with $g(z_0) = 0$. If m is the order of the zero, $g(z) = (z-z_0)^m h(z)$, where h is a holomorphic function with $h(z_0) \neq 0$ by Theorem 3.4.1 2. Hence $(z-z_0)^m f(z) = (z-z_0)^m \frac{1}{(z-z_0)^m h(z)} = \frac{1}{h(z)}$ for $z \neq z_0$, so $(\cdot - z_0)^m f$ has a removable singularity at z_0. (We also see that the order of the pole is the order of the zero of $\frac{1}{f}$ after the singularity has been removed.) □

Corollary 4.1.4 (TODO (Tut VI))
There does not exists an entire function such that $f(\frac{1}{n}) = \frac{n}{2n-1}$ for all $n \in \mathbb{N}_{>0}$.

Proof. Suppose there exists such a function f Define $g(z) = \frac{\frac{1}{z}}{2\frac{1}{z}-1} = \frac{1}{2-z}$ for $z \in \mathbb{C} \backslash \{2\}$. Then $f(z) = g(z)$ for all $z \in \{\frac{1}{n} : n \in \mathbb{N}_{>0}\}$. As $(\frac{1}{n})_{n \in \mathbb{N}_{>0}}$ has a limit point, by Theorem 3.4.2, we must have $f = g$. But as $\lim_{z \to 2} |g(z)| = \infty$, so $z = 2$ is not a removable singularity of g, so it cannot be continued to a holomorphic function f, which is a contradiction to f being entire. □

We prove a theorem about the behaviour of a function at an essential singularity, the most mysterious type of singularity.

Theorem 4.1.3: Casorati-Weierstrass

If z_0 is an essential singularity of a holomorphic function f on U, then the set of values that f takes on any open punctured neighbourhood of z_0 is dense in $\mathbb{C}$.

Whereas for poles, where the function values tend to infinity when approaching a singularity, near an essential singularity, the set of values of the function is dense, that is, no matter how small a neighbourhood of the singularity we choose, we can come arbitrarily close to any complex number.

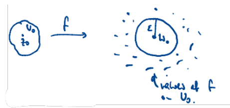

Fig. 40: According to Theorem 4.1.3, any small neighbourhood of the essential singularity gets "splatted" over the whole complex plane.

Proof. We will show: if there is a neighbourhood U_0 of z_0 such that $f(U_0\backslash\{z_0\})$ is not dense in $\mathbb{C}$, then z_0 is a removable singularity or a pole of f. By assumption, there is a complex number $w_0 \in \mathbb{C}$ that is not a a limit point of $f(U_0\backslash\{z_0\})$. Hence there is a $\varepsilon > 0$ such that $|f(z) - w_0| > \varepsilon$ for all $z \in U_0\backslash\{z_0\}$. This implies that

$$g(z) := \frac{1}{f(z) - w_0}$$

is holomorphic on $U_0\backslash\{z_0\}$ and bounded. Hence g has a removable singularity at z_0 by Theorem 4.1.1. Hence

$$f(z) = \frac{1}{g(z)} + w_0$$

has a removable singularity at z_0 or a pole by Theorem 4.1.2, depending on whether $\lim_{z\to z_0} g(z) \neq 0$ (then z_0 is removable) or not (then z_0 is a pole). □

In fact, an even stronger statement is true, whose proof is more complicated and hence omitted.

Theorem 4.1.4: Great Picard's Theorem

In any neighbourhood of an essential singularity, a holomorphic function takes all values in $\mathbb{C}$ or all values in $\mathbb{C}$ except for one.

We have already defined the order of a zero. It makes sense to extend this definition of the order of a zero to poles and assign poles negative numbers.

Definition 4.1.5 (Order of any point)
Let f be holomorphic on U and let z_0 be an isolated singularity of f or a just $z_0 \in U$. The order of f at z_0 is

$$\operatorname{ord}(f, z_0) := \sup\left\{m \in \mathbb{Z} : z \mapsto \frac{f(z)}{(z-z_0)^m} \text{ has a removable sing. at } z_0\right\} \in \mathbb{Z} \cup \{\pm\infty\}$$

with the convention $\sup(\mathbb{Z}) = \infty$ and $\sup(\varnothing) = -\infty$.

Remark 4.1.6 (Consistency of the Definition of the order) This Definition agrees with the previous Definition: if $\operatorname{ord}(f, z_0) = m \geqslant 0$, then f has at most a removable singularity at z_0. After removing the singularity (if necessary), f has a zero of order m at z_0.

If $\operatorname{ord}(f, z_0) = m < 0$ and $m \neq -\infty$, then f has a pole of order $-m > 0$.

If $\operatorname{ord}(f, z_0) = -\infty$, then f has an essential singularity at z_0. ○

Remark 4.1.7 For holomorphic functions f and g we have $\operatorname{ord}(f \cdot g, z_0) = \operatorname{ord}(f, z_0) + \operatorname{ord}(g, z_0)$, where $\infty + (-\infty) := \infty$ (as if f has a zero of infinite order and g has a essential singularity, then f is zero in a neighbourhood of z_0 and thus so is $f \cdot g$). We also have $\operatorname{ord}(\frac{1}{f}, z_0) = -\operatorname{ord}(f, z_0)$ and hence $\operatorname{ord}(\frac{f}{g}, z_0) = \operatorname{ord}(f, z_0) - \operatorname{ord}(g, z_0)$.

In particular: if f has a pole of order n at z_0 and g has a zero of order m at z_0, then there are three cases:

- if $n > m$, then $f \cdot g$ has a pole of order $n - m$ at z_0.
- if $n < m$, then $f \cdot g$ has a zero of order $m - n$.
- if $n = m$, then $f \cdot g$ has a removable singularity at z_0 and $\lim_{z \to z_0} f(z) \cdot g(z) \neq 0$. ○

Example 4.1.8 (Three types of singularities)
The function $f(z) := \frac{1}{1-z^2} = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right)$ has poles of order 1 and $z_0 = \pm 1$, as if we multiply f by $1 \pm z$, then this product can be bounded, so the singularity ∓ 1 can be removed.

The function $f(z) := \frac{1}{\sin(z)}$ has poles of order 1 at the points $z_k := 2\pi k$ for $k \in \mathbb{Z}$, as

$$\frac{z}{\sin(z)} = \frac{z}{\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!} z^{2k+1}} = \frac{1}{\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!} z^{2k}} \xrightarrow{z \to 0} = \frac{1}{1-0} = 1.$$

For the other singularities we use that $\sin(z + k\pi) = (-1)^k \sin(z)$ for all $k \in \mathbb{Z}$, which can be deduced from the power series of sin.

The function $g(z) := e^{\frac{1}{z}}$ has an essential singularity at $z_0 := 0$, as

$$\lim_{\substack{z \to 0 \\ z > 0}} e^{\frac{1}{z}} = \infty \neq 0 = \lim_{\substack{z \to 0 \\ z < 0}} e^{\frac{1}{z}}$$

and, even worse, if $\Re(z) = 0$, then $|e^{\frac{1}{z}}| = 1$, so $\lim\limits_{\substack{z \to 0 \\ \Re(z) = 0}} e^{\frac{1}{z}}$ does not exist. ◇

Example 4.1.9 (Singularities and their orders (Tut VII))
For $n \in \mathbb{Z}$ we find the singularities and their order for the following functions: $f(z) := \frac{\cos(z)}{z^n}$, $g(z) := \frac{\sin(z)}{z^n}$ and $h(z) := \frac{1-\cos(z)}{\sin(z)}$.

- We have $\operatorname{ord}(f, 0) = \operatorname{ord}(\cos, 0) - \operatorname{ord}((\cdot)^n, 0) = 0 - n = -n$ and 0 is the only singularity, as $(\cdot)^n$ is holomorphic on $\mathbb{C}^*$.
- We have $\operatorname{ord}(g, 0) = \operatorname{ord}(\sin, 0) - \operatorname{ord}((\cdot)^n, 0) = 1 - n$ and 0 is the only singularity, as $(\cdot)^n$ is holomorphic on $\mathbb{C}^*$.
- We have $\sin(z) = 0$ if and only if $z = z_k := k\pi$. Hence $\operatorname{ord}(h, z_{2k}) = \operatorname{ord}(1 - \cos, z_{2k}) - \operatorname{ord}(\sin, z_{2k}) = 0 - 1 = -1$ and $\operatorname{ord}(h, z_{2k+1}) = \operatorname{ord}(1 - \cos, z_{2k+1}) - \operatorname{ord}(\sin, z_{2k+1}) = 2 - 1 = 1$ for $k \in \mathbb{Z}$. ◇

4.2 Meromorphic functions

A meromorphic function is a function which is holomorphic except for poles.

Definition 4.2.1 (Meromorphic / holomorphic except for ...)
Let $U \subset \mathbb{C}$ be an open subset. A function f is holomorphic on U except for isolated singularities if f is holomorphic on $U \backslash S$ for some subset $S \subset U$ and all points in S are isolated singularities of f. If all points in S are removable singularities or poles, then f is holomorphic on U except for poles or meromorphic. meromorphic

Meromorphic is Greek and means "fraction-like". We know that holomorphic function behave similar to polynomials (power series expansion, infinitely often differentiable, ...) and we will see that meromorphic functions behave similarly to rational functions.

Lemma 4.2.2 (Quotient of holomorphic functions is meromorphic)
If f and g are holomorphic on a domain $U \subset \mathbb{C}$ and $h \not\equiv 0$, then $\frac{f}{g}$ is meromorphic on U.

This is the generalisation of the statement "if f and g are polynomials and $g \not\equiv 0$, then $\frac{f}{g}$ is a rational function".

Proof. The function $h := \frac{f}{g}$ is holomorphic on $\{z \in U : g(z) \neq 0\}$. Since the zeros of g are isolated by Theorem 3.4.1 (here we need the connectedness of U: otherwise g could be zero on one connected component and nonzero on the other but then the quotient will not be meromorphic on the first connected component. Hence h is holomorphic on U except for isolated singularities. If z_0 is a zero of g of order m and also a zero of f if order n, then z_0 is a removable singularity of h if $n \geqslant 0$ and a pole of order $m-n$ of h otherwise. $\square$

Corollary 4.2.3 (Meromorphic functions on a domain are a field)
If $U \subset \mathbb{C}$ is a domain, then the set of meromorphic functions on U is a field (depending on U), where the operators are defined pointwise (after removing the removable singularities).

This shows that there is a close connection between Complex Analysis and Algebra. Before we show another such connection, let us define zeros and isolated singularities of a function at $z = \infty \in \hat{\mathbb{C}}$.

We had this idea that we can map the complex plane to the sphere and then we get all points except ∞. So ∞ is one point in $\hat{\mathbb{C}}$ and it makes sense to say that ∞ is an isolated singularity of a holomorphic function on some domain.

Definition 4.2.4 (Isolated singularity at ∞)
Let f be holomorphic on some domain U. Then $\infty \in \hat{\mathbb{C}}$ is an isolated singularity of f if there is a number $R \geqslant 0$ such that $\{z \in \mathbb{C} : |z| > R\} \subset U$.

equivalently: if $\mathbb{C} \backslash U$ is bounded and hence compact

Motivation. To classify the isolated singularities at ∞, note the following. If $z_0 \in \mathbb{C}^*$ is a removable singularity, a pole of order m or a essential singularity of f, then $\frac{1}{z_0}$ is a singularity of the same type of the function $g(z) := f\left(\frac{1}{z}\right)$.

Definition 4.2.5 (Singularity at ∞)
If ∞ is an isolated singularity of a holomorphic f, then we say that f has a

- removable singularity at ∞ if $z \mapsto f\left(\frac{1}{z}\right)$ has a removable singularity at 0.
- pole of order m at ∞ if $z \mapsto f\left(\frac{1}{z}\right)$ has a pole of order m at 0.
- essential singularity at ∞ if $z \mapsto f\left(\frac{1}{z}\right)$ has a essential singularity at 0.

Using the results about isolated singularities in $\mathbb{C}$ obtained before we obtain the following characterisation of isolated singularities at ∞: The function f has a removable singularity at ∞ if f is defined and bounded on $\{z \in \mathbb{C} : |z| > R\}$ for some $R \geqslant 0$ and f has a pole at ∞ if $\lim_{z\to\infty} |f(z)| = \infty$. This is case, there is a $m \in \mathbb{Z}_{>0}$ such that $z \mapsto z^{-m} f(z)$ is bounded on $\{z \in \mathbb{C} : |z| > R\}$ for some $R \geqslant 0$. The smallest such m is the order of the pole at ∞. Otherwise, an isolated singularity at ∞ is essential.

Example 4.2.6 (Singularity at ∞)
The entire functions exp, sin and cos have essential singularities at ∞. Any polynomial of degree d has a pole of order d at ∞. (Check for yourselves!) ◇

Lemma 4.2.7 (∞ is 1st-order pole of bijective f (Tut VIII))
Let f be a bijective entire function. Then the isolated singularity at ∞ is a pole of order one.

Proof. 1 We first show that if f is injective and z_0 is an isolated singularity of f, then z_0 is not essential. Assume that z_0 is essential, then by Theorem 4.1.3 $A := f(B_r(z_0)) \subset \mathbb{C}$ is dense. The set $B_R(z_0)\backslash \overline{B}_r(z_0)$ is open and by Theorem 3.5.2 $B := f(B_R(z_0)\backslash \overline{B}_r(z_0))$ is open, too. As A is dense and B is open, there exists a $y \in A \cap B$ and thus there exist $x_1 \in A$ and $x_2 \in B$ such that $f(x_1) = y = f(x_2)$, which is contradiction to the injectivity of f.

2 We now show that if $f\colon \mathbb{C}^* \to \mathbb{C}$ is holomorphic and injective, then the isolated singularity $z_0 = 0$ is removable or a pole of first order. If z_0 is not removable, consider the injective map $g(z) := \frac{1}{f(z)}$. If $\operatorname{ord}(f, 0) = -k$ for $k \in \mathbb{N}$, then $\operatorname{ord}(g, 0) = k$, that is 0 is a k-th order zero of g. By lemma 3.4.5 there exists a holomorphic function h on some neighbourhood of 0 such that $h^k = g$ and $\operatorname{ord}(h, 0) = 1$. There exists a $\varepsilon > 0$ such that $B_\varepsilon(0) \subset h(\mathbb{C}^*)$. We have $\tilde{e}_k := \frac{\varepsilon}{2} e_k \in B_\varepsilon(0)$ for $k \in \{1, \ldots, k\}$, where $e_1, \ldots, e_k$ are the k-th roots of unity. But we have $\tilde{e}_k^k = \frac{\varepsilon^k}{2^k}$, so g cannot be injective and hence f can't be injective provided $k \geqslant 2$.

3 We now show that for an entire bijective function f, ∞ is a pole of order 1. Consider $g(z) := f\left(\frac{1}{z}\right)$ on $\mathbb{C}^*$, which is injective and has an isolated singularity at 0. From 2 we know that 0 is either removable or a pole of first order of g. If 0 were removable for g, then g would be bounded in a neighbourhood of 0, but then f would be bounded. If f is bounded, then it is constant by Theorem 3.2.5, which is a contradiction to the bijectivity of f. Hence 0 is a pole of first order of g and thus ∞ is a pole of first order of f. □

The following Theorem also illustrates the close connection between Algebra and Complex Analysis as rational functions are essentially algebraically defined: rational functions are exactly the functions of one variable that can be defined in terms of finitely many elementary operators ($+$, $-$, $\cdot$, $/$).

Theorem 4.2.1: Meromorphic on $\hat{\mathbb{C}}$ $\iff$ rational

The meromorphic functions on $\hat{\mathbb{C}}$ are precisely the rational functions.

The rational functions are of the form $f = \frac{p}{q}$, where p and q are polynomials and $q \not\equiv 0$.

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Proof. "$\Longleftarrow$": We show that rational functions are meromorphic on $\hat{\mathbb{C}}$. On $\mathbb{C}$, a rational function has only removable singularities and poles. For $z \to \infty$, a rational function f either has a finite limit or $\lim_{z\to\infty} |f(z)| = \infty$. Hence ∞ is a removable singularity or a pole.

"$\Longrightarrow$":

1 Assume that f is meromorphic on $\hat{\mathbb{C}}$. So f has a pole or a removable singularity at ∞. In the first case let $m > 0$ be the order of the pole at ∞, in the second case, let $m = 0$. In either case, there are numbers $R, M \geqslant 0$ such that

$$|z^{-m} f(z)| \leqslant M \qquad \forall z \in \mathbb{C} \text{ with } |z| > R. \tag{20}$$

In particular, there are no further poles in the region $\{z \in \mathbb{C} : |z| > R\}$. All poles, if any, lie in the closed disk $\{z \in \mathbb{C} : |z| \leqslant R\}$. Since the poles are isolated, they cannot have a limit point, the function f can only have finitely many poles $z_1, \ldots, z_n \in \mathbb{C}$. Let $m_1, \ldots, m_n$ be their orders.

(2) Then the function

$$g(z) := f(z) \prod_{k=1}^{n} (z - z_k)^{m_k}$$

has only removable singularities in $\mathbb{C}$. After removal, we obtain an entire function $\hat{g}$. We want to apply corollary 3.2.4 to $\hat{g}$, as then $\hat{g}$ is a polynomial of degree at most m and hence $f(z) = \frac{\hat{g}(z)}{(z-z_1)^{m_1} \cdot \ldots \cdot (z-z_n)^{m_n}}$ is a rational function.

(3) By (20) we have that $|f(z)| \leqslant M|z|^m$ for all $z \in \mathbb{C}$ with $|z| > R$. Thus

$$|\hat{g}| \leqslant M|z - z_1|^{m_1} \cdot \ldots \cdot |z - z_n|^{m_n} |z|^k$$

for all $z \in \mathbb{C}$ with $|z| > R$. Note that $|z| > R$ and $|z_j| \leqslant R$ implies

$$|z - z_j| \leqslant |z| + |z_j| \leqslant |z| + R \leqslant 2|z|.$$

So

$$|\hat{g}| \leqslant M 2^{\tilde{m}} |z|^{m+\tilde{m}},$$

where $\tilde{m} := \sum_{k=1}^{n} m_k$. □

4.3 Laurent series

Laurent series are not power series.

Definition 4.3.1 (Laurent series)
A Laurent series with centre z_0 is a series of the form

Laurent series

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k. \tag{21}$$

More precisely, a Laurent series is composed of two ordinary series:

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{(nonsingular part)}$$

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k. \qquad \text{(principal part)}$$

If both series converge, then the expression (21) also denotes the sum of the limits.

The nonsingular part of a Laurent series is an ordinary power series centred in z_0, so it has a radius of convergence $R \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$. The principal part is a power series in $w := \frac{1}{z-z_0}$ with centre zero. Let its radius of convergence be $\frac{1}{r} \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ (if the radius is 0, then $r = \infty$ and if the radius is ∞, then $r = 0$). Hence the nonsingular part diverges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$ and the principal part converges for ($|w| = \frac{1}{|z-z_0|} < \frac{1}{r}$ and thus) $|z - z_0| > r$ and diverges for $|z - z_0| < r$. If $r < R$, then both parts and hence the Laurent series converges on the domain $\{z \in \mathbb{C} : r < |z - z_0| < R\}$. This domain is an annulus if $0 < r < R < \infty$. This domain can also be the complement of a closed disk in $\mathbb{C}$ (if $r > 0$

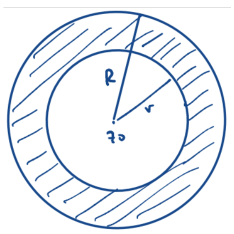


Fig. 41: A definition domain of a Laurent series can be an annulus.

and $R = \infty$) or a punctured disk ($r = 0 < R < \infty$) or the punctured plane ($r = 0$, $R = \infty$). But these are somewhat degenerate cases, in general, one should think of an annulus.

Theorem 4.3.1: Differentiating and integrating Laurent series term by term

If the Laurent series
$$f(z) := \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$
converges on the nonempty domain $U := \{z \in \mathbb{C} : r < |z - z_0| < R\}$ for $r, R \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, then it can be differentiated and integrate term-wise. More precisely
$$f'(z) = \sum_{k=-\infty}^{\infty} k a_k (z - z_0)^{k-1} \qquad \text{and} \qquad \int_\gamma f(z)\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} a_k \int_\gamma (z - z_0)^k\,\mathrm{d}z$$
for any piecewise $\mathcal{C}^1$ curve $\gamma\colon [t_0, t_1] \to U$.

Proof. The statement about differentiation follows from the corresponding statement for the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ and $\sum_{k=1}^{\infty} a_{-k} w^k$ together with the chain rule for $z = \frac{1}{w}$. The statement about integration follows similarly from the fact that power series convergence uniformly on compact subset of the open disks of convergence. □

Lemma 4.3.2 (Cauchy formula for Laurent coefficients)
If the Laurent series $\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ converges on the domain $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ and represents a holomorphic function f there, then
$$a_n = \frac{1}{2\pi i} \int\limits_{|z - z_0| = \varrho} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$
for all $n \in \mathbb{N}$ and any $\varrho \in (r, R)$.

Proof. For simplicity let us assume $z_0 = 0$. Then by Theorem 4.3.1 for $\xi \in (0, R)$ we get
$$\int\limits_{|z| = \xi} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z = \int\limits_{|z| = \xi} \sum_{k=-\infty}^{\infty} a_k \frac{z^k}{z^{n+1}}\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} a_k \int\limits_{|z| = \xi} z^{k-n-1}\,\mathrm{d}z,$$
In example 2.1.3 we showed that
$$\int\limits_{|z| = \xi} z^{k-n-1}\,\mathrm{d}z = \begin{cases} 0, & \text{if } k - n - 1 \neq -1, \\ 2\pi i, & \text{if } k - n - 1 = -1, \end{cases}$$
so every summand except the n-th one vanishes and we get
$$\int\limits_{|z| = \xi} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z = 2\pi i a_n.$$
□

We can now state the Laurent series equivalent of Theorem 3.2.4.

Theorem 4.3.2: Laurent series theorem

Let f be holomorphic on the domain $U := \{z \in \mathbb{C} : r < |z - z_0| < R\}$. Then for all $z \in U$ we have

$$f(z) = \sum_{k=-\infty}^{\infty} a_n (z - z_0)^k,$$

where

$$a_n = \frac{1}{2\pi i} \int\limits_{|z-z_0|=\varrho} \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z \tag{22}$$

for all $n \in \mathbb{Z}$ and any $\varrho \in (r, R)$.

The value of the integral in (22) does not depend on the choice of ϱ by Cauchy's Theorem for Annuli.

01.06.2021

The proof is similar to the proof of Theorem 3.2.4 but requires a few more steps. First, we prove Cauchy's Integral Formula for Annuli and then prove the above theorem. First, a warm-up.

Let us consider two concentric circles, the inner one being centred at 0 and a circle tangent to both circles, centred at 1, with radius $r \in (0, 1)$. Now consider the region enclosed by the largest circle without the area enclosed by the two smaller circles

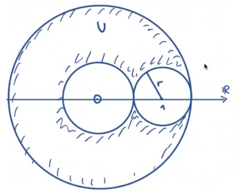


Fig. 42: The setup described above.

$$U := \{z \in \mathbb{C} : 1 - r < |z| < 1 + r, |z - 1| > r\}.$$

We claim that the closure $\bar{U}$ of U is the $\mathcal{C}^1$-image of a closed rectangle. One way to do this is the following. Consider the angles α and β as in figure 43. Then $\tan(\beta) = \frac{r\sin(\alpha)}{1-r\cos(\alpha)}$ and thus $\beta(\alpha) = \arctan\left(\frac{r\sin(\alpha)}{1-r\cos(\alpha)}\right)$. Furthermore, $\varrho(\alpha) = \sqrt{(r\sin(\alpha)^2 + (1 - r\cos(\alpha))^2} = \sqrt{1 - 2r\cos(\alpha) + r^2}$. Define

$$\Phi\colon [0, \pi] \times [0, 1] \to \mathbb{C}, \qquad (\alpha, t) \mapsto \varrho(\alpha) = e^{i\theta(\alpha,t)},$$

where

$$\theta(\alpha, t) := \beta(\alpha) + (2\pi - 2\beta(\alpha))t$$

is a linear interpolation.

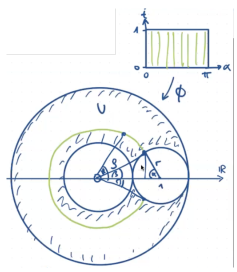


Fig. 43: TODO

We can now easily prove Cauchy's Integral Formula for Annuli.

Theorem 4.3.3: Cauchy's Integral Formula for Annuli

Let $z_0 \in \mathbb{C}$ and let f be holomorphic on the annulus

$$A := \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

for $0 \leqslant r < R \leqslant \infty$. If $z \in \mathbb{C}$ is such that $r < \varrho_1 < |z - z_0| < \varrho_2 < R$, then

$$f(z) = \frac{1}{2\pi i}\left(\int\limits_{|z-z_0|=\varrho_2} \frac{f(u)}{u - z}\,\mathrm{d}u - \int\limits_{|z-z_0|=\varrho_1} \frac{f(u)}{u - z}\,\mathrm{d}u\right)$$

So again, we can represent the value of f anywhere between the two circles $\{z \in \mathbb{C} : |z - z_0| = \varrho_i\}$, $i \in \{1, 2\}$, by integrating the function along those circles, so the values of f between the two circles are completely determined by the values of f along the two circles.

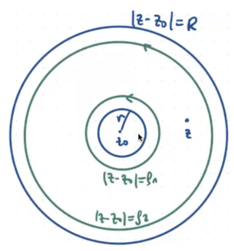


Fig. 44: The setup of Theorem 4.3.3.

Proof. Choose $\varepsilon > 0$ small enough such that the closed ε-disk around z lies completely between the two circles, that is $|z - z_0| + \varepsilon < \varrho_2$ and $|z - z_0| - \varepsilon > \varrho_1$. By Theorem 3.1.1 we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{|u-z|=\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u. \tag{23}$$

Now consider two circles centred at z_0 which touch the disk around z as in figure 45. Since the closed region bounded between the pink circles and outside the blue circle is the $\mathcal{C}^1$-image of a rectangle (see warm-up), Theorem 2.3.1 implies

$$-\int\limits_{|u-z|=r} \frac{f(u)}{u-z}\,\mathrm{d}u + \int\limits_{|u-z_0|=|z-z_0|+\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u - \int\limits_{|u-z_0|=|z-z_0|-\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u = 0.$$

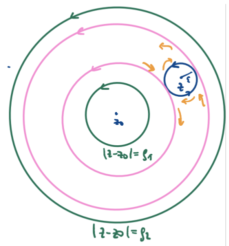


Fig. 45: The orange arrows indicate the orientation of the rectangle, whose $\mathcal{C}^1$-image is the region bounded between the pink circles and outside the blue circle.

Hence we get

$$f(z) \overset{(23)}{=} \frac{1}{2\pi i} \int\limits_{|u-z|=\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u = \frac{1}{2\pi i}\left(\int\limits_{|u-z_0|=|z-z_0|+\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u - \int\limits_{|u-z_0|=|z-z_0|-\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u \right).$$

Finally, by CAUCHY's Integral Theorem for Annuli, we have

$$\int\limits_{|u-z_0|=|z-z_0|+\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u = \int\limits_{|u-z_0|=\varrho_2} \frac{f(u)}{u-z}\,\mathrm{d}u$$

and

$$\int\limits_{|u-z_0|=|z-z_0|-\varepsilon} \frac{f(u)}{u-z}\,\mathrm{d}u = \int\limits_{|u-z_0|=\varrho_1} \frac{f(u)}{u-z}\,\mathrm{d}u.$$

□

Proof. (of Theorem 4.3.2) Assume that $r < \varrho_1 < |z - z_0| < \varrho_2 < R$. By Theorem 4.3.3 we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{|u-z_0|=\varrho_2} \frac{f(u)}{u-z}\,\mathrm{d}u - \frac{1}{2\pi i} \int\limits_{|u-z_0|=\varrho_1} \frac{f(u)}{u-z}\,\mathrm{d}u =: I_1 - I_2.$$

We have

$$\frac{1}{u-z} = \frac{1}{(u-z_0)-(z-z_0)} = \frac{1}{u-z_0}\,\frac{1}{1-\frac{z-z_0}{u-z_0}} = \frac{1}{u-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{u-z_0}\right)^k,$$

by the properties of the geometric series as on the larger circle we have $|u-z_0| = \varrho_2 > |z-z_0|$ and thus $\left|\frac{z-z_0}{u-z_0}\right| < 1$. Using the uniform convergence of power series on compact domains inside the domain of convergence (U) we get

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int\limits_{|u-z_0|=\varrho_2} f(u)\frac{1}{u-z}\,\mathrm{d}u = \frac{1}{2\pi i} \int\limits_{|u-z_0|=\varrho_2} \frac{f(u)}{u-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{u-z_0}\right)^k \mathrm{d}u \\ &\overset{(\mathrm{U})}{=} \frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(\int\limits_{|u-z_0|=\varrho_2} \frac{f(u)}{(u-z_0)^{k+1}}\,\mathrm{d}u\right)(z-z_0)^k \\ &= \sum_{k=0}^{\infty}\underbrace{\frac{1}{2\pi i}\left(\int\limits_{|u-z_0|=\varrho} \frac{f(u)}{(u-z_0)^{k+1}}\,\mathrm{d}u\right)}_{=a_k,\ k\geqslant 0}(z-z_0)^k \end{aligned}$$

for any $\varrho \in (r, R)$.

Similarly, we have

$$-\frac{1}{u-z} = \frac{1}{(z-z_0)-u(-z_0)} = \frac{1}{z-z_0}\frac{1}{1-\frac{u-z_0}{z-z_0}} = \frac{1}{z-z_0}\sum_{m=0}^{\infty}\left(\frac{u-z_0}{z-z_0}\right)^m,$$

as on the smaller circle we have $|u-z_0| = \varrho_1 < |z-z_0|$ and thus $\left|\frac{u-z_0}{z-z_0}\right| < 1$. Hence (with $k = -m-1$)

$$\begin{aligned}
-I_2 &= -\frac{1}{2\pi i}\int_{|u-z_0|=\varrho_1}\frac{f(u)}{u-z}\,\mathrm{d}u = \frac{1}{2\pi i}\int_{|u-z_0|=\varrho_1}\frac{f(u)}{z-z_0}\sum_{m=0}^{\infty}\left(\frac{u-z_0}{z-z_0}\right)^m\mathrm{d}u \\
&= \sum_{m=0}^{\infty}\frac{1}{2\pi i}\left(\int\limits_{|u-z_0|=\varrho_1} f(u)(u-z_0)^m\,\mathrm{d}u\right)\frac{1}{(z-z_0)^{m+1}} \\
&= \sum_{k=-\infty}^{-1}\underbrace{\left(\frac{1}{2\pi i}\int\limits_{|u-z_0|=\varrho}\frac{f(u)}{(u-z_0)^{k+1}}\,\mathrm{d}u\right)}_{=a_k,\ k<0}(z-z_0)^k.
\end{aligned}$$

□

Remark 4.3.3 (Principal part of f at an isolated singularity; classification)
If f is holomorphic except for an isolated singularity at z_0, then for small enough $r > 0$, one can represent f on the punctured disk $\{z \in \mathbb{C} : 0 < |z-z_0| < R\}$ by a LAURENT series. The principal part of it, $\sum_{k=-\infty}^{-1}(z-z_0)^k$ is the principal part of f at the isolated singularity z_0. The principal part of the LAURENT series describes the singular behaviour of f at z_0. There are three possibilities:

1. $a_k = 0$ for all $k < 0$. Then z_0 is a removable singularity and the LAURENT series is a power series because the principal part vanishes.
2. $a_k \neq 0$ for at least one but finitely many $k < 0$. Then z_0 is a pole of order $n := \max\{m \in \mathbb{Z} : a_{-m} \neq 0\}$.
3. $a_k \neq 0$ for infinitely many $k < 0$ (that is, for all $m < 0$ there exists a $k < m$ such that $a_k \neq 0$). Then z_0 is an essential singularity. ○

Example 4.3.4 (LAURENT series expansion of $\frac{1}{1-z}$)
Let us consider the LAURENT series of the function $f(z) := \frac{1}{1-z}$ around $z_0 := 0$. The function f has a pole of first order at 1, so it is holomorphic on $\mathbb{C}\setminus\{1\}$. The function f is holomorphic on the annuli

$$A_1 := \{z \in \mathbb{C} : 0 < |z| < 1\} \qquad \text{and} \qquad A_2 := \{z \in \mathbb{C} : 1 < |z|\}.$$

For $z \in A_1$ we have

$$f(z) = \sum_{k=0}^{\infty} z^k,$$

so the LAURENT series on A_1 is the geometric series. The principal part vanishes.

For $z \in A_2$ we have

$$f(z) = \frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} = \sum_{n=-\infty}^{-1}(-1)z^n$$

and the nonsingular part vanishes.

Let now $z_0 = 1$. Then the LAURENT series around z_0 is

$$f(z) = -\frac{1}{z-1} = (-1)(z-1)^{-1}.$$

◇

Example 4.3.5 (Laurent series of $\exp(z^{-2})$)
The function $g(z) := e^{\frac{1}{z^2}}$ is holomorphic on $\mathbb{C}^*$ and we have

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-2k} = \sum_{n=-\infty}^{0} a_n z^n$$

for all $z \in \mathbb{C}^*$, where $a_n = \frac{1}{(-\frac{n}{2})!}$ if n is nonpositive and even and 0 if $n > 0$ or n is odd. ◇

Example 4.3.6 (Singularities of Laurent series and their orders (Tut VII))
We characterise the singularities of $f(z) := \cos(\frac{1}{z})$. The only singularity of f is 0. We can write

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{-2k} = \sum_{k=-\infty}^{0} \frac{(-1)^k}{(-2k)!} z^{2k},$$

so for all $m \in \mathbb{Z}$ there exists a $k < m$ such that $a_k \neq 0$ and hence $\operatorname{ord}(f, 0) = -\infty$. ◇

Example 4.3.7 (Laurent series of $\frac{2z}{1-z^2}$ (Tut VII))
Consider the function $f(z) := \frac{2z}{(1+z)(1-z)}$. Partial fraction decomposition yields $\frac{1}{1-z} - \frac{1}{1+z}$. For $|z-1| \in (0, 2)$ we want to find the Laurent series of f. We have (as $\left|\frac{z-1}{2}\right| < 1$)

$$\frac{1}{z+1} = \frac{1}{2+z-1} = \frac{1}{2} \frac{1}{1-\left(-\frac{z-1}{2}\right)} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1-z}{2}\right)^k$$

and thus

$$f(z) = \frac{1}{1-z} - \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1-z}{2}\right)^k.$$

◇

5 Analytic continuation

5.0 Motivation and History

02.06.2021

Any mathematical theory should solve some (not necessarily mathematically defined) problem. In this case, the problem is: many holomorphic functions are in a natural way multi-valued. In the case of $z \mapsto \sqrt[n]{z}$ or the logarithm, both functions are the inverse function of some other function ($z \mapsto z^n$ or exp) and the logarithm is an antiderivative of the completely innocuous function $z \mapsto \frac{1}{z}$. There are n n-th roots of any number and also the real part of the complex logarithm is the logarithm of the absolute value, but the imaginary part of the logarithm is the argument and there is no sensible way to define that globally in a unique way. So far, we have stuck to the pragmatic (and somewhat simple-minded) solution to only ever consider such functions on domains $U \subset \mathbb{C}$, where one can pick at each point $z \in U$ one of the values in a consistent way to obtain a holomorphic function on U.

Technically, any function is single-valued by definition.

This pragmatic solution can always be done. But the problem with this solution is that it involves arbitrary choices, like the choice of U and the choice of the value. The different functions one obtains this way belong together somehow as different "branches" (will be defined later) of one "function". Wouldn't it be important to have a theory for this to make all these concept and notions precise? This is what analytic continuation is for.

Remark 5.0.1 (Analytic continuation according to Weierstrass)
Consider the power series

$$z \mapsto \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

with positive radius of convergence $R := (\limsup_{n\to\infty} \sqrt[n]{a_n})^{-1} > 0$. Then this power series defines a holomorphic function on a disk with radius R and centre z_0. Now consider the power series expansion of this function around some $z_1 \in \mathbb{C}$ with $|z_1 - z_0| < R$. We know that the radius of convergence of that new power series (which is said to be a direct analytic continuation) is at least the radius of the largest disk that is contained in the disk of convergence of the power series around z_0. But it can happen that we get a power series that converges in a disk that reaches outside the previous domain. In this case, we can extend the domain, on which the first function is defined, onto the union of the two disks by the Identity Theorem for Holomorphic Functions because both power series must agree on the intersection of the two disks. After performing some iterations of this procedure, it may happen that we obtain a power series that converges on a disk that overlaps with the first disk. Then it may or may not happen that the power series agree on the intersection of both disks.

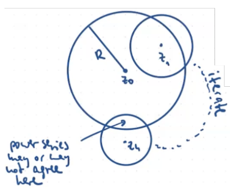


Fig. 46: Iterated direct analytic continuation.

> The idea of Weierstrass and then Riemann was to take a power series and its disk of convergence and then consider all the power series and their disks of convergence that can be obtained by this iterated process of direct analytic continuation. According to Weierstrass, all these power series together describe one "global analytic function".

global analytic function

One can imagine cutting out these disks of convergence out of paper and gluing them together. When the power series agree on the intersection, then we can image getting back to the same point where we already were. If the power series don't agree on the intersection, we don't glue them together. We hence obtain a multiple covering of a part of the complex plane by this process by glued-together disks, called the Riemann surface of the global analytic function. ○

What is nice about this approach is that the power series representations always exist and they give canonical domains for these "function elements". However, always dealing with power series makes some things more complicated and unnecessarily so. For example: the power series around z_1 is a direct analytic continuation of the power series around z_0 but the converse is not true because z_0 is not contained in the disk centred around z_1. So just showing that the equivalence relation $f \sim g$ if f is a direct analytic continuation of g is symmetric requires some work which we don't want to do.

There are different ways not to use power series and we follow a compromise between the book of Jähnich and Alfohrs.

5.1 Analytic Continuation of Function Elements

Instead of power series we will consider holomorphic functions defined on domains. Let us begin with some definitions.

Definition 5.1.1 (Function element)
A function element is a pair (f, U) consisting of a domain $U \subset \mathbb{C}$ and a holomorphic function f on U. function element

Every function already determines its domain but it is useful to have a notation where we have the function and its domain indicated. Furthermore, it is important that U is connected.

Definition 5.1.2 (Direct analytic continuation)
Function elements (f, U) and $(\tilde{f}, \tilde{U})$ are direct analytic continuations of each other if $U \cap \tilde{U} \neq \varnothing$ and $f \equiv \tilde{f}$ on $U \cap \tilde{U}$. direct analytic continuation

This definition of direct analytic continuation is inherently symmetric.

Remark 5.1.3 (Existence/Uniqueness of direct analytic continuation)
If (f, U) is a function element and $\tilde{U}$ is a domain, then there may not exist a direct analytic continuation $(\tilde{f}, \tilde{U})$ because $U \cap \tilde{U}$ may be empty or because $U \cap \tilde{U} \neq \varnothing$ but there exists no holomorphic function $\tilde{f}$ on $\tilde{U}$ such that $\tilde{f} = f$ on $U \cap \tilde{U}$. If $\tilde{U} \subset U$, then there exists a direct analytic continuation $(f|_{\tilde{U}}, \tilde{U})$. Beyond that, not much can be said.

In any case, if there is a direct analytic continuation $(\tilde{f}, \tilde{U})$, then it is unique because if $(g, \tilde{U})$ is also a direct continuation of (f, U), then

$$\tilde{f}|_{\tilde{U} \cap U} = f|_{\tilde{U} \cap U} = g|_{\tilde{U} \cap U},$$

so $\tilde{f} = g$ by the Identity Theorem for Holomorphic Functions. Note that here we use that $\tilde{U}$ is connected. ○

Having defined *direct* analytic continuation, we can define analytic continuation.

Definition 5.1.4 (Analytic continuation along a sequence of domains)
Function elements (f, U) and $(\tilde{f}, U)$ are analytic continuations of each other, if there exists a finite sequence analytic continuation

$$(f, U) = (f_1, U_1),\ (f_2, U_2),\ \ldots,\ (f_n, U_n) = (\tilde{f}, \tilde{U}_n)$$

of function elements such that (f_j, U_j) and (f_j, U_{j+1}) are direct analytic continuations of each other for all $j \in \{1, \dots, n-1\}$.

In this case we say that $(\tilde{f}, \tilde{U})$ is an analytic continuation of (f, U) along the sequence of domains $U_1, \dots, U_n$.

This defines an equivalence relation on the set of function elements, where $(f, U) \sim (\tilde{f}, \tilde{U})$ if and only if (f, U) and $(\tilde{f}, \tilde{U})$ are analytic continuations of each other.

Definition 5.1.5 (Global analytic function, branch)
An equivalence class of $\sim$ as described above is a global analytic function. A function element of an equivalence class is a branch of the global analytic function. branch

5.2 Example: The Complex Logarithm

In Definition 1.5.4, we defined the principal branch of the complex logarithm function as the holomorphic function

$$\log\colon\ \mathbb{C}\setminus\mathbb{R}_{\leqslant 0} \to \mathbb{C}, \qquad z \mapsto \log(|z|) + i\arg(z),$$

where the log on the left side is the real logarithm and $\arg(z) \in (-\pi, \pi)$. We will now define some other branches of the complex logarithm function.

The principal value logarithm is a locally defined inverse of the exponential function. The exponential function is $2\pi i$-periodic and therefore the inverse is not uniquely defined.

Definition 5.2.1 (Non-principal branches of the logarithm)
For $k \in \mathbb{Z}$, let

$$\log_k\colon\ \mathbb{C}\setminus\mathbb{R}_{\leqslant 0} \to \mathbb{C}, \qquad z \mapsto \log(z) + 2\pi i k.$$

and

$$\log_{k+\frac{1}{2}}\colon\ \mathbb{C}\setminus\mathbb{R}_{\geqslant 0} \to \mathbb{C}, \qquad z \mapsto \log_k(-z) + i\pi,$$

where we slit the complex plane along the nonnegative axis.

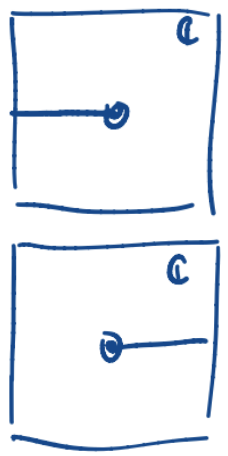


Fig. 47: The domains of $\log_k$ and $\log_{k+\frac{1}{2}}$, where $k \in \mathbb{Z}$.

The function $\Im(\log_k)$ takes values in the open interval $(-\pi + 2\pi k, \pi + 2\pi k)$ and $\Im(\log_{k+\frac{1}{2}})$ takes values in the open interval $(2\pi k, 2\pi(k+1))$. For all $k \in \mathbb{Z}$, $\log_k$ and $\log_{k+\frac{1}{2}}$ agree on the upper half plane $H^+ := \{z \in \mathbb{C} : \Im(z) > 0\}$ and $\log_k$ and $\log_{k-\frac{1}{2}}$ agree on the lower half plane $H^- := \{z \in \mathbb{C} : \Im(z) < 0\}$. In particular, $\log_k$ and $\log_{k+1}$ agree nowhere and their difference is everywhere $2\pi i k$.

So $(\log_k, \mathbb{C}\setminus\mathbb{R}_{\leqslant 0})$ and $(\log_{k+\frac{1}{2}}|_{H^+}, H^+)$ are direct continuations of each other and so are $(\log_{k+\frac{1}{2}}|_{H^+}, H^+)$ and $(\log_{k+\frac{1}{2}}, \mathbb{C}\setminus\mathbb{R}_{\leqslant 0})$. Hence $(\log_k, \mathbb{C}\setminus\mathbb{R}_{\leqslant 0})$ and $(\log_{k+\frac{1}{2}}, \mathbb{C}\setminus\mathbb{R}_{\leqslant 0})$ are only *indirect* analytic continuations of each other. So we can go from the slit complex plane $\mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$ to the top half plane and then from the top half plane to the other slit plane $\mathbb{C}\setminus\mathbb{R}_{\geqslant 0}$ (and vice versa).

Hence all function elements $(\log_k, \mathbb{C}\setminus\mathbb{R}_{\leqslant 0})$ and $(\log_{k+\frac{1}{2}}, \mathbb{C}\setminus\mathbb{R}_{\leqslant 0})$ are different branches of a global analytic function, the global complex logarithm.

Motivated by this example, we add some remarks.

The exponential function is not injective. The global complex logarithm function (sort of the inverse of the exponential function) is not a normal (i.e. single-valued) function. But for each $z \in \mathbb{C}^*$ ($\mathbb{C}^*$ is the image of the exponential function), there is a branch of the global logarithm defined at z_0. (In fact, there are infinitely many branches on different domains.)

We will now see that if a function element is a local inverse of some function f, then any analytic continuation is also a local inverse of f.

Lemma 5.2.2 (Analytic continuation of local inverse of a holomorphic function) *Let f be an entire function and (g, U) be a function element such that $f(g(z)) = z$ for all $z \in U$. If $(\tilde{g}, \tilde{U})$ is a analytic continuation of (g, U), then $f(\tilde{g}(z)) = z$ for all $z \in \tilde{U}$.*

Proof. The general case follows directly from the special case that $(\tilde{g}, \tilde{U})$ is a *direct* analytic continuation of (g, U), because any non-direct analytic continuation is a sequence of direct analytic continuations and if the property of being a local inverse of f is preserved from one direct continuation to the other, then it is preserved for all steps. So assume $(\tilde{g}, \tilde{U})$ is a direct analytic continuation of (g, U), that is $U \cap \tilde{U} \neq \varnothing$ and $g \equiv \tilde{g}$ on $U \cap \tilde{U}$.

Hence for $z \in U \cap \tilde{U}$ we have $f(\tilde{g}(z)) = f(g(z)) = z$. So $f \circ g$ and the identity function $z \mapsto z$ agree on $U \cap \tilde{U} \subset \tilde{U}$. By Identity Theorem for Holomorphic Functions $f \circ \tilde{g}$ and $z \mapsto z$ agree of the domain $\tilde{U}$. □

Hence if f is holomorphic on $U \subset \mathbb{C}$, $z_0 \in U$ is a point and $f'(z_0) \neq 0$, then by the Inverse Function Theorem, there exists a small neighbourhood U_0 of z_0 and a local inverse function g. We can now analytically continue this local inverse function to a global analytic function and all the branches of this global analytic function will be local inverse functions of f.

In the same way we can prove the following statements:

Lemma 5.2.3 (Analytic continuation and algebraic or differential equations) *Suppose the coefficient functions $a_0, \ldots, a_n, b$ are entire and that the function elements (f, U) and $(\tilde{f}, \tilde{U})$ are analytic continuations of each other.*

- *If f satisfies the pointwise polynomial equation*

$$a_n f^n + a_{n-1} f^{n-1} + a_1 f + a_0 = b$$

on U, then $\tilde{f}$ satisfies the pointwise polynomial equation

$$a_n \tilde{f}^n + a_{n-1} \tilde{f}^{n-1} + a_1 \tilde{f} + a_0 = b$$

on $\tilde{U}$.

- *If f satisfies the linear differential equation*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots a_1 y = b$$

on U, then $\tilde{f}$ satisfies the same differential equation on $\tilde{U}$.

Proof. Homework 8.2 □

Hence if we define a function locally implicitly by a polynomial equation, where the coefficients are functions of z and continue this function analytically, then the other branches also satisfy this polynomial equation and something similar holds for differential equations.

08.06.2021

Lemma 5.2.4 (??)
Let (f, U) and (g, W) be function elements. The following statements are equivalent:

1. *$f \equiv g$ on $U \cap W$.*
2. *There exists a holomorphic function h on $U \cup W$ such that $h|_U = f$ and $h|_W = g$.*

Proof. "2 $\implies$ 1": For $z \in U \cap W$ we have $f(z) = h(z) = g(z)$.
1 $\implies$ 2": Define

$$h: U \cup W \to \mathbb{C}, \qquad z \mapsto \begin{cases} f(z), & \text{if } z \in U, \\ g(z), & \text{if } z \in W, \end{cases}$$

which is well-defined by 1 and holomorphic as f and g are. □

5.3 Analytic continuation along curves

We discussed the direct analytic continuation of function elements and we discussed the indirect analytic continuation, which is just the repeated process. But we need to put some order into the function elements that can be obtained by analytic continuation and it turns out that it is important to have the notion of continuation along curves to get a grip on this.

Definition 5.3.1 (Analytic continuation along curves)
Let $\gamma: [t_0, t_1] \to \mathbb{C}$ be a continuous curve. A function element $(\tilde{f}, \tilde{U})$ is an analytic continuation of a function element (f, U) along γ if there is a family of function elements $((f_t, U_t))_{t \in [t_0, t_1]}$ such that

1. $(f_{t_0}, U_{t_0}) = (f, U)$ and $(f_{t_1}, U_{t_1}) = (\tilde{f}, \tilde{U})$,
2. $\gamma(t) \in U_t$ for all $t \in [t_0, t_1]$ and there exists a $\varepsilon > 0$ such that for each $t' \in [t_0, t_1]$ with $|t - t'| < \varepsilon$ we have $\gamma(t') \in U_t$ and $f_{t'}$ agrees with f_t on $U_t \cap U_{t'}$.

In contrast to analytic continuation, the family of function elements is a continuous and not a discrete set.

Lemma 5.3.2 (From direct continuation to continuation along a curve)
Suppose there is a finite family

$$(f, U) = (f^{(0)}, U^{(0)}), (f^{(1)}, U^{(1)}) \dots (f^{(n)}, U^{(n)}) = (\tilde{f}, \tilde{U})$$

such that

1. *$(f^{(j)}, U^{(j)})$ and $(f^{(j+1)}, U^{(j+1)})$ are direct analytic continuations of each other for every $j \in \{0, \dots, n-1\}$,*
2. *there is a subdivision*
$$t_0 = \tau_0 < \tau_1 < \dots < \tau_n = t_1$$
such that $\gamma(\tau_j) \in U^{(j)}$ for all $j \in \{0, \dots, n\}$ and $\gamma([\tau_j, \tau_{j+1}]) \subset U^{(j)} \cup U^{(j+1)}$ for all $j \in \{0, \dots, n-1\}$.

Then $(\tilde{f}, \tilde{U})$ is an analytic continuation of (f, U) along γ.

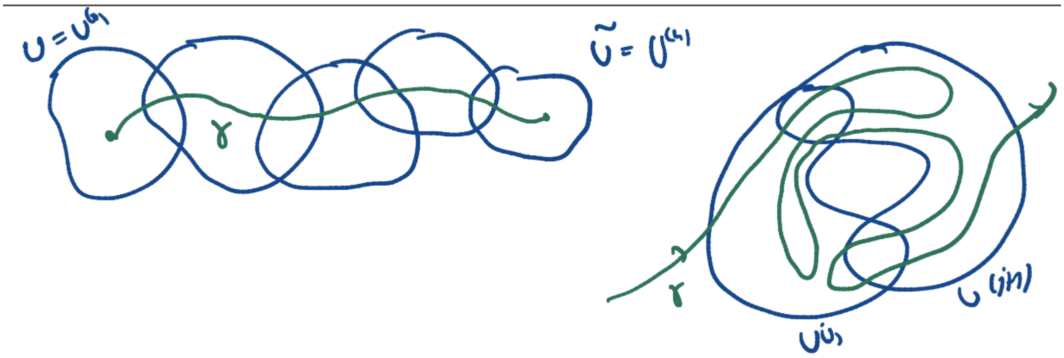


Fig. 48: An analytic continuation along a curve via a finite family of domains (left: simple version, right: more complicated transition from $U^{(j)}$ to $U^{(j+1)}$).

Proof. Define the family of function elements $((f_t, U_t))_{t \in [t_0, t_1]}$ as follows

- If $t = \tau_j$ for some $j \in \{0, \ldots, n\}$, then $(f_t, U_t) = (f^{(j)}, U^{(j)})$.
- If $t \in (\tau_j, \tau_{j+1})$ for some $j \in \{0, \ldots, n-1\}$, let $U_t = U^{(j)} \cup U^{(j+1)}$ and let f_t be the holomorphic function on U_t that is equal to $f^{(j)}$ on $U^{(j)}$ and equal to $f^{(j+1)}$ on $U^{(j+1)}$. □

So what we really do is to chose the f_t and U_t to be "piecewise constant": in the open intervals between the subdivision points they remain constant, in the end point the domain becomes smaller and then it becomes constant again.

We have defined what it means to continue a function element analytically along a curve and now we talk about what happens if we continue essentially the same function element. The following lemma roughly states that analytic continuation along curves is uniquely determined - it depends only on the curve - but we have to be satisfied with only neighbourhoods around the starting and endpoint.

Lemma 5.3.3 (If function elements agree, their continuations do, too)
Let $\gamma \colon [t_0, t_1] \to \mathbb{C}$ be a continuous curve, let $(\tilde{f}, \tilde{U})$ be an analytic continuation of (f, U) along γ and let $(\tilde{g}, \tilde{W})$ also be an analytic continuation of (g, W) along γ. If f and g agree on some open neighbourhood $V_0 \subset U \cap W$ of $\gamma(t_0)$, then $\tilde{f}$ and $\tilde{g}$ are equal on some open neighbourhood $V_1 \subset \tilde{U} \cap \tilde{W}$ of $\gamma(t_1)$.

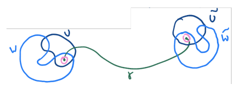


Fig. 49: We have extended the function from (some neighbourhood of) the starting point to (some neighbourhood of) the ending point.

Proof. 1 By assumption, there exists families of function elements $((f_t, U_t))_{t \in [t_0, t_1]}$ and $((g_t, W_t))_{t \in [t_0, t_1]}$ such that

- $(f_t, U_{t_0}) = (f, U)$, $(f_{t_1}, U_{t_1}) = (\tilde{f}, \tilde{U})$, $(g_t, W_{t_0}) = (g, W)$, and $(g_{t_1}, W_{t_1}) = (\tilde{g}, \tilde{W})$,
- $\gamma(t) \in U_t \cap W_t$ for all $t \in [t_0, t_1]$,
- for each $t \in [t_0, t_1]$ there is a number $\varepsilon > 0$ such that for all $t' \in [t_0, t_1]$ with $|t - t'| < \varepsilon$ we have $\gamma(t') \in U_t \cap W_t$ and also $f_{t'} \equiv f_t$ on $U_t \cap U_{t'}$ and $g_{t'} \equiv g_t$ on $W_t \cap W_{t'}$.

We actually have two εs, one for each f and g, but we can just take ε as their minimum.

2 Let

$$A := \{t \in [t_0, t_1] : f_t \text{ and } g_t \text{ agree on some open neighbourhood of } \gamma(t).\}$$

We want to show that $t_1 \in A$ by showing that $A = [t_0, t_1]$.

1. It is $A \neq \varnothing$ because $t_0 \in A$ by assumption.
2. A is open in $[t_0, t_1]$, that is, for each $t \in A$ there is a number $\varepsilon > 0$ such that $t' \in A$ if $t' \in [t_0, t_1]$ and $|t - t'| < \varepsilon$.
3. A is closed.

Together, these three imply that $A = [t_0, t_1]$. We have to show 2 and 3.

2: Suppose $t \in A$. Then f_t and g_t agree in some open neighbourhood V_t of $\gamma(t)$. Also, there is a number $\varepsilon > 0$ such that the following holds for all $t' \in [t_0, t_1]$ with $|t - t'| < \varepsilon$:

- $\gamma(t') \in V_t$ (because γ is continuous and V_t is an open set),
- $f_{t'}$ and $f_{t'}$ agree on $U_t \cap U_{t'}$,
- g_t and $g_{t'}$ agree for all $W_t \cap W_{t'}$.

Then ft' and $g_{t'}$ agree on $V_t \cap U_t \cap U_{t'} \cap W_t \cap W_{t'}$, which is an open neighbourhood of $\gamma(t')$. So $t' \in A$.

Fig. 50: The pink circle is the disk D.

3: Suppose $t \in [t_0, t_1]$ is a limit point of A. Let D be an open disk around $\gamma(t)$ that is contained in $U_t \cap W_t$.

By assumption, there is a sequence $(s_k)_{k \in \mathbb{N}} \subset A \cap D$ with limit t. Hence g_{s_k} and f_{s_k} agree in some open neighbourhood of $\gamma(s_k)$ for all $k \in \mathbb{N}$. If k is large enough, then g_{s_k} agrees with g_t on $W_{s_k} \cap W_t$ and f_{s_k} agrees with f_t on $U_{s_k} \cap U_t$ (local compatibility condition). So for k large enough, the following functions agree on some neighbourhood of $\gamma(s_k)$: $g_{s_k}, g_t, f_{s_k}, f_t$. Since the holomorphic functions g_t and f_t agree on a nonempty open subset of the domain D, they are equal on the whole of D by the Identity Theorem for Holomorphic Functions. Hence $t \in A$. □

Back to something more concrete and less technical, we will look at what kind of non-obvious things can happen with analytic continuation.

Example 5.3.4 (The dilogarithm function)
Start with the geometric series

$$1 + z + z^2 + z^3 + \ldots = \frac{1}{1 - z}$$

for $|z| < 1$. If we continue this function on the unit disk analytically along arbitrary paths, then we get (restrictions of) the holomorphic function $z \mapsto \frac{1}{1-z}$ on $\mathbb{C} \setminus \{1\}$.

Integrating this power series (and setting the constant of integration to zero), we obtain

$$z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots = -\log(1 - z)$$

for $|z| < 1$. Dividing by z yields

$$1 + \frac{1}{2}z + \frac{1}{3}z^2 + \ldots = -\frac{\log(1 - z)}{z}$$

for $|z| < 1$. This is also holomorphic on the unit disk because the singularity at $z = 0$ is removable.

Integrating again, we obtain the dilogarithm function: dilogarithm

$$z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \ldots = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1 - u)}{u}\, du =: \mathrm{Li}_2(z)$$

for $|z| < 1$.

What happens to these function if we continue them analytically? The domain of $\log(1-z)$ can be extended to $\mathbb{C}\setminus[1,\infty)$. The singularity of $\frac{\log(1-z)}{z}$ at 0 is removable, s the domain of $\frac{\log(1-z)}{z}$ can also be extended to $\mathbb{C}\setminus[1,\infty)$. And this is also the domain of the principal branch of the dilogarithm function.

If we continue $-\frac{\log(1-z)}{z}$ along γ as in figure 51, we go to another branch of the logarithm, which does not have a zero a 1 anymore (the value is $\pm 2\pi i$). So the analytic continuation of $-\frac{\log(1-z)}{z}$ has a singularity at 0, which is not removable but a pole of order 1.

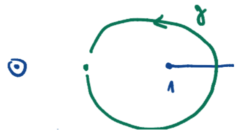


Fig. 51: TODO

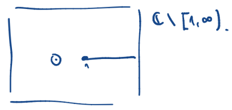
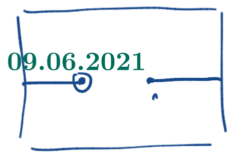


Fig. 52: $\mathbb{C}\setminus[1,\infty)$.

So the picture of the global dilogarithm is this: the principal branch is defined on $\mathbb{C}\setminus[1,\infty)$. The next branches are obtained by analytic continuation along curves that cross the cut. These can be defined on a doubly slit plane. Each side of one of the cuts is glued to cuts of other branches.

In particular, if we take the principal branch of the dilogarithm and continue it analytically around zero, then nothing happens because the function is well defined at 0. If we continue around γ, we go to a different branch, which cannot be continued around zero anymore. ◇



Sometimes, Mathematics is difficult because there are contradicting objectives. For one thing, we need absolutely precise definitions. If our concepts are not precisely defined, all statements about them are mathematically meaningless because it is not clear what they mean. On the other hand, often it is the case that we can view certain things from different perspectives and they may look very different. If often happens that these diverse points of view are really useful because somethings seem clearer from one point of view but not all things are clear from one single point of view. Maybe we would even like the different perspectives to have different definitions for the same concept. This is particularly true for analytic continuation, explaining why in many textbooks the definitions are slightly different.

Hence we want to back up a little and provide some material which allows us to better translate between different points of view. We will use the different points of view because it is very useful to be able to translate between multiple points of view.

Definition 5.3.5 (Loc. compatible function elements along curve) A family of locally compatible function elements along a curve $\gamma\colon [t_0,t_1] \to \mathbb{C}$ is a family of function elements $((f_t,U_t))_{t\in[t_0,t_1]}$ with the following property that for every $t \in [t_0,t_1]$ there exists a number $\varepsilon > 0$ such that all $t' \in [t_0,t_1]$ with $|t-t'| < \varepsilon$ we have

family of locally compatible function elements

1. $\gamma(t') \in U_t$ (in particular $\gamma(t) \in U_t$),
2. (f_t,U_t) and $(f_{t'},U_{t'})$ are direct continuations of each other, i.e. $f_t \equiv f_{t'}$ on $U_t \cap U_{t'}$.

Remark 5.3.6 If $((f_t,U_t))_{t\in[t_0,t_1]}$ is a family of locally compatible function elements along $\gamma\colon [t_0,t_1] \to \mathbb{C}$, then the following statements are easy to check:

1. One can make the domain smaller, that is, if for each $t \in [t_0 t_1]$, W_t is a domain contained in U_t and containing $\gamma(t)$, then $(f_t|_{W_t}, W_t)$ is also a family of locally compatible function elements along γ. In particular, we may choose W_t to be an open disk with centre $\gamma(t)$ that is small enough.
2. the parametrisation is not so important, that is, if $\Phi\colon [s_0,s_1] \to [t_0,t_1]$ is a continuous function and $\tilde\gamma := \gamma \circ \varphi$, then $((f_{\varphi(s)},U_{\varphi(s)}))_{s\in[s_0,s_1]}$ is a family of locally compatible function elements along $\tilde\gamma\colon [s_0,s_1] \to \mathbb{C}$. ○

The Diversity Lemma allows us to translate between different points of view.

Lemma 5.3.7 (Diversity Lemma)
Let $\gamma\colon [t_0,t_1] \to \mathbb{C}$ be a continuous curve and let (f,U) and $(\tilde{f},\tilde{U})$ be function elements. Then the following statements are equivalent.

1. *There exists a family of locally compatible function elements $((f_t,U_t))_{t\in[t_0,t_1]}$ along γ such that $(f_{t_0},U_{t_0}) = (f,U)$ and $(f_{t_1},U_{t_1}) = (\tilde{f},\tilde{U})$ (that is, $(\tilde{f},\tilde{U})$ is an analytic continuation of (f,U)) along γ.*

2. *There exists a subdivision*
$$t_0 = \tau_0 < \ldots < \tau_n = t_1$$
of $[t_0,t_1]$ and function elements $((f_j,U_j))_{j=0}^n$ such that
   - *consecutive function elements (f_j,U_j) and (f_{j+1},U_{j+1}) are direct analytic continuations of each other for all $j \in \{0,\ldots,n-1\}$,*
   - *$\gamma(\tau_j) \in U_j$ for all $j \in \{0,\ldots,n\}$,*
   - *$\gamma([\tau_j,\tau_{j+1}]) \subset U_j \cup U_{j+1}$ for all $j \in \{0,\ldots,n-1\}$,*
   - *$(f_0,U_0) = (f,U)$ and $(f_n,U_n) = (\tilde{f},\tilde{U})$.*

3. *There exists a subdivision*
$$t_0 = \tau_0 < \ldots < \tau_n = t_1$$
of $[t_0,t_1]$ and function elements $((f_j,U_j))_{j=0}^n$ such that
   - *consecutive function elements (f_j,U_j) and (f_{j+1},U_{j+1}) are direct analytic continuations of each other for all $j \in \{0,\ldots,n-1\}$,*
   - *$\gamma([\tau_j,\tau_{j+1}]) \subset U_j \cap U_{j+1}$ for all $j \in \{0,\ldots,n-1\}$ (implying $\gamma(\tau_j) \in U_j$ for all $j \in \{0,\ldots,n\}$),*
   - *$(f_0,U_0) = (f,U)$ and $(f_n,U_n) = (\tilde{f},\tilde{U})$.*

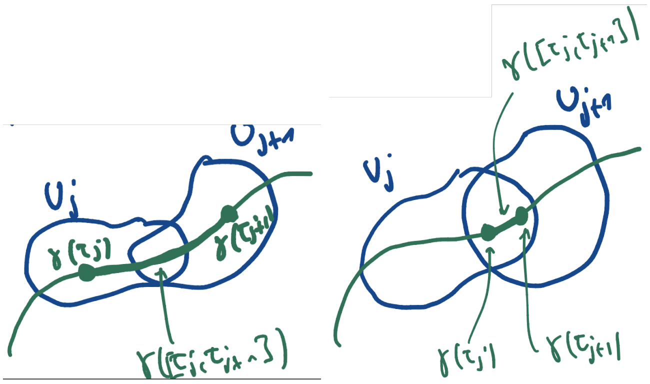


Fig. 53: Comparing the third bullet point in 2 and 3.

Proof. "3 $\implies$ 2" Follows from $U_j \cap U_{j+1} \subset U_j \cup U_{j+1}$.

"2 $\implies$ 1" is a lemma.

"1 $\implies$ 3":

a. By assumption there is for each $t \in [t_0,t_1]$ a number $\varepsilon_t > 0$ such that for each $t' \in (t-\varepsilon_t, t+\varepsilon_t) \cap [t_0,t_1]$
   - $\gamma(t') \in U_t$,
   - (f_t,U_t) and $(f_{t'},U_{t'})$ are direct analytic continuations of each other.

For each $t \in [t_0, t_1]$ let $I_t := (t - \frac{1}{2}\varepsilon_t, t + \frac{1}{2}\varepsilon_t) \cap [t_0, t_1]$. Applying lemma 5.3.8 to the open cover $(I_t)_{t\in[t_0,t_1]}$ of the compact metric space $[t_0, t_1] \subset \mathbb{R}$ (equipped with the subspace topology of $\mathbb{R}$) yields that there is a number $\delta > 0$ such that any interval in $[t_0, t_1]$ of length smaller than δ is contained in one of the I_t.

b Choose $n \in \mathbb{N}_{>0}$ large enough such that $\frac{1}{n}(t_1 - t_0) < \min(\frac{1}{2}\delta, \varepsilon_{t_0}, \varepsilon_{t_1})$ and let $\tau_j := t_0 + \frac{j}{n}(t_1 - t_0)$ and consider the subdivision

$$t_0 = \tau_0 < \tau_1 \ldots < \tau_n = t_1. \tag{24}$$

For each $j \in \{1, \ldots, n-1\}$, the length of the interval $[\tau_{j-1}, \tau_{j+1}]$ is $< \delta$, so there is a point $\theta_j \in [t_0, t_1]$ such that $[\tau_{j-1}, \tau_{j+1} \subset I_{\theta_j}$. This implies in particular (we will need this later)

$$\begin{cases} |\tau_{j-1} - \theta_j| < \frac{1}{2}\varepsilon_{\theta_j}, & \\ |\tau_j - \theta_j| < \frac{1}{2}\varepsilon_{\theta_j}, & \forall j \in \{1, \ldots, n-1\} \\ |\tau_{j+1} - \theta_j| < \frac{1}{2}\varepsilon_{\theta_j}. & \end{cases}$$

Now the subdivision $\tau_0 < \tau_1 < \ldots < \tau_n$ and the sequence of function elements

$$(\{_0, \mathcal{U}_0) := (f_{t_0}, U_{t_0}) = (f, U), \qquad (\{_j, \mathcal{U}_j) := (f_{\theta_j}, U_{\theta_j}), \quad (j \in \{1, \ldots, n-1\}),$$
$$(\{_n, \mathcal{U}_n) := (f_{t_1}, U_{t_1}) = (\tilde{f}, \tilde{U})$$

have the desired properties:

- $(\{_0, \mathcal{U}_0)$ and $(\{_1, \mathcal{U}_1)$ are direct analytic continuation because $|\tau_0 - \theta_{t_1}| < \frac{1}{2}\varepsilon_{\theta_1} < \varepsilon_{\theta_1}$.
- $(\{_j, \mathcal{U}_j)$ and $(\{_{j+1}, \mathcal{U}_{j+1})$ are directly analytic continuations for $j \in \{1, \ldots, n-2\}$ because
$$|\theta_{j+1} - \theta_j| \leqslant |\theta_{j+1} - \tau_j| + |\tau_j - \theta_j| < \frac{1}{2}\varepsilon_{\theta_j} + \frac{1}{2}\varepsilon_{\theta_j} = \varepsilon_{\theta_j}.$$
- $(\{_{n-1}, \mathcal{U}_{n-1})$ and $(\{_n, \mathcal{U}_n)$ are direct analytic continuations of each other because $|\tau_n - \theta_{\tau_{n-1}} < \frac{1}{2}\varepsilon_{\theta_{n-1}} < \varepsilon_{\theta_{n-1}}$.
- $\gamma([\tau_0, \tau_1]) \subset \mathcal{U}_0$ because $|\tau_1 - \tau_0| < \varepsilon_{t_0}$.
- $\gamma([\tau_{j-1}, \tau_j]) \subset \mathcal{U}_j$ for $j \in \{1, \ldots, n-1\}$ because $[\tau_{j-1}, \tau_j] \in I_{\theta_j}$.
- $\gamma([\tau_j, \tau_{j+1}]) \subset \mathcal{U}_j$ for $j \in \{1, \ldots, n-1\}$ because $[\tau_j, \tau_{j+1}] \in I_{\theta_j}$.
- $\gamma([\tau_{n-1}, \tau_n]) \subset \mathcal{U}_n$ because $|\tau_n - \tau_{n-1}| < \varepsilon_{t_1}$. □

Lemma 5.3.8 (Lebesgue Number Lemma)
If (X, d) is a compact metric space and $\{U_i\}_{i\in I}$ is an open cover of X, that is a set of open subsets such that $\bigcup_{i\in I} U_i = X$, then there exists a number ε (the Lebesgue number *of the cover) such that any subset $W \subset X$ of diameter smaller than ε is contained in one of the open sets U_i.* Lebesgue *number*

Proof. Homework 9.1.

First show that the function

$$\varrho\colon X \to \mathbb{R}, \qquad x \mapsto \sup\{r \geqslant 0 : \exists i \in I \text{ such that } B_r(x) \subset U_i\}$$

is continuous (Homework 8.2). □

We have now achieved the diverse points of view of analytic continuation.

> A function element $(\tilde{f}, \tilde{U})$ is an analytic continuation of (f, U) along γ if one (and hence all) of the conditions 1 - 3 of the Diversity Lemma are satisfied.

Lemma 5.3.9 (TODO (Tut VIII))
Let $f\colon D\backslash\{0\} \to \mathbb{C}$ be holomorphic with $|f(z)| \leqslant M|z|^t$ for some $M > 0$ and $t > -1$ and for all $z \in \mathbb{D}\backslash\{0\}$. Then the isolated singularity of f at 0 is removable.

Proof. For $z \in D\backslash\{0\}$ we have $|z| < 1$ and thus $|z|^t \leqslant |z|^{-1}$ for all $t > -1$. Hence $|f(z)| \leqslant M|z|^t$ implies that $|zf(z)| \leqslant M$ and thus $\operatorname{ord}(f,0) \leqslant\geqslant -1$. For all $z \in \mathbb{C}$ there exists $(c_k)_{k\geqslant -1}$ such that

$$f(z) = c_{-1}\frac{1}{z} + \underbrace{\sum_{k=0}^{\infty} c_k z^k}_{=:h(z)}.$$

There exists a $C > 0$ such that for all $z \in \mathbb{C}$ with $|z| \leqslant \frac{1}{2}$ we have $|h(z)| \leqslant C$, since h is continuous and thus bounded on a compact set. For all $z \in D\backslash\{0\}$ we have

$$\left|\frac{c_{-1}}{z}\right| \leqslant |f(z)| + |h(z)|$$

and thus

$$|c_{-1}| \leqslant M|z|^{t+1} + C|z| \leqslant (M + C)|z|^a \xrightarrow{z\to 0} 0,$$

where $a := \min(t + 1, 1) > 0$. Hence $c_{-1} = 0$ and thus $f = h$ on D, that is, the singularity of f is removable. $\square$

Lemma 5.3.10 (No local square root if $g'(0) \neq 0$ (Tut VIII))
Let $g\colon D \to D$ be holomorphic with $g(0) = 0$ and $g'(0) \neq 0$. Then there does not exist a holomorphic function $h\colon D\backslash\{0\} \to \mathbb{C}$ such that $h^2 = g|_{D\backslash\{0\}}$.

Proof. Towards contradiction assume such a function. Then $h(D) \subset D$ and 0 is a removable singularity of h by lemma 5.3.9, as $|g(z)| \leqslant |z|$ implies $|h(z)| \leqslant |z|^{\frac{1}{2}}$. Hence we can represent h locally as

$$h(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then $g(0) = h(0)^2 = a_0^2 = 0$, so $a_0 = 0$ and thus $g'(0) = 2h(0)h'(0) = 2a_0a_1 = 0$, which is a contradiction. $\square$

Intuitive approach: let $a := g'(0)$. Let $\tilde{g} := \frac{1}{|a|}g$, then $|\tilde{g}'(0)| = 1$, so $\tilde{g}(z) = bz$ for some $b \in \mathbb{S}^1$ and thus $g(z) = |a|bz$. There does not exists a square root of z and thus also not of any multiples.

Remark 5.3.11 Let $f(z) := \sum_{k=0}^{\infty} z^k$ for $z \in D$. The direct analytic continuation of f with the largest domain is $g(z) := \frac{1}{1-z}$ for all $z \in \mathbb{C}\backslash\{1\}$. The power series of g around $z_0 \in \mathbb{C}$ is

$$\frac{1}{1-z} = \frac{1}{1 - z_0 - (z - z_0)} = \frac{1}{1 - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{1 - z_0}\right)^k = \sum_{k=0}^{\infty}(1 - z_0)^{-k-1}(z - z_0)^k$$

for all $z \in \mathbb{C}$ with $0 < |z - z_0| < |1 - z_0|$ and its convergence radius is $R = |1 - z_0|$. ○

5.4 Analytic continuation and integration along continuous curves

The goal of this section is to define integrals of holomorphic function along any continuous curve in the domain. This is really remarkable because in Real Analysis, there is no such thing; one needs some kind of regularity conditions on the curve to define an integral. Since holomorphic functions are so extremely well-behaved, we can extend the definition to integration along arbitrary continuous curves.

Lemma 5.4.1 (Analytic continuation of the derivative)
If the derivative (f', U) of a function element (f, U) can be analytically continued along a curve $\gamma\colon [t_0, t_1] \to \mathbb{C}$, then (f, U) can be analytically continued along γ.

Proof. As (f', U) can be analytically continued, there exists a family $((g_t, U_t))_{t \in [t_0, t_1]}$ of locally compatible function elements such that $(g_{t_0}, U_{t_0}) = (f', U)$.

Without loss of generality we can (by remark 5.3.6 1) assume that for all $t \in [t_0, t_1]$, U_t is an open disk with centre $\gamma(t)$.

By the Discretisation Lemma, there exists a subdivision

$$t_0 = \tau_0 < \ldots < \tau_n = t_1$$

such that (g_{τ_j}, U_{τ_j}) and $(g_{\tau_{j+1}}, U_{\tau_{j+1}})$ are direct analytic continuations of each other and $\gamma([\tau_j, \tau_{j+1}]) \subset U_{\tau_j} \cap U_{\tau_{j+1}}$. How we can define function elements $((f_j, U_j))_{j=0}^n$ recursively as follows:

- $(f_0, U_{\tau_0}) = (f, U)$. (Then $(f_0', U_{\tau_u}) = (f', U) = (g_{\tau_0}, U_{\tau_0})$.)
- If (f_j, U_{τ_j}) has been defined such that $(f_j', U_{\tau_j}) = (g_{\tau_j}, U_{\tau_j})$ define f_{j+1} and $U_{\tau_{j+1}}$ as follows: Since $U_{\tau_{j+1}}$ is a disk (!) and $g_{\tau_{j+1}}$ is holomorphic on $U_{\tau_{j+1}}$, the function $g_{\tau_{j+1}}$ is represented by a power series on $U_{\tau_{j+1}}$. Since power series can be integrated term by term, there exists an antiderivative f_{j+1} of $g_{\tau_{j+1}}$ on $U_{\tau_{j+1}}$, which is uniquely determined up to an additive constant. We choose this constant of integration such that $f_{j+1}(\tau_j) = f_j(\tau_j)$. Then f_{j+1} and f_j agree on $U_{\tau_j} \cap U_{\tau_{j+1}}$. Indeed, on $U_{\tau_j} \cap U_{\tau_{j+1}}$ we have
$$(f_{j+1} - f_j)' = f_{j+1}' - f_j' = g_{\tau_{j+1}} - g_{\tau_j} = 0. \tag{25}$$
Since U_{τ_j} and $U_{\tau_{j+1}}$ are disks (!), their intersection is connected and hence $f_{j+1} - f_j$ is constant on $U_{\tau_j} \cap U_{\tau_{j+1}}$. Since $(f_{j+1} - f_j)(\tau_j) = 0$, this constant is zero.

So we have a sequence (f_j, U_{τ_j}) of function elements such that (f_j, U_{τ_j}) and $(f_{j+1}, U_{\tau_{j+1}})$ are direct analytic continuations of each other and $\gamma(\tau_j) \in U_{\tau_j}$ and $\gamma([\tau_j, \tau_{j+1}]) \subset U_{\tau_j} \cap U_{\tau_{j+1}}$.

Now we use the Diversity Lemma in the other direction: By the Diversity Lemma, $(f_n, U\tau_n)$ is an analytic continuation of $(f, U) = (f_0, U_0)$ along γ. □

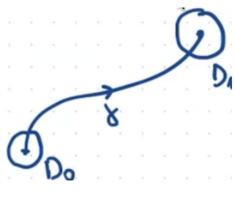


Fig. 54: (F_1, D_1) is the continuation of (F_0, D_0) along γ.

Definition 5.4.2 (Integral along a continuous curve)
Let f be a holomorphic function on U and let $\gamma\colon [t_0, t_1] \to U$ be a continuous function, that is, a curve in U. Let $D_0 \subset U$ be an open disk around $\gamma(t_0)$ and let F_0 be an antiderivative of f on D_0 (which exists because f is represented by a power series on D_0). Let (F_1, D_1) be an analytic continuation of (F_0, D_0) along γ (which exists by lemma 5.4.1 because $(F_0', D_0) = (f|_{D_0}, D_0)$ can be trivially continued along γ). Define the integral of f along γ by

$$\int_\gamma f(z)\,\mathrm{d}z := F_1(\gamma(t_1)) - F_0(\gamma(t_0)).$$

The RHS does not depend on any choice involved in the construction.

Theorem 5.4.1: Equivalence of Integral Definitions

If γ is piecewise continuously differentiable, then the integral from Definition 5.4.2 agrees with our original Definition 2.1.1, that is, in that case

$$F_1(\gamma(t_1)) - F_0(\gamma(t_0)) = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

Proof. Homework 9.3. □

Analytic continuation and arcsin

Consider the function $f(z) := z^2$. Then $f'(1) = 2 \neq 0$, so f is locally invertible, but as it is not injective, it is not globally invertible. The principal branch of the inverse is $\mathrm{sqrt}\colon \mathbb{C}\setminus\mathbb{R}_{\leqslant 0} \to \mathbb{C}$, $re^{i\varphi} \mapsto \sqrt{r}e^{i\frac{\varphi}{2}}$ for $r > 0$ and $\varphi \in (-\pi, \pi)$.

Now consider $f(z) := \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$. As $\sin'(0) = \cos(0) = 1$, sin is locally invertible, but as it is not injective, it is not globally invertible.

Definition 5.4.3 ((Minimal) period)
Let $f\colon \mathbb{C} \to \mathbb{C}$. Then $w \in \mathbb{C}^*$ is a period of f if period

$$f(z + kw) = f(z) \qquad \forall z \in \mathbb{C}, k \in \mathbb{Z}\,.$$

and minimal if rw is not a period of f for all $r \in (-1, 1)$.

A period of sin is 2π as $z \mapsto e^{iz}$ is 2π-periodic. As 2π is the minimal period of $\sin|_{\mathbb{R}}$, it also is the minimal period of sin.

Let arcsin be the local inverse of sin around $z_0 := 0$. Then

$$\arcsin'(z) = \frac{1}{f'(f^{-1}(z))} = \frac{1}{\cos(\arcsin(z))} \overset{(\star)}{=} \frac{1}{\sqrt{1 - \sin^2(\arcsin(z))}} = \frac{1}{\sqrt{1 - z^2}},$$

where in $(\star)$ we pick the principal branch sqrt of the square root function because $\cos(0) = 1 > 0$.

What is the domain of definition of $\arcsin'$? We have $1 - z^2 = (1 - z)(1 + z)$ and $1 - x \leqslant 0$ if and only if $x \geqslant 1$ and $1 + x \leqslant 0$ if and only if $x \geqslant 1$, so the domain of $\arcsin'$ is $\mathbb{C}\setminus\big((-\infty, -1] \cup [1, \infty)\big)$. We can now write

$$\arcsin(z) = \int_{[0,z]} \arcsin(\tilde{z})\,\mathrm{d}\tilde{z}$$

as $\arcsin(0) = 0$.

5.5 Homotopy of curves

Another tool for bringing order into the different analytic continuations of a function is a subfield of topology called homotopy. For a more detailed view consider [1] and [2]. **15.06.2021**

Definition 5.5.1 (Curve)
A curve in a topological space X is a continuous map $c\colon [t_0, t_1] \to X$.

The topological spaces we will be interested in are open subsets of $\mathbb{C}$.

Definition 5.5.2 (Homotopy)
Two curves $c_0, c_1\colon [0,1] \to X$ in a topological space X are homotopic (in X) if there exists a homotopy between them, that is, a continuous map

homotopy

$$H\colon [0,1] \times [0,1] \to X$$

for which

$$H(\cdot, 0) = c_0 \qquad \text{and} \qquad H(\cdot, 1) = c_1$$

as well as

$$H(0, \cdot) = c_0(0) = c_1(0) \qquad \text{and} \qquad h(1, \cdot) = c_0(1) = c_1(1).$$

In particular, the curves have the same starting point $c_0(0) = c_1(0)$ and the same endpoint $c_0(1) = c_1(1)$.

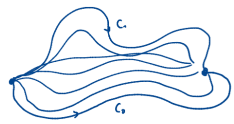


Fig. 55: A homotopy from the curve c_1 to the c_2

Definition 5.5.3 (Null homotopic)
A closed curve $c\colon [0,1] \to X$ is null homotopic if it is homotopic to the constant curve at $c_1(t) = c(0) = c(1)$.

null homotopic

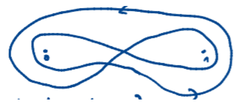
Example 5.5.4 The curve $c\colon [0,1] \to \mathbb{C}$, $t \mapsto e^{2\pi i t}$ is null homotopic in $\mathbb{C}$. A homotopy to the constant curve 1 is (the linear interpolation)

$$H(t, \tau) := (1 - \tau)c(t) + \tau.$$

But this curve is not null homotopic in $\mathbb{C}^*$, which is harder to prove. ◇

Fig. 56: The homotopy of the unit circle and a constant path in $\mathbb{C}$.

Example 5.5.5 Consider the following curve in $X := \mathbb{C} \setminus \{0, 1\}$. Is it nullhomotopic? ◇



Example 5.5.6 (To hang a picture)
We want to find a closed curve $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0,1\}$ such that γ is nullhomotopic in $\mathbb{C} \setminus \{1\}$ and in $\mathbb{C} \setminus \{0\}$ but not in $\mathbb{C} \setminus \{0,1\}$.

Consider the counterclockwise loops at $z_0 \in \mathbb{C} \setminus \{0,1\}$, one going around 0 and one around 1, named a and b. Then $\gamma = aba^{-1}b^{-1}$ is such a curve. In $\mathbb{C} \setminus \{0\}$, $a \simeq e$ and then $\gamma \simeq ebe^{-1}b = e$ and analogously for $\mathbb{C} \setminus \{0, 1\}$.

What if we instead take n points $\{0, 1, \ldots, n-1\}$ and consider the same problem. We can recursively construct such an solution (it is not the solution with the fewest characters, though) by consider $cwc^{-1}w^{-1}$, where w is the word that worked for k points and c is the loop around the $(k+1)$-th point. ◇

Definition 5.5.7 (Composition/Concatenation of curves)
The composition of two curves $c_1, c_2\colon [0,1] \to X$ with $c_1(1) = c_2(0)$ is the curve

$$c_1 c_2\colon [0,1] \to X, \qquad t \mapsto \begin{cases} c_1(2t), & \text{for } t \in [0, \frac{1}{2}], \\ c_2(2t-1), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Fig. 57: Concatenation of two curves c_1 and c_2.

Notice that the order is opposite to function composition, we first go along c_1 and then along c_2.

Definition 5.5.8 (Inverse of a curve)
The inverse of a curve $c\colon [0,1] \to X$ is the curve $c^{\text{inv}}\colon [0,1] \to X$, $t \mapsto c(1-t)$.

Lemma 5.5.9 (Neutral element, reparametrisation invariant under homotopy)
Let $c\colon [0,1] \to X$ be a curve in X and let $\varphi\colon [0,1] \to [0,1]$ be a continuous (reparametrisation) map with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then

1. *cc^{inv} is null homotopic in X.*
2. *c and $c \circ \varphi$ are homotopic.*

Proof. 1 We have $c^{\text{inv}}(t) = c(1-t)$ and thus $c^{\text{inv}}(2t-1) = c(1-(2t-1)) = c(2-2t)$, so

$$cc^{\text{inv}}(t) = \begin{cases} c(2t), & \text{if } t \in [0, \frac{1}{2}], \\ c(2-2t), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Hence $cc^{\text{inv}} = c \circ \psi$, where

$$\psi\colon [0,1] \to [0,1], \qquad t \mapsto \begin{cases} 2t, & \text{if } t \in [0, \frac{1}{2}], \\ 2-2t, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

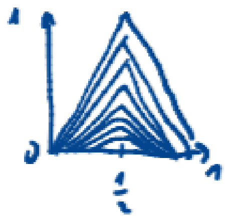


Fig. 58: The function ψ

A homotopy of cc^{inv} to the constant curve $c(0)$ is

$$H(t,\tau) := c((1-\tau)\psi(t)) = \begin{cases} c((1-\tau)2t), & \text{if } t \in [0, \frac{1}{2}], \\ c((1-\tau)(2-2t)), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

2 A homotopy is

$$H(t,\tau) := c((1-\tau)t + \tau\varphi(t)).$$

□

Lemma 5.5.10 (Associativity of concatenation up to homotopy)
Let $c_1, c_2, c_3\colon [0,1] \to X$ be curves with $c_1(1) = c_2(0)$ and $c_2(1) = c_3(0)$. Then $(c_1c_2)c_3$ is homotopic to $c_1(c_2c_3)$.

Proof. We have

$$c_1c_2(t) = \begin{cases} c_1(2t), & \text{for } t \in [0, \frac{1}{2}], \\ c_2(2t-1), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

and thus

$$(c_1c_2)c_3(t) = \begin{cases} c_1c_2(2t), & \text{for } t \in [0, \frac{1}{2}], \\ c_3(2t-1), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} = \begin{cases} c_1(4t), & \text{for } t \in [0, \frac{1}{4}], \\ c_2(4t-1), & \text{for } t \in [\frac{1}{4}, \frac{1}{2}], \\ c_3(2t-1), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Similarly,

$$c_1(c_2c_3)(t) = \begin{cases} c_1(2t), & \text{for } t \in [0, \frac{1}{2}], \\ c_2(4t-2), & \text{for } t \in [\frac{1}{2}, \frac{3}{4}], \\ c_3(4t-3), & \text{for } t \in [\frac{3}{4}, 1]. \end{cases}$$

Check yourself that

$$c_1(c_2c_3) = (c_1c_2)c_3 \circ \psi,$$

where

$$\psi(t) := \begin{cases} \frac{1}{2}t, & \text{if } t \in [0, \frac{1}{2}], \\ t - \frac{1}{4}, & \text{if } t \in [\frac{1}{2}, \frac{3}{4}], \\ 2t - 1, & \text{if } t \in [\frac{3}{4}, 1]. \end{cases}$$

The claim follows by lemma 5.5.9 2. □

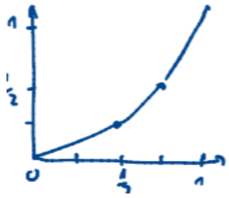


Fig. 59: The function ψ.

Theorem 5.5.1: Fundamental group

Let X be a topological space and $x_0 \in X$ a (base)point. A curve $c\colon [0,1] \to X$ is a loop at x_0 if $c(0) = x_0 = c(1)$. Then homotopy is an equivalence relation on the set of loops at x_0. The set of equivalence classes, $\pi_1(X, x_0)$, together with the well-defined operation

$$[c_1 c_2] = [c_1][c_2], \tag{26}$$

where c_1 and c_2 are loops at x_0, is the fundamental group of X with base point x_0. The neutral element is the class of constant curves $[x_0]$, i.e. the set of null-homotopic loops at x_0. The inverse of $[c]$ is $[c^{\text{inv}}]$.

fundamental group

Proof. 1 First we show that homotopy is an equivalence relation on the set of loops at x_0.

- Homotopy is reflexive because a homotopy between c and itself is $H(t, \tau) = c(t)$.
- Homotopy is symmetric: if H is a homotopy from c_1 to c_2, then a homotopy from c_2 to c_1 is $(t, \tau) \mapsto H(t, 1 - \tau)$.
- Homotopy is transitive: if H_{12} is a homotopy from c_1 to c_2 and H_{23} is a homotopy from c_2 to c_3, then a homotopy from c_1 to c_3 is

  $$H_{13}(t, \cdot) = H_{12}(t, \cdot) H_{23}(t, \cdot)$$

  seen as concatenation of curves, that is

  $$H_{13} = \begin{cases} H_{12}(t, 2\tau), & \text{if } \tau \in [0, \frac{1}{2}], \\ H_{23}(t, 2\tau - 1), & \text{if } \tau \in [\frac{1}{2}, 1]. \end{cases}$$

2 We now show that the group operation is well defined. If c_1 and $\tilde{c_1}$ are homotopic, and c_2 and $\tilde{c_2}$ are homotopic, then $c_1 c_2$ and $\tilde{c_1}\tilde{c_2}$ are homotopic. Indeed, if H_1 is a homotopy from c_1 to $\tilde{c_1}$ and H_2 is a homotopy from c_2 to $\tilde{c_2}$, then a homotopy from $c_1 c_2$ to $\tilde{c_1}\tilde{c_2}$ is

$$H(t, \tau) := \begin{cases} H_1(2t, \tau), & \text{if } t \in [0, \frac{1}{2}], \\ H_2(2t - 1, \tau), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

3 The group operation is associative: if c_1, c_2 and c_3 are loops at x_0, then

$$([c_1][c_2])[c_3] = [(c_1 c_2) c_3] \overset{\text{lemma 5.5.10}}{=} [c_1(c_2 c_3)] = [c_1]([c_2][c_3]).$$

4 The neural element is $[x_0]$ and $[c]^{-1} = [c^{\text{inv}}]$. This follows from lemma 5.5.9. □

Remark 5.5.11 (Commutativity) The fundamental group is in general not commutative. Therefore, the group operation is written is as multiplication. Instead of null homotopic it would be more correct to say "one homotopic" but nobody does that. ○

Example 5.5.12 (Non-Abelian Fundamental Group) Consider $X \backslash \{0,1\}$. Consider a loop c_0 around 0 based at $x_0 \in X$ concatenated with a loop c_1 around 1 based at z_0. The curve $c_0 c_1 c_0^{\text{inv}} c_1^{\text{inv}}$ is not null homotopic, so $[c_0][c_1][c_0]^{-1}[c_1]^{-1}$ is not the neutral element of $\pi_1(X, z_0)$, so $[c_0][c_1] \neq [c_1][c_0]$. ◇

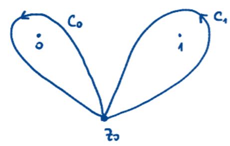


Fig. 60: A non null homotopic loop.

16.05.2021

Remark 5.5.13 (Dependence on the base point)
The fundamental group $\pi_1(X, y)$ depends not only on the space X but also on the base point $y \in X$. However, if X is path-connected, then two base points y and $x \in X$ can always be connected via a curve c and one obtains a group isomorphism

$$\pi(X, x) \to \pi(X, y), \qquad [c] \mapsto [c^{-1}\gamma c^{\text{inv}}].$$

as detailed in lemma 5.5.14 Hence for path-connected space, fundamental groups with different base points are isomorphic. However, the isomorphism is in general not unique but depends on the choice of connecting path γ. For example for the path-connected space $X := \mathbb{C} \backslash \{0,1\}$ one choose different in X non-homotopic paths. ○

Lemma 5.5.14 (Isomorphism of fundamental groups along curve (Tut IX))
Let X be a topological space, $c\colon [0,1] \to X$ be a curve and $x := c(0)$ and $y := c(1)$. Then $\pi(X, x) \cong \pi(X, y)$.

Proof. Consider the map

$$\Phi\colon \pi(X, x) \to \pi(X, y), \qquad [\gamma] \mapsto [c^{-1}\gamma c].$$

Let $[\gamma] = [\hat{\gamma}]$. We have to show that $[c^{-1}\gamma c] = [c^{-1}\hat{\gamma} c]$. We have

$$\Phi([\gamma\hat{\gamma}]) = [c^{-1}\gamma\hat{\gamma}c] = [c^{-1}\gamma c c^{-1}\hat{\gamma} c] \overset{(26)}{=} [c^{-1}\gamma c][c^{-1}\hat{\gamma} c],$$

so Φ is a homomorphism.

The map

$$\Psi\colon \pi(X, y) \to \pi(X, x), \qquad [\hat{\gamma}] \mapsto [c\hat{\gamma}c^{-1}]$$

is the inverse of Φ:

$$(\Psi \circ \Phi)([\gamma]) = \psi([c^{-1}\gamma c]) = [cc^{-1}\gamma cc^{-1}] = [\gamma]$$

for any $[\gamma] \in \pi(X, x)$. □

Remark 5.5.15 We are only interested in open subsets of $\mathbb{C}$. More generally, for open subset of $\mathbb{R}^n$, connectedness and path-connectedness are equivalent. ○

Theorem 5.5.2: Characterisation of trivial fundamental group

Let X be a nonempty path-connected topological space, e.g. a domain. Then the following are equivalent:

1. Every closed curve $c\colon [0,1] \to X$ is null homotopic in X.
2. For every $x_0 \in X$, $\pi_1(X, x_0) = \{1\}$.
3. There is a point $x_0 \in X$ such that $\pi_1(X, x_0) = \{1\}$.
4. Any curves $c_1, c_2\colon [0,1] \to X$ with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$ are homotopic.

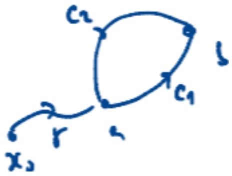
Definition 5.5.16 (Simply connected)
If one (and hence all) of the statements in Theorem 5.5.2 hold, X is simply connected.

simply connected

Proof. "1 $\implies$ 2": is clear from Theorem 5.5.1.

"2 $\implies$ 3": is trivial.

"3 $\implies$ 4": Let $c_1(0) = c_2(0) =: a$ and $c_1(1) = c_2(1) =: b$. Choose a curve γ from x_0 to a. By assumption, $[\gamma c_1 c_2^{\mathrm{inv}} \gamma^{\mathrm{inv}}] = 1$, so $\gamma c_1 c_2^{\mathrm{inv}} \gamma^{\mathrm{inv}}$ is null homotopic. If we write $\cong$ for the homotopy relation, we obtain $\gamma c_2 \cong \gamma c_1 c_2^{\mathrm{inv}} \gamma^{\mathrm{inv}} \gamma c_2 \cong \gamma c_1$. This means that γc_1 and γc_2 are homotopic. Then $\gamma^{\mathrm{inv}} \gamma c_1$ and $\gamma^{\mathrm{inv}} \gamma c_2$ are homotopic, so c_1 and c_2 are homotopic.



"4 $\implies$ 1": If c is a closed curve, consider $c_1 = c$ and c_2 to be the constant curve $c(0) = c(1)$. □

Example 5.5.17 (Convex domain is simply connected)
A convex domain $U \subset \mathbb{C}$ is simply connected: if c_1 and c_2 are curve in U with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$, then a homotopy in U from c_1 to c_2 is the linear interpolation

$$H(t, \tau) := (1 - \tau)c(t) + \tau c_2(t).$$

◇

Fig. 61: As U is convex, the linear interpolation between $c_1(t_0)$ and $c_2(t_0)$ lies in U for any $t_0 \in [0, 1]$.

Example 5.5.18 (Starshaped domains have trivial fundamental group)
If a domain $U \subset \mathbb{C}$ is starshaped with respect to a point $z_0 \in U$, then $\pi_1(U, z_0) = \{1\}$: for any loop c at z_0, a homotopy from c to the constant curve is the linear interpolation

$$H(t, \tau) := (1 - \tau)c(t) + \tau z_0.$$

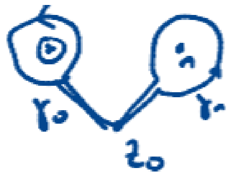
Hence starshaped domains are simply connected by Theorem 5.5.2.

The slit complex plane $U := \mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$ is starshaped with respect to $1 \in U$, so it is simply connected. ◇

Fig. 62: As U is starshaped with respect to z_0, the linear interpolation between $c(t_0)$ and z_0 lies in U for any $t_0 \in [0, 1]$.

Example 5.5.19 (Fundamental group of $\mathbb{C}^*$)
The unit circle $c\colon [0, 1] \to \mathbb{C}$, $t \mapsto e^{2\pi i t}$ is not null homotopic, so $\pi_1(\mathbb{C}^*, 1) \neq \{1\}$. One can show that $\pi_1(\mathbb{C}^*, z)$ is isomorphic to $(\mathbb{Z}, +)$ for any $z \in \mathbb{C}^*$. ◇

Example 5.5.20 (Fundamental group of the twice punctured plane)
The fundamental group of the twice punctured plane $U := \mathbb{C} \setminus \{0, 1\}$ is not Abelian. One can show that it is isomorphic to the free group of rank 2; $\pi_1(U, z_0)$ is generated by $[\gamma_0]$ and $[\gamma_1]$. ◇



5.6 The Monodromy Theorem

following Ferus' notes and Ahlfors

The Monodromy Theorem, briefly, states that analytic continuations along homotopic curves give the same result.

Theorem 5.6.1: Monodromy

Let $U \subset \mathbb{C}$ be a domain and let (f_0, U_0) be a function element, $z_0 \in U \cap U_0$ and suppose (f_0, U_0) can be continued analytically along every curve in U starting at z_0. If c and $\tilde{c}$ are homotopic curves starting at z_0 and (f_1, U_1) and $(\tilde{f}_1, \tilde{U}_1)$ are analytic continuations of (f_0, U_0) along c and $\tilde{c}$ respectively, then f_1 and $\tilde{f}_1$ agree in some open neighbourhood of $z_1 := c(1) = \tilde{c}(1)$.

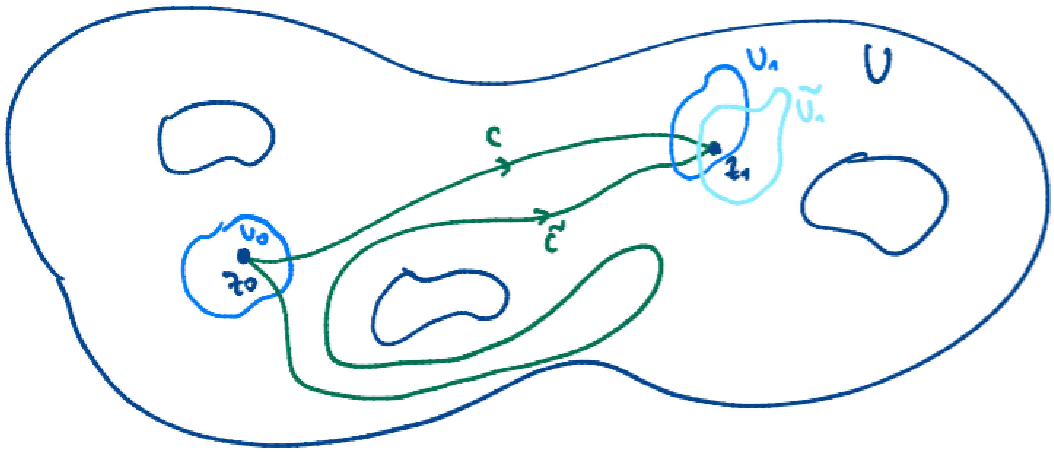
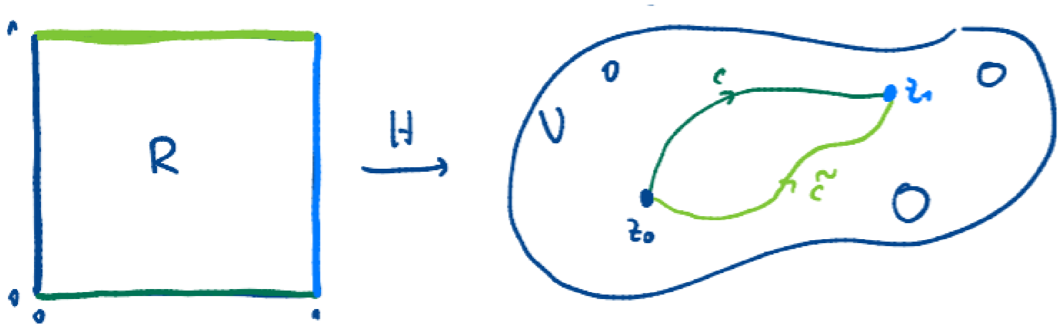


Fig. 63: The Monodromy Theorem.

Proof. (From Ahlfors) Let $H\colon [0,1]^2 \to U$ be a homotopy from c to $\tilde{c}$.



1. Any curve $\gamma\colon [0,1] \to [0,1]^2 =: R$ with $\gamma(0) = (0,0)$ corresponds to a curve $H \circ \gamma$ in U starting at z_0, along which there exists an analytic continuation of (f_0, U_0).

Fig. 64: The boundary of R corresponds to the closed curve from z_0 via c to z_1 and back via $\tilde{c}$.

2. It is enough to show that (f_0, U_0) is an analytic continuation of itself along the boundary curve of R. If we go along c first, we get some analytic continuation, which agrees with (f_1, U_1) in some neighbourhood of z_1, because analytic continuation is locally unique. Now continue this analytic continuation which agrees locally with (f_1, U_1) around z_1 back to z_0, If we get back to (f_0, U_0), then if we go back in the other direction, we get something again which agrees in a neighbourhood of z_1. And this is what we will show.

3. We prove this by contradiction and the "method of dissection". Suppose an analytic continuation of (f_0, U_0) along $H \circ \gamma$ results in a function element that does not agree with (f_0, U_0) in any neighbourhood of z_0. We say that analytic continuation along $H \circ \gamma$ is not the identity. Now we construct a sequence of curves $\gamma_1, \gamma_2, \ldots$ as follows: γ_1 is one of the curves in the plot to the left. Whichever rectangle we circle we call R_1. We choose between both option such that analytic continuation along γ_1 is not the identity. If the analytic continuation along both options would be the identity, one can show that the continuation along the boundary of R would be the identity, because if we go first along the first curve, ending in the bottom left point and then

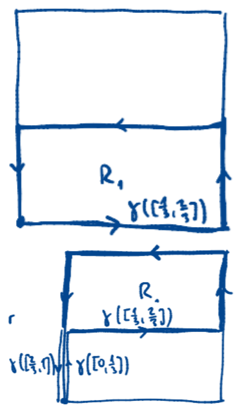


Fig. 65: The two possibilities for γ_1.

along the second curve, we undo the analytic continuation so at the top right point of the lower rectangle we reach a function element which agrees in a neighbourhood of this point, continuing to trace this curve, we end up again at the bottom left point and have traced an analytic continuation along the boundary curve of R.

For the future construction it is important to note that in either case, γ_1 goes along the boundary of R_1 for $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

4. In the next step, divide vertically: so γ_2 is either one of the possibilities in figure 66, where the boundary of R_2 is traced out in the interval $\left[\frac{1}{3} + \frac{1}{3^2}, 1 - \left(\frac{1}{3} + \frac{1}{3^2}\right)\right]$. We again choose such that the analytic continuation along γ_2 is not the identity.

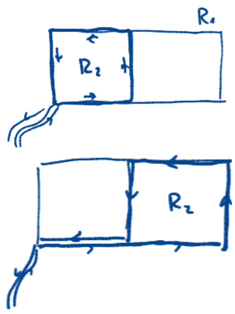


Fig. 66: The two possibilities for γ_2.

5. We continue this process to obtain a sequence of $(\gamma_k)_{k\in\mathbb{N}}$ along which analytic continuation is not the identity. This sequence of curves converges to a curve γ^* from $(0,0)$ to $x^* \in \bigcap_{k\in\mathbb{N}} R_k$ (which is unique) and back.

   Let (f^*, U_*) be the function element obtained by from (f_0, U_0) by analytic continuation along $H \circ \gamma^*|_{[0,\frac{1}{2}]}$ from $(0,0)$ to $z^* := H(x^*)$.

   If $k \in \mathbb{N}$ is large enough, then $R_k \subset H^{-1}(U^*)$ and $H^{-1}(U^*)$ is an open neighbourhood of x^*. Then analytic continuation along $H \circ \gamma_k$ would give function elements agreeing in a neighbourhood of z_0. But continuation along $H \circ \gamma^*$ is the identity. This is a contradiction. □

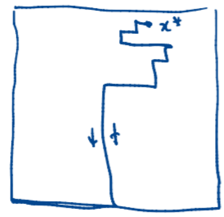


Fig. 67: The closed loop γ and the point x^*.

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The first corollary states that under certain conditions we can uniquely continue a function element (f, U) to all other points in the domain and we can in this way define a holomorphic function.

Corollary 5.6.1

Let $U \subset \mathbb{C}$ be a simply connected domain, (f, U_0) be a function element, and $U_0 \subset U$. If (f, U_0) can be continued analytically along every curve U starting in U_0 (e.g. because f is an antiderivative of a holomorphic function on U), then f is the restriction to U_0 of a holomorphic function on U.

Recall that we extended the Definition of the Integral along $\mathcal{C}^1$ curves to continuous curves via analytic continuation.

Corollary 5.6.2 ($\int_c f(z)\,\mathrm{d}z = 0$ if c null homotopic)

If f is holomorphic on $U \subset \mathbb{C}$ and c_1 and c_2 are homotopic curve in U, then

$$\int_{c_1} f(z)\,\mathrm{d}z = \int_{c_2} f(z)\,\mathrm{d}z.$$

In particular, $\int_c f(z)\,\mathrm{d}z = 0$ if c is null homotopic.

Corollary 5.6.3 (Cauchy's Integral Theorem for continuous images of rectangles)

Let f be holomorphic on U, $R \subset \mathbb{R}^2$ be a rectangle with boundary curve γ and $\varphi\colon R \to U$ be a continuous map. Then

$$\int_{\varphi\circ\gamma} f(z)\,\mathrm{d}z = 0.$$

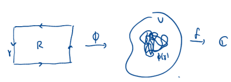


Fig. 68: The setup of corollary 5.6.3.

Proof. Show that $\varphi \circ \gamma$ is null homotopic in U. □

6 The winding number version of CAUCHY's Integral Theorem

6.1 Towards a definitive version of Cauchy's Integral Theorem

For a domain $U \subset \mathbb{C}$, what is a sufficient and necessary condition that a closed curve γ in U has to fulfil such that

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for every holomorphic function f on U?

The most general sufficient condition we know is that γ is null homotopic:

THEOREM 6.1.1: HOMOTOPY-VERSION OF CAUCHY'S INTEGRAL THEOREM

If f is holomorphic on the domain U and γ is a null homotopic closed curve in U, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Proof. See corollary 5.6.2. □

But this condition is not necessary:

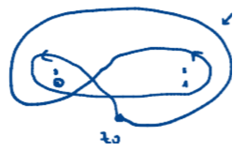


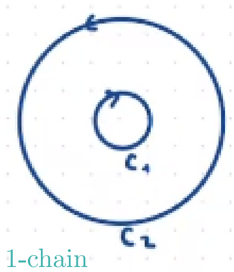
Fig. 69: This curve is homotopic to c.

Example 6.1.1 Consider the loops from figure 60. Then the curve $c := c_0 c_1 c_0^{\mathrm{inv}} c_1^{\mathrm{inv}}$ is not null homotopic in $U := \mathbb{C}\setminus\{0,1\}$. But for every holomorphic function f on U we have

$$\int_c f(z)\,\mathrm{d}z = \int_{c_0} f(z)\,\mathrm{d}z + \int_{c_1} f(z)\,\mathrm{d}z - \int_{c_0} f(z)\,\mathrm{d}z - \int_{c_1} f(z)\,\mathrm{d}z = 0.$$

◇

We want to characterise closed curves along which the integrals of holomorphic functions vanish. What we get is some sort of commutative version of homotopy. To this end it is convenient to extend integration to more general objects than curves: 1-chains. We would like to integrate over collections of curves such as the two 1-chains in figure 70.



1-chain

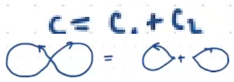


Fig. 70: We want to consider the sum of curves.

DEFINITION 6.1.2 (1-CHAIN, C_1)
A 1-chain c in an open set $U \subset \mathbb{C}$ is a formal linear combination

$$c = n_1 \odot c_1 \oplus \ldots \oplus n_k \odot c_k \tag{27}$$

of curves $c_j\colon [0,1] \to U$, where $n_j \in \mathbb{Z}$ for $j \in \{1,\ldots,k\}$. The ABELIAN group of 1-chains in U is $C_1(U)$.

Aside. Since the curves are maps to $\mathbb{C}$, one might be tempted to thinking that $\oplus$ and $\odot$ denote pointwise addition and multiplication, respectively. This is not what is meant. Think of a formal linear combination with integer coefficients as a shopping list with finitely many items. All that matters is how often an item appears in the list.

Definition 6.1.3 (Integral over a 1-chain)
For a 1-chain (27) in U for curves $c_j\colon [0,1] \to U$ for $j \in \{1,\dots,k\}$ and a holomorphic function f on U, the integral of f along c is

$$\int_c f(z)\,\mathrm{d}z := \sum_{j=1}^{k} n_j \int_{c_j} f(z)\,\mathrm{d}z.$$

A more formal definition.

Definition 6.1.4 (Free Abelian group)
If B is some set, then one can define the free Abelian group generated by B as the group $(\mathbb{Z}^{(B)}, +)$, where $\mathbb{Z}^{(B)}$ is the set of functions $B \to \mathbb{Z}$ (mapping a shopping item to its multiplicity), which are zero for all but finitely many elements and $+$ means pointwise addition.

The confusing part: interpret an element $b_0 \in B$ also as the characteristic function

$$\varphi_{b_0}\colon B \to \mathbb{Z}, \qquad b \mapsto \begin{cases} 1, & \text{if } b = b_0, \\ 0, & \text{if } b \neq b_0. \end{cases}$$

Then we can write any element in the free Abelian group generated by B as a finite "formal" linear combination $\sum_{j=1}^{k} n_j b_j$ for $(n_j)_{j=1}^{k} \subset \mathbb{Z}$.

Definition 6.1.5 (0-chain, C_0)
A 0-chain in U is a formal linear combination $\bigoplus_{j=1}^{k} n_j \odot z_j$ of points $(z_j)_{j=1}^{k} \subset U$ with integer coefficients $(n_j)_{j=1}^{k} \subset \mathbb{Z}$. The Abelian group of 0-chains in U is $C_0(U)$. 0-chain

Definition 6.1.6 (Boundary map)
The boundary map $\partial\colon C_1(U) \to C_0(U)$ is the group homomorphism defined as follows: the 1-chain (27) is mapped to boundary

$$\partial c := \bigoplus_{j=1}^{k} n_j \oplus \big(c_j(1) \ominus c_j(0)\big).$$

Definition 6.1.7 (Closed 1-chain, Cycle, Support)
A 1-chain c is closed if $\partial c = 0$. A closed 1-chain in U is also called a cycle in U. The support $|c|$ of a 1-chain (27) in U is the subset $\bigcup_{\substack{j=1 \\ n_j \neq 0}}^{k} c_j([0,1]) \subset U$. cycle support

Example 6.1.8 (Closed curves are closed 1-chains)
If $c\colon [0,1] \to U$ is a closed curve in U, then, viewed as a 1-chain, c is also closed: $\partial c = c(1) \ominus c(0) = (1-1) \odot c(0) = 0$. ◇

Example 6.1.9 (All coefficients being 1)
Consider a formal linear combination

$$c = \bigoplus_{j=1}^{k} c_j,$$

where all coefficients are one. Then

$$\partial c = \sum_{j=1}^{k} \big(c_j(1) \oplus (-1) \odot c_j(0)\big).$$

Hence $\partial c = 0$ if and only if any point for any $z \in \mathbb{C}$ the same number of curve start at z and end at z. ◇

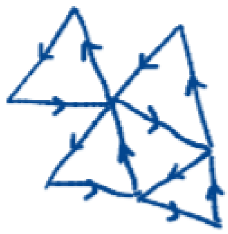


Fig. 71: The formal sum of these twelve straight line segments is a closed 1-chain

Example 6.1.10 The formal sum of the twelve straight line segments in figure 71 is a closed 1-chain. We could also find a closed curve that traces out all these line segments but the point is that it doesn't matter how we assemble them to a closed curve, it is good enough to consider them as ABELIAN sum (the order doesn't matter). ◇

Remark 6.1.11 (Cycles) The set of cycles in U is the kernel of the the boundary homomorphism $\partial\colon C_1(U) \to C_0(U)$, so this set is a subgroup of $C_1(U)$.

If c_1 and c_2 are cycles, then $c_1 \oplus c_2$ and $c_1 \ominus c_2$ are also cycles. ○

In the homotopy group, we can only concatenate curves where the endpoint of the first curve is the starting point of the second curve. Now, we have defined a different ABELIAN group, where we can add any kind of curves, and we can figure out a certain class of curves that are closed.

6.2 The winding number

We will define our classification of paths over which all integrals of holomorphic functions are zero in terms of the winding number.

DEFINITION 6.2.1 (WINDING NUMBER)
The winding number or winding index of a closed curve $\gamma\colon [0,1] \to \mathbb{C}$ around a point $z_0 \in \mathbb{C} \setminus \gamma([0,1])$ is

$$\nu_\gamma(z_0) := \operatorname{Ind}_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,\mathrm{d}z.$$

winding number

Example 6.2.2 Consider the curve $\gamma\colon [0,1] \to \mathbb{C}$, $t \mapsto z_0 + re^{2\pi nit}$, which circles the point z_0 $n \in \mathbb{N}$ times at distance $r > 0$ has winding index

$$\operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,\mathrm{d}z = \frac{1}{2\pi i}\int_0^1 \cancel{\frac{1}{re^{2\pi nit}}}\,2\pi ni\cancel{re^{2\pi nit}}\,\mathrm{d}t = \frac{\cancel{1}}{\cancel{2\pi i}}\int_0^1 \cancel{2\pi i}n\,\mathrm{d}t = n.$$ ◇

THEOREM 6.2.1: WINDING NUMBER IS A INTEGER

We have $\operatorname{Ind}_\gamma(z_0) \in \mathbb{Z}$.

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Proof. We will show that $\exp\left(\int_\gamma \frac{1}{z-z_0}\right) = 1$. (This is true because this integral is essentially one branch of the logarithm function of $z - z_0$ plus some constant of integration at the endpoint minus the initial point of the curve.) This implies $\int_\gamma \frac{1}{z-z_0} = 2\pi i n$ for some $n \in \mathbb{Z}$ and hence the statement. If F is a locally defined antiderivative of $z \mapsto \frac{1}{z-z_0}$, that is $F'(z) = \frac{1}{z-z_0}$, then

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{e^{F(z)}}{z-z_0} = \frac{e^{F(z)}}{z-z_0}F'(z) - \frac{e^{F(z)}}{(z-z_0)^2} = e^{F(z)}\left(\frac{1}{(z-z_0)^2} - \frac{1}{(z-z_0)^2}\right) = 0.$$

Hence $e^{F(z)} = A(z - z_0)$ for some constant $A \in \mathbb{C}$.

Suppose F_0 is an antiderivative of $z \mapsto \frac{1}{z-z_0}$ defined on a domain U_0 around $\gamma(0)$ satisfying $e^{F_0(z)} = A(z - z_0)$. Analytic continuation of (F_0, U_0) along γ (which is possible since the derivative $\frac{1}{z-z_0}$ is defined on $\mathbb{C}\backslash\{z_0\}$ and can therefore be trivially continued) leads to a function element (F_1, U_1) which also satisfies $e^{F_1(z)} = A(z - z_0)$ because (e^{F_1}, U_1) is the trivial continuation of (e^{F_0}, U_0) because the function $e^{F_0(z)} = A(z - z_0)$ is defined on the whole complex plane.

Hence

$$\int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z = F_1(\gamma(1)) - F_0(\gamma(0))$$

and

$$\exp\left(\int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z\right) = \frac{A(\gamma(1) - z_0)}{A(\gamma(0) - z_0)} = 1$$

because γ is closed. □

Lemma 6.2.3 (Ind$_\gamma$ constant on connected components)
The winding number Ind_γ *is constant on connected components of* $\mathbb{C}\backslash\gamma([0, 1])$.

Proof. For $z_0 \in \mathbb{C}\backslash\gamma([0, 1])$, the winding number depends continuously on z_0 and takes integer values. Hence it is constant on connected components of its image.

To see continuity, note that

$$\left|\frac{1}{z - z_0} - \frac{1}{z - z_1}\right| = \frac{|z_0 - z_1|}{|z - z_0||z - z_1|}$$

and that $\frac{1}{|\gamma - z_0||\gamma - z_1|}$ is bounded on $[0, 1]$. □

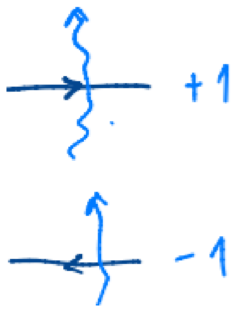


Fig. 72: Rule for increasing/decreasing the winding number when crossing the curve.

Example 6.2.4 To find the winding numbers of the regions that a closed curve γ separates the complex plane into we can proceed as follows. The image of the curve is compact, so there is a disk containing it, and outside of this disk the winding number is zero. Whenever we cross the curve and the curve comes from the left, then the winding number increasing by one and it decreases by one if the curve comes from the right.

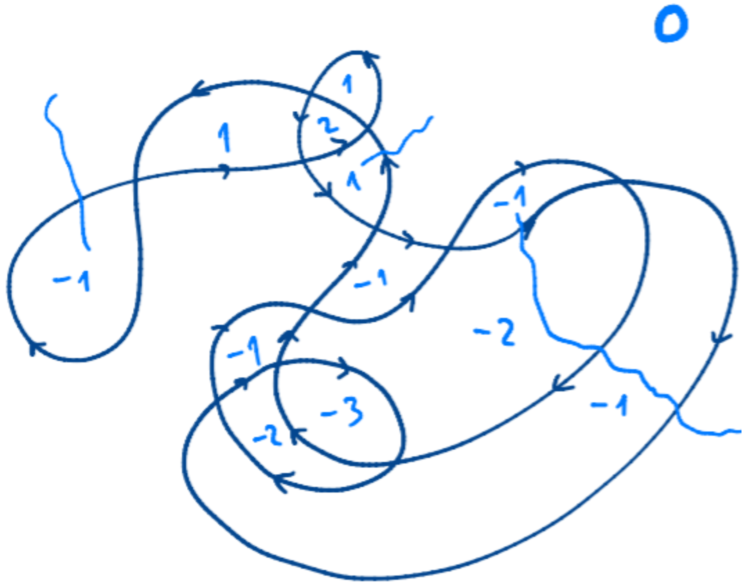


Fig. 73: A curve and the winding numbers of the regions that a closed curve γ separates the complex plane into.

We can and want to extend the concept of the winding number to cycles.

Definition 6.2.5 (Winding number of a cycle)
The winding number $\operatorname{Ind}_c(z_0)$ of a cycle c in $\mathbb{C}$ around a point $z_0 \in \mathbb{C}\setminus|c|$ is

$$\operatorname{Ind}_c(z_0) = \frac{1}{2\pi i}\int_c \frac{1}{z-z_0}\,\mathrm{d}z.$$

Theorem 6.2.2: Winding number is an integer

The winding number of a cycle is an integer.

Proof. Let c be as in (27) for curves $c_j\colon [0,1] \to \mathbb{C}\setminus\{z_0\}$. For $j \in \{1,\dots,k\}$ let $(F_j^{(0)}, U_j^{(0)})$ be a local antiderivative of $\frac{1}{z-z_0}$ defined on a domain U_j around $c_j(0)$ and let $(F_j^{(1)}, U^{(1)})$ be its analytic continuation along c_j.

As before,

$$e^{F_j^{(0)}(z)} = A_j(z-z_0) = e^{F_j^{(1)}(z)}.$$

Hence

$$\exp\left(\int_{c_j}\frac{1}{z-z_0}\,\mathrm{d}z\right) = \exp\left(F_j^{(1)}(c_j(1)) - F_j^{(0)}(c_j(0))\right) = \frac{\cancel{A_j}(c_j(1)-z_0)}{\cancel{A_j}(c_j(0)-z_0)},$$

for all $j \in \{1,\dots,k\}$ and therefore

$$\begin{aligned}\exp\left(\int_c\frac{1}{z-z_0}\,\mathrm{d}z\right) &= \exp\left(\sum_{j=1}^k n_j\int_{c_j}\frac{1}{z-z_0}\,\mathrm{d}z\right)\\ &= \frac{(c_1(1)-z_0)^{n_1}(c_2(1)-z_0)^{n_2}\dots(c_k(1)-z_0)^{n_k}}{(c_1(0)-z_0)^{n_1}(c_2(0)-z_0)^{n_2}\dots(c_k(0)-z_0)^{n_k}} = 1,\end{aligned}$$

because c is closed and thus every point occurs the same number of times as an endpoint $c_j(1)$ as it occurs as a starting point $c_{\tilde{j}}(0)$, taking the weights n_j into account. More precisely, for each $z \in \mathbb{C}$

$$\sum_{\substack{j=1\\ c_j(1)=z}}^k n_j = \sum_{\substack{j=1\\ c_j(1)=z}}^k n_j.$$

As before, this implies $\int_c \frac{1}{z-z_0}\,\mathrm{d}z \in 2\pi i\,\mathbb{Z}$ and thus the claim. □

6.3 The winding number / homology version of Cauchy's Integral Theorem

Definition 6.3.1 (Zero homogolous cycle)
A cycle c in an open set $U \subset \mathbb{C}$ is zero homogolous in U if $\operatorname{Ind}_c(z) = 0$ for all $z \in \mathbb{C}\setminus U$. zero homogolous

Example 6.3.2 A circle c around z_0 is not zero homologous in $U := \mathbb{C}\setminus\{z_0\}$ because $z_0 \notin U$, but $\operatorname{Ind}_c(z_0) = 1$. ◇

Remark 6.3.3 (null-homotopic $\implies$ null-homologous) If γ is null-homotopic, then γ is null-homologous. The converse does not hold, consider $\gamma := aba^{-1}b^{-1}$, which is not null-homotopic but

$$\operatorname{Ind}_\gamma(z_0) = \operatorname{Ind}_a(z_0) + \operatorname{Ind}_b(z_0) + \operatorname{Ind}_{a^{-1}}(z_0) + \operatorname{Ind}_{b^{-1}}(z_0) = 0,$$

as $\operatorname{Ind}_{a^{-1}}(z_0) = -\operatorname{Ind}_a(z_0)$ for any $z_0 \in ???$. ○

Lemma 6.3.4 (Has to be somewhere else)
Let $U \subset \mathbb{C}$ be open and $\mathbb{C} \setminus U$ unbounded and connected. Then a closed curve in U is null-homologous.

Proof. Let γ be a closed curve in U. Without loss of generality we can assume that γ is piecewise $\mathcal{C}^1$. Then

$$|\operatorname{Ind}_\gamma(z_0)| = \frac{1}{2\pi}\left|\int_\gamma \frac{1}{z-z_0}\,\mathrm{d}z\right| \leqslant \frac{1}{2\pi}\operatorname{len}(\gamma)\underbrace{\sup_{z\in|\gamma|}\left|\frac{1}{z-z_0}\right|}_{\xrightarrow{|z_0|\to\infty}0} \xrightarrow{|z_0|\to\infty} 0.$$

Hence there exists a $z_0 \in \mathbb{C}\setminus U$ such that

$$|\operatorname{Ind}_\gamma(z_0)| < \frac{1}{2}.$$ □

As $\operatorname{Ind}_\gamma \in \mathbb{Z}$, $\operatorname{Ind}_\gamma(z_0) = 0$ for all $z_0 \in \mathbb{C}\setminus U$, as $\mathbb{C}\setminus U$ is connected.

Theorem 6.3.1: Cauchy's Integral Theorem (Winding number / homology version)

Let U be a domain in $\mathbb{C}$ and c be a cycle in U. The following statements are equivalent.
1. c is zero homologous in U
2. $\int_c f(z)\,\mathrm{d}z = 0$ for all holomorphic functions f on U.

Proof. $\neg$ 1 $\implies$ $\neg$ 2: There is a point $z_0 \in \mathbb{C}\setminus U$ such that $\operatorname{Ind}_c(z_0) \neq 0$. Then the function $f(z) := \frac{1}{z-z_0}$ is holomorphic on U and

$$\int_c f(z)\,\mathrm{d}z = 2\pi i \operatorname{Ind}_c(z_0) \neq 0.$$

1 $\implies$ 2: Assume c is a zero homologous cycle in U.

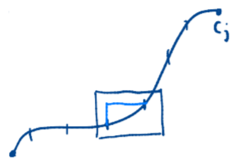
1 First we will construct a cycle $\tilde{c}$ that is a formal linear combination of horizontal or vertical line segment traversed at constant speed such that

$$\int_c f(z)\,\mathrm{d}z = \int_{\tilde{c}} f(z)\,\mathrm{d}z$$

for all holomorphic functions f on U.

To this end, consider on curve c_j occurring in the cycle c.

We want to cover the curve by rectangles contained in U and then we take a subdivision of c_j such that the portions in one subinterval are contained in a rectangle that is contained in U and the we replace this portion to a vertical and a horizontal line segment. Because the old and the new portion are homotopic, the integral doesn't change. Let $\mathcal{R}$ be the set of open rectangular regions with sides parallel to the axes that are contained in U. Applying the LEBESGUE number lemma to the open cover $\{c_j^{-1}(R)\}_{R\in\mathcal{R}}$ at $[0,1]$, we see that there exists a subdivision



$$0 = t_0 < t_1 < \ldots < t_M = 1$$

such that $c_j([t_\ell, t_{\ell+1}])$ is contained in some open rectangle $R \subset U$.

Consider the curves

$$\begin{aligned} c^v_{j,\ell} &:= c_j(t_\ell) + t \cdot i\Im(c_j(t_{\ell+1}) - c_j(t_\ell)) \\ c^h_{j,\ell} &:= c_j(t_\ell) - (1-t)\cdot i\Re(c_j(t_{\ell+1}) - c_j(t_\ell)). \end{aligned}$$

Since the rectangle is convex and hence simply connected, the curves $c_j|_{[t_\ell,t_{\ell+1}]}$ and $c^v_{j,\ell}c^h_{j,\ell}$ are homotopic.

So for all holomorphic functions f on U,

$$\int_{c_j|_{[t_\ell,t_{\ell+1}]}} f(z)\,\mathrm{d}z = \int_{c^v_{j,\ell}} f(z)\,\mathrm{d}z + \int_{c^h_{j,\ell}} f(z)\,\mathrm{d}z.$$

Let $c = \bigoplus_{j=1}^k (n_j \odot c_j)$ and define

$$\tilde{c} = \bigoplus_{j=1}^{k} n_j \cdot \bigoplus_{\ell=1}^{M-1} c^v_{j,\ell} \oplus c^h_{j,\ell} = \oplus_{j=1}^k \oplus_{\ell=1}^{M-1} \left(n_j \odot c^v_{j,\ell}\right) \oplus \left(n_j \odot c^h_{j,\ell}\right).$$

Hence for the rest of the proof we may assume without loss of generality that c is a cycle consisting of horizontal and vertical line segments.

29.06.2021

2 We will now construct a rectangular grid by taking all the endpoints of the vertical and horizontal line segments and making a grid out of these endpoints.

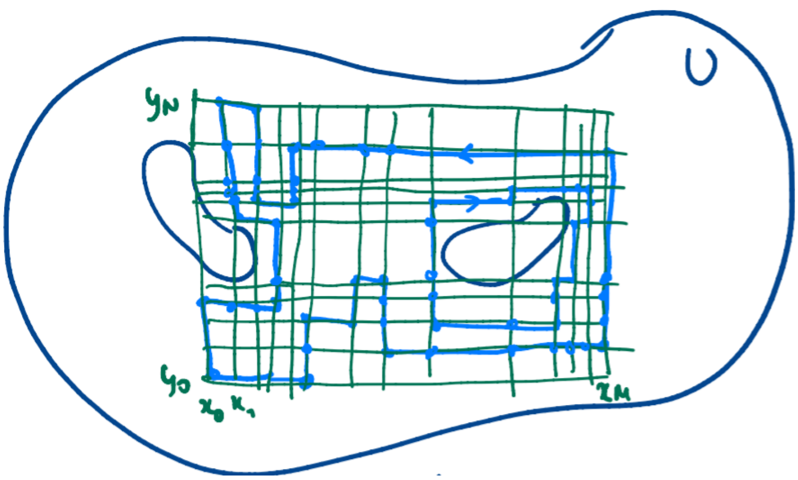


Fig. 74: The not necessarily simply connected domain and a zero homologous cycle consisting only of horizontal and vertical line segment and the grid constructed from the endpoints of the vertical and horizontal line segments.

Let

$$x_0 < x_1 < \ldots, x_N$$

be the real parts of the segments' endpoints and let

$$y_0 < y_1 < \ldots < y_M$$

be the imaginary parts. Now we can subdivide the horizontal and vertical line segments further such that all the horizontal and vertical line segments are really edges of this non-uniform rectangular grid: by subdividing the line segments of which the cycle c consists further if necessary, we may arrive at a cycle whose curves are edges of the non-uniform rectangle grid with vertices

$$z_{j,k} := x_j + iy_k.$$

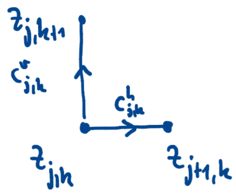


Fig. 75: The horizontal or vertical straight line segments $c^h_{j,k}$ and $c^v_{j,k}$.

So without loss of generality we may assume that the cycle c is of the form

$$c = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \left(n^h_{j,k} c^h_{j,k} + n^v_{j,k} c^v_{j,k} \right),$$

where $c^h_{j,k}$ is the horizontal straight line segment from $z_{j,k}$ to $z_{j+1,k}$ and $c_{j,k}$ is the vertical straight line segment from $z_{j,k}$ to $z_{j,k+1}$, both parametrised on $[0,1]$. The coefficients $n^h_{j,k}$ and $n^v_{j,k}$ are integers

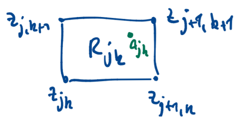


Fig. 76: The rectangle $R_{j,k}$.

3 In the interior of each rectangle

$$R_{j,k} := [x_j, x_{j+1}] + i[y_k, y_{k+1}]$$

choose an arbitrary point a_{jk} and let

$$\nu_{j,k} := \mathrm{Ind}_c(a_{j,k}).$$

Since the interior of $R_{j,k}$ and $|c|$ are disjoint, Ind_c is constant on $R_{j,k}$, so $\nu_{j,k}$ is independent of the choice of $a_{j,k}$.

The assumption that c is null-homogolous enters here! If $R_{j,k} \backslash U \neq \varnothing$, then $\nu_{j,k} = 0$. (This is true even if the point of $R_{j,k}$ outside U are all boundary points of $R_{j,k}$.)

Fig. 77: Suppose that the bottom left corner is the only point not contained in U. Then the winding number is zero around that point. But as the winding number is locally constant - the adjacent edges can not be traced out by the curve - there is a neighbourhood around this point, where $\mathrm{Ind}_c = 0$.

Claim. We claim that the coefficients of the horizontal / vertical straight line segments are the differences of winding numbers of adjacent rectangles, in particular

a $n^h_{j,k} = \nu_{j,k} - \nu_{j,k-1}$,

b $n^v_{j,k} = \nu_{j-1,k} - \nu_{j,k}$.

4 We now prove a, the proof of b is similar. Let

$$r_{j,k} := c^h_{j,k} + c^v_{j+1,k} - c^h_{j,k+1} - c^v_{j,k}.$$

Then $\mathrm{Ind}_{r_{j,k}}(a_{j,k}) = 1$ $(\star)$.

Consider the cycle

$$\tilde{c} = c - n^h_{j,k} r_{j,k}.$$

Then the support $|\tilde{c}|$ does not contain the edge $|c^h_{j,k}|$ because the coefficient of $c^h_{j,k}$ is zero. That means that the $\mathrm{Ind}_{\tilde{c}}$ is constant on the interior of $R_{j,k} \cup R_{j,k-1}$. Hence (by the linearity of the integral (L) in the definition of the winding index)

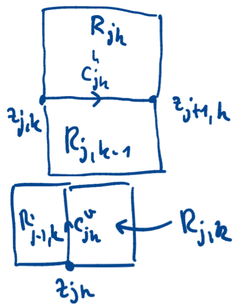


Fig. 78: Adjacent rectangles.

$$\begin{aligned}
\nu_{j,k-1} &= \underbrace{\mathrm{Ind}_c(a_{j,k-1})}_{\nu_{j,k-1}} - n^h_{j,k} \underbrace{\mathrm{Ind}_{r_{j,k}}(a_{j,k-1})}_{=0} = \mathrm{Ind}_{\tilde{c}}(a_{j,k-1}) \\
&= \mathrm{Ind}_{\tilde{c}}(a_{j,k}) \overset{(L)}{=} \underbrace{\mathrm{Ind}_c(a_{j,k})}_{\nu_{j,k}} - n^h_{j,k} \underbrace{\mathrm{Ind}_{r_{j,k}}(a_{j,k})}_{=1} \overset{(\star)}{=} \nu_{j,k} - n^h_{j,k}.
\end{aligned}$$

(5) We claim that

$$c = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \nu_{j,k} r_{j,k}.$$

This can be shown using the previous claim and comparing coefficients of the edges.

But this implies

$$\int_c f(z)\,\mathrm{d}z = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \nu_{j,k} \int_{r_{j,k}} f(z)\,\mathrm{d}z.$$

Note that $R_{j,k} \subset U$ if $\nu_{j,k} \neq 0$, so all the integrals in the sum vanish due to CAUCHY's Integral Theorem for Rectangles. □

6.4 Cauchy's Integral Formula & the Residue Theorem

THEOREM 6.4.1: CAUCHY'S INTEGRAL FORMULA (WINDING NUMBER VERSION)

Let f be a holomorphic function on $U \subset \mathbb{C}$, let $a \in U$ and let c be a cycle in $U \backslash \{a\}$ that is zero-homologous in U. Then

$$\frac{1}{2\pi i} \int_c \frac{f(z)}{z-a}\,\mathrm{d}z = \mathrm{Ind}_c(a) \cdot f(a).$$

Fig. 80: The proof of Theorem 6.4.1.

Proof. Choose $r > 0$ so small that $\overline{B}_r(a) \subset U$. Let $\gamma\colon [0,1] \to U$, $t \mapsto a + re^{2\pi i t}$. Then $\mathrm{Ind}_\gamma(a) = 1$, so for $\tilde{c} := c \ominus \mathrm{Ind}_c(a) \odot \gamma$ we have

$$\mathrm{Ind}_{\tilde{c}}(a) = \mathrm{Ind}_c(a) - \mathrm{Ind}_c(a) \underbrace{\mathrm{Ind}_\gamma(a)}_{=1} = 0.$$

It follows that $\tilde{c}$ is zero homologous in $U \backslash \{a\}$, because $\mathrm{Ind}_{\tilde{c}}(a) = 0$ and $\mathrm{Ind}_{\tilde{c}}(z) = 0$ for any $z \notin U$, because $\mathrm{Ind}_c(z) = 0$ for any $z \notin U$ and the winding number of $\mathrm{Ind}_\gamma(z) = 0$ for any $|z-a| > r$, so $\tilde{c}$ is not only zero homologous in U but also in $U \backslash \{a\}$.

Since $z \mapsto \frac{f(z)}{z-a}$ is holomorphic on $U \backslash \{a\}$, the winding number version of CAUCHY's Integral Theorem implies

$$0 = \int_{\tilde{c}} \frac{f(z)}{z-a}\,\mathrm{d}z = \int_c \frac{f(z)}{z-a}\,\mathrm{d}z - \mathrm{Ind}_c(a) \int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z \overset{(\star)}{=} \int_c \frac{f(z)}{z-a}\,\mathrm{d}z - \mathrm{Ind}_c(a) 2\pi i f(a),$$

where in $(\star)$ we use CAUCHY's Integral Formula for Disks. □

Let us now introduce a notion

Definition 6.4.1 (Bounding cycle)
Let $K \subset \mathbb{C}$ be a compact set. A cycle c bounds K if $|c| \subset \partial K$ and if

$$\operatorname{Ind}_c(z) = \begin{cases} 1, & \text{if } z \in \mathring{K}, \\ 0, & \text{if } z \notin K. \end{cases}$$

Remark 6.4.2 We need not have $|c| = \partial K$ (but in most cases we do): consider for example $K := \overline{D} \cup [1,2]$. Then the unit circle c is a bounding cycle for K, but the ∂K contains $(1,2] \not\subset |c|$. The point is: We did not assume that K is the closure of its interior. ○

We can now formula Theorem 6.4.1 more simply:

Corollary 6.4.3
If f is holomorphic on U and the cycle c bounds the compact subset $K \subset U$, then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} \,\mathrm{d}z$$

for all $a \in \mathring{K}$.

Proof. In this case, the winding number is 1. □

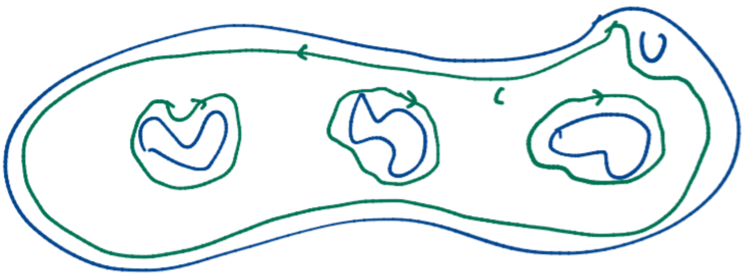


Fig. 81: Corollary 6.4.3 is perfectly suited for dealing with a domain U with holes and a cycle as shown.

Residues

Theorem 6.4.2: Residue

(1) Suppose the holomorphic function f has an isolated singularity at z_0 (or is holomorphic at z_0, too). The residue of f at z_0 is residue

$$\operatorname{Res}_{z_0}(f) := \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z) \,\mathrm{d}z,$$

where $\varepsilon > 0$ is so small that $\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant \varepsilon\} \subset U$.

(2) Equivalently, if the Laurent series around z_0 representing f is $\sum_{k\in\mathbb{Z}} a_k(z - z_0)^k$, then $\operatorname{Res}_{z_0}(f) = a_{-1}$.

Proof. Of (2): use the formula for the coefficients of the LAURENT series expansion or argue directly:

$$\begin{aligned}\operatorname{Res}_{z_0}(f) &= \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon}\sum_{k\in\mathbb{Z}} a_k (z-z_0)^k\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\sum_{k\in\mathbb{Z}} a_k \underbrace{\int_{|z-z_0|=\varepsilon} (z-z_0)^k\,\mathrm{d}z}_{=\mathbb{1}_{-1}(k)}.\end{aligned}$$

□

THEOREM 6.4.3: RESIDUE THEOREM

Let f be holomorphic on U except for a set $S \subset U$ of isolated singularities. Let c be a zero homologous cycle in U with $|c| \cap S \neq \varnothing$. Then

$$\frac{1}{2\pi i}\int_c f(z)\,\mathrm{d}z = \sum_{a\in S}\operatorname{Ind}_c(a)\operatorname{Res}_a(f),$$

where the sum is finite because $\operatorname{Ind}_c(a) \neq 0$ only for finitely many $a \in S$.

Corollary 6.4.4
If c bounds a compact subset $K \subset U$, then

$$\frac{1}{2\pi i}\int_c f(z)\,\mathrm{d}z = \sum_{a\in S\cap\mathring{K}}\operatorname{Res}_a(f).$$

Proof. **(of Theorem 6.4.3)** Let us assume that $\operatorname{Ind}_c(a) \neq 0$ for finitely many singularities $a \in S$. **30.06.2021**

For each $a \in S$, let γ_a be a circle around a with radius small enough such that the closed disk that it bounds in contained in U and doesn't contain any other singularities.

Fig. 82: The set U, a closed curve c and some circles γ_a.

Let $\tilde{c} := c \ominus \left(\bigoplus_{a\in S}\operatorname{Ind}_c(a)\odot\gamma_a\right)$. For every $a_0 \in S$,

$$\operatorname{Ind}_{\tilde{c}}(a_0) = \operatorname{Ind}_c(a_0) - \sum_{a\in S}\operatorname{Ind}_c(a)\cdot\operatorname{Ind}_{\gamma_a}(a_0) = 0,$$

as $\operatorname{Ind}_{\gamma_a}(a_0) = 1$ if $a = a_0$ and 0 else. Hence $\tilde{c}$ is zero homologous in $U\backslash S$.

Also, f is holomorphic on $U\backslash S$. By the winding number version of CAUCHY's Integral Theorem,

$$0 = \int_{\tilde{c}} f(z)\,\mathrm{d}z = \int_c f(z)\,\mathrm{d}z - \sum_{a\in S}\operatorname{Ind}_c(a)\cdot\underbrace{\int_{\gamma_a} f(z)\,\mathrm{d}z}_{=2\pi i\operatorname{Res}_f(a)}.$$

It remains to show the assumption made at the beginning. Towards contradiction assume that $S_c := \{a \in S : \mathrm{Ind}_c(a) \neq 0\}$ is infinite.

1 The set S_c is bounded. To see this, note that $|c|$ is compact, hence bounded. Suppose $|c| \subset B_R(0)$. Then for all points $z \in \mathbb{C} \backslash B_R(0)$, we have $\mathrm{Ind}_c(z) = 0$. Hence $|a| \leqslant R$ for all $a \in S_c$.

2 The set S_c has a limit point a^*. But this cannot be contained in U, since the set of singularities is isolated in U and hence a^* can't be the limit point in U of a sequence in U. Because c is zero homologous, $\mathrm{Ind}_c(a^*) = 0$. But the winding number Ind_c is constant on the connected components of $\mathbb{C} \backslash |c|$, which are open. So $\mathrm{Ind}_c(a^*) = 0$ in an open neighbourhood of a^*, contradicting the claim that a^* is a limit point of S_c. □

7 The calculus of residues

7.1 Computing integrals using the residue theorem (preliminary remarks)

The "calculus of residues" is a bag of tricks that are helpful to compute some definite integrals, in particular real integrals. To use the Residue Theorem, we have to be able to compute residues. This is easy if we know the LAURENT series for some reason.

Example 7.1.1 (Computing residues of $\exp(z^{-1})$) By the power series expansion of the exponential function, we have

$$e^{\frac{1}{z}} = \sum_{k=-\infty}^{\infty} \frac{1}{(-k)!} z^k$$

for $z \neq 0$ and this function has one singularity at $z = 0$, which is essential. We have $\operatorname{Res}_{\exp(z^{-1})}(0) = \frac{1}{1!} = 1$. ◇

For poles it is also straightforward to compute the residue.

Example 7.1.2 (Computing residues at poles) ① The simplest case are poles of order 1. If f has a poles of order 1 at z_0, then the LAURENT series at z_0 is

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots,$$

so

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \ldots$$

and

$$\lim_{z\to z_0} (z - z_0)f(z) = a_{-1} = \operatorname{Res}_f(z_0).$$

② If $f = \frac{g}{h}$, where h has a simple zero at z_0 and $g(z_0) \neq 0$, then f has a first order pole at z_0 and

$$\operatorname{Res}_f(z_0) = \lim_{z\to z_0} \frac{z - z_0}{h(z)} g(z) = \frac{g(z_0)}{h'(z_0)}.$$

③ If f has a pole of order n at z_0, then the LAURENT series expansion is

$$f(z) = a_{-n}\frac{1}{(z - z_0)^n} + \ldots + a_{-1}\frac{1}{(z - z_0)} + a_0 + a_1(z - z_0) + \ldots$$

and thus

$$(z - z_0)^n f(z) = a_{-n} + \ldots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + \ldots$$

is a TAYLOR series in a neighbourhood of z_0. Hence

$$\operatorname{Res}_f(z_0) = a_{-1} = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \Big[(z - z_0)^n f(z)\Big].$$ ◇

Example 7.1.3 (Computing residues with cot (Tut XII))
Consider $f(z) := \frac{\cot(z)}{z+1}$. Let $g(x) := \frac{\cos(z)}{z+1}$ and $h(z) := \sin(z)$. Then h has a simple zero at 0 and π and g is holomorphic with $g(0) \neq 0 \neq g(\pi)$ and thus by example 7.1.2 2

$$\operatorname{Res}_f(0) = \frac{g(0)}{h'(0)} = 1 \qquad \text{and} \qquad \operatorname{Res}_f(\pi) = \frac{g(\pi)}{h'(\pi)} = \frac{1}{\pi+1}.$$

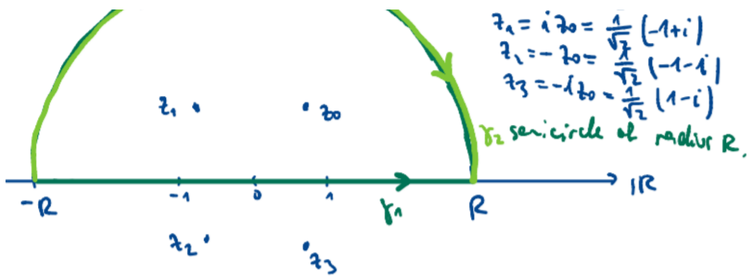
◇

7.2 Integrals along the whole real axis

Example 7.2.1 (Computing $\int_{\mathbb{R}}(1+x^4)^{-1}\,\mathrm{d}x$)
One can calculate $\int_{\mathbb{R}}(1+x^4)^{-1}\,\mathrm{d}x$ with partial fraction decomposition (the antiderivative is very complicated: $c + \frac{-\log\left(x^2-\sqrt{2}x+1\right)+\log\left(x^2+\sqrt{2}x+1\right)-2\tan^{-1}\left(1-\sqrt{2}x\right)+2\tan^{-1}\left(\sqrt{2}x+1\right)}{4\sqrt{2}}$), but also with the Residue Theorem. We can write

$$\int_{\mathbb{R}} \frac{1}{1+x^4}\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,\mathrm{d}x$$

The idea is to consider the function $f(z) := \frac{1}{1+z^4}$, which is holomorphic except for first order poles at the four square roots of $-1 = e^{i\pi}$, which are $z_0 := e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$, $z_1 := iz_0 = \frac{1}{\sqrt{2}}(-1+i)$, $z_2 := -z_0 = \frac{1}{\sqrt{2}}(-1-i)$ and $z_3 := \frac{1}{\sqrt{2}}(1-i)$.



By the Residue Theorem,

$$\int_{\gamma_1 \ominus \gamma_2} \frac{1}{1+z^4}\,\mathrm{d}z = 2\pi i \left(\operatorname{Res}_f(z_0) + \operatorname{Res}_f(z_1)\right).$$

Also,

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{1+x^4}\,\mathrm{d}x.$$

Hence

$$\int_{-R}^{R} \frac{1}{1+x^4}\,\mathrm{d}x = \int_{\gamma_2} \frac{1}{1+z^4}\,\mathrm{d}z + 2\pi i \left(\operatorname{Res}_f(z_0) + \operatorname{Res}_f(z_1)\right).$$

1 We have

$$\left|\int_{\gamma_2} \frac{1}{1+z^4}\,\mathrm{d}z\right| =\leqslant \pi R \frac{1}{R^4} = \pi R^{-3} \xrightarrow{R\to\infty} 0,$$

so we don't have to care about γ_2 in the limit.

2 By example 7.1.2 3 we have

$$\operatorname{Res}_f(z_0) = \frac{1}{4z_0^3} = -\frac{1}{4}z_0 = -\frac{1}{4\sqrt{2}}(1+i)$$

as $z_0^4 = -1$ and analogously

$$\operatorname{Res}_f(z_1) = \frac{1}{4z_1^3} = -\frac{1}{4\sqrt{2}}(-1+i)$$

and hence

$$2\pi i\left(\operatorname{Res}_f(z_0)+\operatorname{Res}_f(z_1)\right)=2\pi i\left(\frac{-1}{4\sqrt{2}}(2i)\right)=\frac{\pi}{\sqrt{2}}.$$

In conclusion

$$\int_{\mathbb{R}}\frac{1}{1+x^4}\,\mathrm{d}x=\frac{\pi}{\sqrt{2}}.$$

For this to work it was important that

- The integral over the large half-circle tends to zero.
- there are no poles on the real axis. ◇

7.3 Integrals over $\mathbb{R}$ with poles on the real axis

Example 7.3.1
We can set $\frac{\sin(0)}{0}=1$ and then

$$\int_{\mathbb{R}}\frac{\sin(x)}{x}\,\mathrm{d}x$$

becomes the integral of a continuous function (without singularities). We can also write the integrand as

$$\frac{\sin(x)}{x}=\Im\left(\frac{e^{ix}}{x}\right)$$

and hence our idea is to integrate $f(z):=\frac{e^{iz}}{z}$ along some cycle. The function f has a simple pole at 0 with $\operatorname{Res}_f(0)=1$.

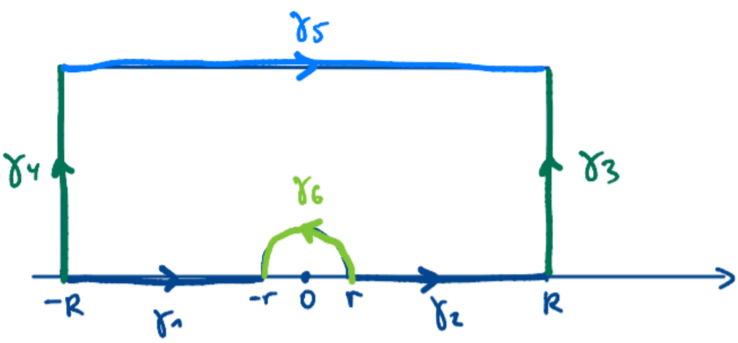
How do we find an appropriate contour? We have

$$|e^{iz}|=|e^{i(x+iy)}|=|e^{ix}e^{-y}|=e^{-y},$$

which tends to zero for $y\to\infty$, and thus

$$|e^{iz}|=\frac{e^{-y}}{|z|}.$$

Consider the following contour for $0<r<R$.



The function f is holomorphic in the region bounded by the cycle

$$\gamma_1\oplus\gamma_2\oplus\gamma_3\ominus\gamma_4\ominus\gamma_5\ominus\gamma_6,$$

so by CAUCHY's Integral Theorem,

$$\int_c f(z)\,\mathrm{d}z=0. \tag{28}$$

Also,

$$\int_{-R}^{-r} \frac{\sin(x)}{x} \, \mathrm{d}x + \int_{r}^{R} \frac{\sin(x)}{x} \, \mathrm{d}x = \Im \left(\int_{\gamma_1 \oplus \gamma_2} f(z) \, \mathrm{d}z \right).$$

We have by (28)

$$\int_{\gamma_1 \oplus \gamma_2} f(z) \, \mathrm{d}z = \underbrace{- \int_{\gamma_3} f(z) \, \mathrm{d}z + \int_{\gamma_4} f(z) \, \mathrm{d}z + \int_{\gamma_5} f(z) \, \mathrm{d}z}_{\xrightarrow{R \to \infty} 0} + \int_{\gamma_6} f(z) \, \mathrm{d}z.$$

- First let us consider the integral along γ_6. Near 0,

$$f(z) = \frac{\mathrm{Res}_f(0)}{z} + g(z),$$

where g is holomorphic at 0. Hence

$$\int_{\gamma_6} f(z) \, \mathrm{d}z = \underbrace{\mathrm{Res}_f(0)}_{=1} \underbrace{\int_{\gamma_6} \frac{1}{z} \, \mathrm{d}z}_{i\pi} + \underbrace{\int_{\gamma_6} g(z) \, \mathrm{d}z}_{\xrightarrow{r \searrow 0} 0}.$$

and thus

$$\lim_{r \to 0} \int_{\gamma_6} f(z) \, \mathrm{d}z = i\pi.$$

- Let us now consider the integral along γ_2. We have

$$\begin{aligned} \left| \int_{\gamma_2} f(z) \, \mathrm{d}z \right| &= \left| \int_{R}^{R+iR} \frac{e^{iz}}{z} \, \mathrm{d}z \right| = \left| \int_0^R \frac{e^{i(R+it)}}{R+it} i \, \mathrm{d}t \right| \\ &\leqslant \int_0^R \frac{e^{-t}}{|R+it|} \, \mathrm{d}z \leqslant \int_0^R \frac{e^{-t}}{R} \, \mathrm{d}z = \frac{1}{R}(-e^{-R}+1) \xrightarrow{R \to \infty} 0. \end{aligned}$$

- The integral along γ_4 also goes to 0 to $R \to \infty$ by an analogous argument.
- For the integral along γ_5 observe

$$\left| \int_{\gamma_5} f(z) \, \mathrm{d}z \right| = \left| \int_{-R+iR}^{R+iR} \frac{e^{iz}}{z} \, \mathrm{d}z \right| \leqslant 2R \frac{e^{-R}}{R} = 2e^{-R} \xrightarrow{R \to \infty} 0.$$

We get

$$\int_{\mathbb{R}} \frac{\sin(x)}{x} \, \mathrm{d}x = \lim_{r \searrow 0} \Im \left(\int_{\gamma_6} f(z) \, \mathrm{d}z \right) = \Im(i\pi) = \pi.$$

◇

Example 7.3.2 (What is $\int_{\mathbb{R}} \frac{1}{(x^2+1)(x^2+4)} \, \mathrm{d}x$? (Tut XII))
Define $f \colon \mathbb{C} \setminus \{\pm i, \pm 2i\} \to \mathbb{C}$, $z \mapsto \frac{1}{(z^2+1)(z^2+4)}$ and $c_R \colon [0, \pi] \to \mathbb{C}$, $t \mapsto Re^{it}$ for $R > 0$.

First we show that if $|f(z)| \leqslant \frac{M}{|z|^a}$ for some $a > 1$ and $M \geqslant 0$ holds for all large z, then $\int_{c_R} f(z) \, \mathrm{d}z \xrightarrow{R \to \infty} 0$. We have

$$\left| \int_{c_R} f(z) \, \mathrm{d}z \right| \leqslant \mathrm{len}(c_R) \frac{M}{R^a} = \frac{\pi M}{R^{a-1}} \xrightarrow{R \to \infty} 0.$$

Why do we have $|f(z)| \leqslant \frac{M}{|z|^a}$ in our case? For $|z| \geqslant 4$, we have $|z|^2 \leqslant |z^2+1||z^2+4|$, so $\int_{c_R} f(z) \, \mathrm{d}z \xrightarrow{R \to \infty} 0$.

Now let $d_R \colon [-R, R] \to \mathbb{R}$, $t \mapsto t$. Then $d_R'(t) = 1$ and thus

$$\int_{d_R} f(z) \, \mathrm{d}z = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)} \, \mathrm{d}t.$$

By Theorem 6.4.3,

$$\int_{d_R} f(z)\,\mathrm{d}z - \int_{c_R} f(z)\,\mathrm{d}z = 2\pi i \sum_{a\in\{i,2i\}} \mathrm{Ind}_{d_R c_R}(a)\,\mathrm{Res}_f(a) = 2\pi i\big(\mathrm{Res}_f(i) + \mathrm{Res}_f(2i)\big).$$

Defining $g_1(z) := \frac{1}{(z+i)(z+2i)(z-2i)}$, $g_2(z) := \frac{1}{(z+i)(z-i)(z+2i)}$, $h_1(z) := z-i$ and $h_2(z) := z-2i$, we find

$$\mathrm{Res}_f(i) = \frac{g_1(i)}{h_1'(i)} = \frac{1}{(2i)(3i)(-i)} = \frac{-i}{6} \qquad \text{and} \qquad \mathrm{Res}_f(2i) = \frac{g_2(i)}{h_2'(i)} = \frac{1}{(3i)(2i)(4i)} = \frac{i}{12}$$

and thus

$$\int_{\mathbb{R}} \frac{1}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \lim_{R\to\infty} \int_{d_R} f(z)\,\mathrm{d}z - \int_{c_R} f(z)\,\mathrm{d}z = 2\pi i\left(\frac{i}{12} - \frac{i}{6}\right) = \frac{\pi}{6}.$$

◇

We omit subsection 7.4

7.5 Integrals of rational function of sin and cos over a period

Example 7.5.1 Consider $\int_0^{2\pi} \frac{1}{3+\cos(x)}\,\mathrm{d}x$. We have

$$\begin{aligned}\int_0^{2\pi} \frac{1}{3+\cos(x)}\,\mathrm{d}x &= \Im\left(\int_0^{2\pi} \frac{1}{3+\frac{1}{2}(e^{ix}+e^{-ix})}\frac{1}{e^{ix}} ie^{ix}\,\mathrm{d}x\right) \\ &= \Im\left(\int_{|z|=1} \frac{1}{(3+\frac{1}{2}(z+\frac{1}{z}))z}\,\mathrm{d}z\right) = \Im\left(\sum_j 2\pi i\,\mathrm{Res}_f(z_j)\right),\end{aligned}$$

by the Residue Theorem, where $f(z) := \frac{1}{(3+\frac{1}{2}(z+\frac{1}{z}))z}$ and the sum is taken over the poles of f inside the unit circle. The result is $\frac{\pi}{\sqrt{2}}$. ◇

7.6 An integral that counts zeros and poles

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If f has an isolated zero (this means f is not zero in a neighbourhood and the order of the zero is finite) or a pole (which implies that it is an isolated singularity) at z_0, then by Theorem 3.4.1 there is a $k \in \mathbb{Z}\backslash\{0\}$ and a holomorphic function g such that

$$f(z) = (z-z_0)^k g(z)$$

in a neighbourhood of z_0, where g is holomorphic at z_0 and $g(z_0) \neq 0$. If f has a zero at z_0, then $k > 0$ and the order of the zero is k. If f has a pole at z_0, then $k < 0$ and the order of the pole is $-k$.

Now consider the logarithmic derivative of f, $z \mapsto \frac{f'(z)}{f(z)}$, which is holomorphic where f is holomorphic except for the zeros of f.

In a neighbourhood of z_0,

$$f'(z) = k(z-z_0)^k g(z) + (z-z_0)^k g'(z),$$

so

$$\frac{f'(z)}{f(z)} = \frac{k(z-z_0)^k g(z) + (z-z_0)^k g'(z)}{(z-z_0)^k g(z)} = \frac{k}{z-z_0} + \frac{g'(z)}{g(z)}.$$

Since g is holomorphic and doesn't have a zero at z_0, its logarithmic derivative $\frac{g'(z)}{g(z)}$ is holomorphic at z_0. By example 7.1.2 1 this implies

$$\operatorname{Res}_{\frac{f'}{f}}(z_0) = \lim_{z\to z_0} (z-z_0)\left(\frac{k}{z-z_0} + \frac{g'(z)}{g(z)}\right) = \lim_{z\to z_0} k + \underbrace{(z-z_0)}_{\to 0} \underbrace{\frac{g'(z)}{g(z)}}_{\to \frac{g'(z_0)}{g(z_0)}} = k.$$

Applying the residue theorem to $\frac{f'}{f}$ yields the following theorem.

Theorem 7.6.1: Zero and pole counting integral

Let f be meromorphic on $U \subset \mathbb{C}$ and c be a cycle that bounds a compact set $K \subset U$ such that ∂K doesn't contain any zero or poles of f. Then

$$\frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} \,\mathrm{d}z = Z - P,$$

where Z is the number of zeros of f in $\mathring{K}$ and P the number of poles, each counted with multiplicity according to their order.

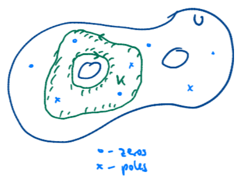

Fig. 83: The setup of Theorem 7.6.1.

There is an topological interpretation in terms of the winding number.

For a 1-chain $c = \bigoplus_j n_j \odot \gamma_j$ we define the image under a map f, defined on $|c|$ by

$$f(c) := \bigoplus_j n_j \cdot (f \circ \gamma_j)$$

Now for some piecewise $\mathcal{C}^1([t_0, t_1])$ curve $\gamma\colon [t_0, t_1] \to U$ that does not pass through any zeros or poles of f

$$\int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \int_{t_0}^{t_1} \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) \,\mathrm{d}t = \int_{f\circ\gamma} \frac{1}{z} \,\mathrm{d}z. \tag{29}$$

This is still true if γ is only continuous, because any continuous curve in $U \backslash \{$poles and zeros of $f\}$ is homotopic to a $\mathcal{C}^1$-curve. Hence, even for a 1-chain c in U whose support $|c|$ does not contain zeros or poles of f, we have

$$= \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} \,\mathrm{d}z = \frac{1}{2\pi i} \int_{f(c)} \frac{1}{z} \,\mathrm{d}z.$$

For the 1-cycle bounding K, we get

$$Z - P = \operatorname{Ind}_{f(c)}(0)$$

We can thus reformulate the above theorem:

Theorem 7.6.2: Zero and pole counting winding number

Under the same assumptions of and using the same notation as in the previous theorem, $\operatorname{Ind}_{f(c)}(0) = Z - P$.

Jänich uses this to prove the following result.

Theorem 7.6.3

A nonconstant rational function has a many zeros as it has poles in $\hat{\mathbb{C}}$ (both counted with multiplicities).

Corollary 7.6.1
A nonconstant rational function takes every value $a \in \hat{\mathbb{C}}$ the same number of times.

Proof. If f is that rational function, apply Theorem 7.6.3 to $f - a$. □

In think that only the zero and pole counting winding number is a bit over the top. If $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials without common zeros, then the number Z of zeros and the number P of poles are

$$Z = \deg(p) + \max(\deg(q) - \deg(p), 0) \qquad \text{and} \qquad P = \deg(q) + \max(\deg(p) - \deg(q), 0),$$

where the second summands correct for zeros and poles at ∞. In any case $Z - P = 0$.

Rouché's theorem

Lemma 7.6.2 (Dog on a leash)
Let $c_1, c_2 \colon [0,1] \to \mathbb{C}$ be two closed curves and $z_0 \in \mathbb{C} \setminus (|c_1| \cup |c_2|)$ Furthermore assume that for all $t \in [0,1]$:

$$|c_1(t) - c_2(t)| < |c_1(t) - z_0|. \tag{30}$$

Then $\operatorname{Ind}_{c_1}(z_0) = \operatorname{Ind}_{c_2}(z_0)$.

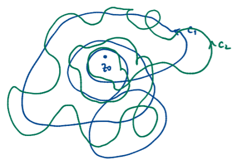

Fig. 84: The setup of lemma 7.6.2.

Proof. Homework 11.3. □

Theorem 7.6.4: Rouché

Let γ be a closed curve bounding a compact region $K \subset U$ and f and g be holomorphic functions on U such that $|g(z)| < |f(z)|$ for all $z \in |\gamma|$. Then f and $f + g$ have the same number of zeros (counted with multiplicities) in $\mathring{K}$.

Proof. Since the functions have no poles, the numbers of zeros are winding numbers of $c_1 := f \circ \gamma$ and $c_2 := f \circ \gamma + g \circ \gamma$ around 0 by (29). But since $|c_1 - c_2| = |g \circ \gamma| < |f \circ \gamma| = |c_1|$, the winding numbers are equal by lemma 7.6.2. □

The following stronger version of Rouché's theorem is also true but more technical to prove (cf. Ferus' lecture notes).

Theorem 7.6.5: Rouché (more general version)

Let c be a cycle bounding a compact region $K \subset U$ and f and g be holomorphic functions on U such that $|g(z)| < |f(z)|$ for all $z \in |c|$. Then f and $f + g$ have the same number of zeros (counted with multiplicities) in $\mathring{K}$.

This is more difficult to prove as the cycle might be a linear combination of not just closed curves, so we can't apply lemma 7.6.2 in a straightforward way.

If we have a region bounded by a curve and suppose on that curve, the function f becomes nonzero - it attains a minimum - and we add to f a holomorphic function g which is smaller in absolute value than f, then it doesn't change the number of zeros in that region.

8 Sequences of holomorphic function

8.1 Uniform convergence on compact sets

If we look at sequences of functions and their convergence in Real Analysis, we know that uniform convergence is a good property, because of its favourable relation to integration. In Complex Analysis, we can also describe derivatives with integrals using Cauchy's formula. Hence in Complex Analysis, also derivatives behave nicely with respect to uniform convergence. In fact, taking derivatives is a local property.

Let $U \subset \mathbb{C}$ be open.

Definition 8.1.1 (Uniform convergence on compact sets)
A sequence $(f_n : U \to \mathbb{C})_{n\in\mathbb{N}}$ of functions converges uniformly on compact sets to a function $f : U \to \mathbb{C}$ if one of the following conditions is satisfied.

converges uniformly on compact sets

- For any compact subset $K \subset U$, we have $f_n \to f$ uniformly on K.
- $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f, that is, for any $z_0 \in U$, there exists an open neighbourhood on which $f_n \to f$ converges uniformly.

Lemma 8.1.2
Both conditions are equivalent.

Proof. " $\implies$ ": Let $z_0 \in U$. If $r > 0$ is small enough, then th compact neighbourhood $\overline{B}_r(z_0)$ is contained in U. On this closed disk, convergence is uniform, therefore, also on $B_r(z_0)$.

" $\impliedby$ ": For $z \in U$, let $U_z \subset U$ be an open neighbourhood of z on which $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Let $K \subset U$ be a compact subset. The open cover $(U_z)_{z\in K}$ has a finite subcover $(U_k)_{k=1}^M$ with $M \in \mathbb{N}$. Given $\varepsilon > 0$, there are numbers $N_1, \dots, N_M$ such that

$$|f_n(z) - f(z)| < \varepsilon$$

if $z \in U_j$ for $n \geqslant N_j$. Let $N := \max(N_1, \dots, N_M)$, then

$$|f_n(z) - f(z)| < \varepsilon$$

if $n \geqslant N$. Hence convergence is uniform □

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Hence uniform convergence on compact sets is the same as locally uniform convergence.

Remark 8.1.3 (Real Analysis reminder) If $(f_n : \mathbb{C} \supset U \to \mathbb{C})_{n\in\mathbb{N}}$ converges to f uniformly on compact sets, then f is also continuous.

If c is a 1-chain in U, then

$$\int_c f_n(z)\,\mathrm{d}z \xrightarrow{n\to\infty} \int_c f(z)\,\mathrm{d}z$$

because the sequence converges uniformly on the compact set $|c|$. ○

Due to Cauchy's integral formula, we can express derivatives as integrals. This is a main ingredient in the proof of the following theorem.

Theorem 8.1.1: Uniform convergence on compact sets (Weierstrass)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on U that converges uniformly on compact sets to the function f. Then f is also holomorphic on U and the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly on compact sets to f'.

Proof. To show that f is holomorphic, it suffices to show that

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$$

for every closed triangular region $\Delta \subset U$ by Morera's Theorem.

Due the remark 8.1.3 $(\star)$, we have

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = \int_{\partial\Delta} \lim_{n\to\infty} f_n(z)\,\mathrm{d}z \overset{(\star)}{=} \lim_{n\to\infty} \underbrace{\int_{\partial\Delta} f_n(z)\,\mathrm{d}z}_{\substack{=0 \text{ by} \\ \text{Cauchy's Theorem}}} = 0.$$

To show that $(f_n')_{n\in\mathbb{N}}$ converges uniformly on compact sets to f', use Cauchy's integral formula for the derivative:

$$\begin{aligned}|f_n'(z) - f'(z)| &= \left|\frac{1}{2\pi i}\int\limits_{|u-z_0|=r} \frac{f_n(u)-f(u)}{(u-z)^2}\,\mathrm{d}u\right| \leqslant \frac{\cancel{2\pi} r}{\cancel{2\pi}} \frac{\max\{|f_n(u)-f(u)| : |u-z_0| = r\}}{\min\{|u-z|^2 : |u-z_0| = r\}} \\ &= \frac{r}{|r-|z-z_0||^2} \max_{\substack{u\in\mathbb{C}: \\ |u-z_0|=r}} |f_n(u)-f(u)|,\end{aligned}$$

where $z_0 \in U$ and $r > 0$ are chosen such that $|z-z_0| < r$ and $\{u \in \mathbb{C} : |u-z_0| = r\} \subset U$. □

This is not true in Real Analysis: $f_n(x) := \frac{1}{n}\sin(nx) \to 0$ uniformly on $\mathbb{R}$, but $f_n(x) = \cos(nx)$ does not converge.

8.2 Multiplicities of values of the limit function

The following Theorem states that limit function can not take a value more often than all the elements of the sequence.

Theorem 8.2.1: Multiplicities of values in the limit (Hurwitz)

Suppose $a \in \mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ is a sequence of holomorphic functions on U that converges uniformly on compact sets to the function f. Suppose further that each function f_n takes the value a at most m times (counting multiplicities). Then f takes the value a at most m times (counting multiplicities) or f is constant.

Corollary 8.2.1

The limit function of a sequence of injective holomorphic functions than converges uniformly on compact sets is also injective or constant.

Proof. (of Theorem 8.2.1) It suffices to treat the case $a = 0$, otherwise apply this case to $\tilde{f} := f - a$. We prove the counterpositive statement: if $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on compact sets and f has more than m zeros (counting multiplicities), then there is an $n \in \mathbb{N}$ for which f_n has more than m zeros.

Let $z_1, \ldots, z_N$ be the distinct zeros of f. Let $r > 0$ be small enough so that the closed disk of radius r around z_j is contained in U, but does not contain any other zeros. Let γ_j be the path tracing out the boundary of those disks for $j \in \{1, \ldots, N\}$.

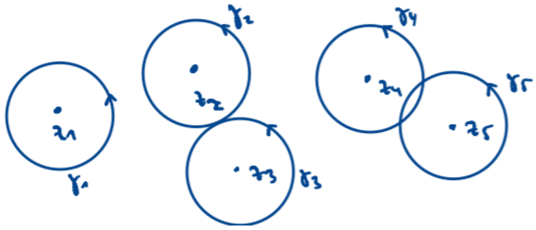

Fig. 85: Closed disks around the distinct zeros.

Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly on the compact set $\Gamma := \bigcup_{j=1}^{N} |\gamma_j|$, for $\varepsilon := \min_{z\in\Gamma} |f(z)| > 0$ there is an $n \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \varepsilon = \min_{u\in\Gamma} |f(u)| < |f(z)|$$

for all $z \in \Gamma$. By ROUCHÉ's Theorem $f_n = f + (f_n - f)$ has the same number of zeros as f in the N open disks, so it also has more than m zeros. □

8.3 Locally bounded function sequences

DEFINITION 8.3.1 (LOCALLY BOUNDED FUNCTION SEQUENCE)
A sequence $(f_n : U \to \mathbb{C})_{n\in\mathbb{N}}$ is locally bounded if every $z_0 \in U$ has an open neighbourhood U_0 so that there is a number $m \in \mathbb{R}$ for which

locally bounded

$$|f_n(z)| \leqslant M \qquad \forall z \in U_0,\ n \in \mathbb{N}\,.$$

The Theorem of BOLZANO-WEIERSTRASS in Real Analysis states that every bounded sequence has a convergent subsequence. MONTEL's Theorem is the analogous Theorem in Complex Analysis.

THEOREM 8.3.1: MONTEL

Every locally bounded sequence of holomorphic functions has a subsequence that converges uniformly on compact sets.

The proof is somewhat involved and hence we prove two lemmas first.

Lemma 8.3.2 (Locally bounded $\implies$ local LIPSCHITZ-equicontinuity)
Let $(f_n : U \to \mathbb{C})_{n\in\mathbb{N}}$ a locally bounded sequence of holomorphic functions. Then for every point in U there is an open neighbourhood $U_0 \subset U$ and a (LIPSCHITZ constant) $M \geqslant 0$ such that for all $n \in \mathbb{N}$ and for all $z_1, z_2 \in U_0$ we have

$$|f_n(z_1) - f_n(z_2)| \leqslant M|z_1 - z_2|.$$

Proof. Let $U_0 = \{z \in \mathbb{C} : |z - z_0| < \frac{r}{2}\}$. By CAUCHY's integral formula (and reverse partial fraction decomposition) we have

$$\begin{aligned}|f_n(z_1) - f_n(z_2)| &= \left| \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f_n(z)}{z - z_1} - \frac{f_n(z)}{z - z_2} \,\mathrm{d}z \right| \\ &= \frac{1}{2\pi} \left| (z_1 - z_2) \int\limits_{|z-z_0|=r} \frac{f_n(z)}{(z - z_1)(z - z_2)} \,\mathrm{d}z \right| \\ &\leqslant \frac{1}{\cancel{2\pi}} |z_1 - z_2| \cancel{2\pi} r \frac{\tilde{M}}{\frac{r^2}{4}} = \underbrace{\frac{4\tilde{M}}{r}}_{=:M} |z_1 - z_2|,\end{aligned}$$

where we use $|z - z_j| \geqslant \frac{r}{2}$ for $z \in U_0$ and $j \in \{1, 2\}$ and that as f_n is locally bounded, there exists the constant $\tilde{M} \geqslant 0$ such that $|f_n|$ are bounded by $\tilde{M}$ on the small neighbourhood U_0 (we can choose $r > 0$ so small that this is true). □

Lemma 8.3.3 (Pointwise on dense subset $\implies$ uniformly on compact sets)
Suppose $(f_n : U \to \mathbb{C})_{n \in \mathbb{N}}$ is a locally bounded sequence of holomorphic functions, which converges pointwise on a dense subset $A \subset U$. Then $(f_n)_{n \in \mathbb{N}}$ converges uniformly on compact sets.

Proof. By lemma 8.1.2 it suffices show local uniform convergence. Let $z_0 \in U$. By lemma 8.3.2, there are numbers $M, r > 0$ so that

$$D_{r,z_0} := \{z \in \mathbb{C} : |z - z_0| < r\} \subset U$$

and for all $n \in \mathbb{N}$ and $z_1, z_2 \in D_{r,z_0}$ we have

$$|f_n(z_1) - f_n(z_2)| \leqslant M |z_1 - z_2|.$$

We show that $(f_n)_{n \in \mathbb{N}}$ converges uniformly on $D_{\frac{1}{3}r,z_0}$, then we are done. To this end we show the CAUCHY-condition: for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $z \in D_{\frac{1}{3}r,z_0}$ and all $n, m \geqslant N$,

$$|f_n(z) - f_m(z)| < \varepsilon.$$

Choose $\varrho > 0$ so that $\varrho < \min(\frac{\varepsilon}{3M}, \frac{r}{3})$ like in figure 86. **13.07.2021**

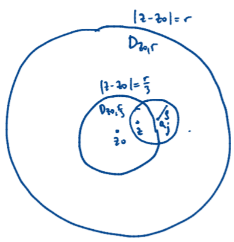

Fig. 86: TODO

Then $(D_{\varrho,a})_{a \in A}$ is an open cover of the compact set $\overline{D_{\frac{r}{3},z_0}}$ and all disks $D_{\varrho,a}$ that have nonempty intersection with $\overline{D_{\frac{r}{3},z_0}}$ are contained in D_{r,z_0}.

Hence there exists a finite subcover. More specifically, there exists numbers $a_1, \ldots, a_k \in A$ such that

$$\overline{D_{\frac{r}{3},z_0}} \subset \bigcup_{j=1}^{k} D_{\varrho,a_j} \subset D_{r,z_0}.$$

Choose $N \in \mathbb{N}$ such that
$$|f_n(a_j) - f_m(a_j)| < \frac{\varepsilon}{3}$$
for all $n, m \geqslant N$ for all $j \in \{1, \ldots, k\}$. Then for each $z \in \overline{D_{\frac{r}{3}, z_0}}$ there is a $j \in \{1, \ldots, k\}$ such that $z \in D_{\varrho, a_j}$, that is, $|z - a_j| < \varrho < \frac{\varepsilon}{3M}$. Hence for $n, m \geqslant N$ we have
$$\begin{aligned}|f_n(z) - f_m(z)| &\leqslant |f_n(z) - f_n(a_j)| + |f_n(a_j) - f_m(a_j)| + |f_m(a_j) - f_m(z)| \\ &\leqslant M|z-a| + \frac{\varepsilon}{3} + M|z-a| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$
□

Proof. (of Theorem 8.3.1 via a classical diagonal argument) Let $(f_n)_{n\in\mathbb{N}}$ be a locally bounded sequence of holomorphic functions on $U \subset \mathbb{C}$. Let $(a_j)_{j\in\mathbb{N}}$ be a sequence in U that is dense in U. For example, arrange the points in U with rational real and imaginary parts in a sequence.

TODO □

9 The RIEMANN mapping theorem

DEFINITION 9.0.1 (CONFORMALLY EQUIVALENT DOMAINS)
Two domains U and $\tilde{U}$ in $\mathbb{C}$ are biholomorphically or conformally equivalent if there is a bijective holomorphic function $f: U \to \tilde{U}$.

conformally equivalent

By corollary 3.4.7, in this case f^{-1} is also holomorphic (hence the term biholomorphically).

Example 9.0.2 The domains $D := \{z \in \mathbb{C} : |z| < 1\}$ and $H := \{z \in \mathbb{C} : \Im(z) > 0\}$ are conformally equivalent, as the MÖBIUS transformation $z \mapsto \frac{z-i}{z+i}$ maps H bijectively onto D by example 1.8.14. ◇

Counterexample 9.0.3 The domains $\mathbb{C}$ and D are *not* conformally equivalent as every holomorphic map on $\mathbb{C}$ with image in D is a bounded entire function and hence constant by LIOUVILLE's theorem. ◇

Counterexample 9.0.4 The domains $D\backslash\{0\}$ and $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$ are not conformally equivalent (Exercise!). ◇

Fig. 87: The two not conformally equivalent and not simply connected domains from counterexample 9.0.4.

The RIEMANN mapping theorem is about simply connected domains (like in example 9.0.2 and counterexample 9.0.3).

THEOREM 9.0.1: RIEMANN MAPPING THEOREM

Every nonempty simply connected domain $U \subsetneq \mathbb{C}$ is conformally equivalent to the open unit disk D.

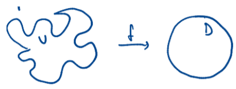

Fig. 88: A biholomorphic map between a simply connected domain $U \subsetneq \mathbb{C}$ and the open disk, which exists by Theorem 9.0.1.

Remark 9.0.5 (Supposedly stronger statement)
Since conformal equivalence is an equivalence relation (the identity is conformal, the inverse of a conformal map is conformal and the composition of conformal maps is conformal), this implies that any nonempty simply connected domain in $\mathbb{C}$ except $\mathbb{C}$ itself is conformally equivalent to any other such domain. ○

Remark 9.0.6 (Horrible simply connected domains) To appreciate how monstrous simply connected domains can be and hence what a strong and remarkable statement Theorem 9.0.1 is, consider the following example:

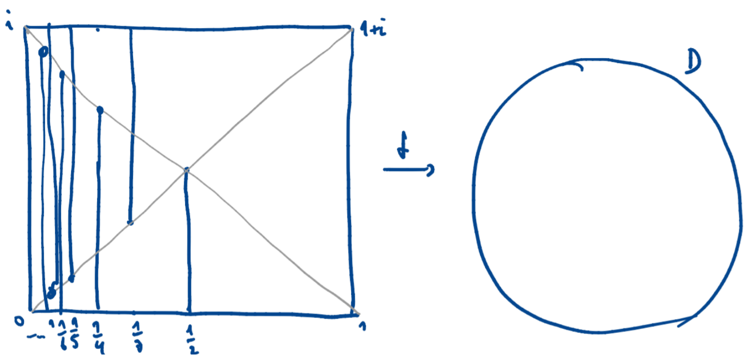

Fig. 89: The set on the right is the interior of the square with endpoints 0, 1, i and $1+i$, which slits at $\{\Re(z) = \frac{1}{k}\}$ with length $1 - \frac{1}{k}$ and alternating starting points for $k \in \mathbb{N}_{>0}$.

There can not be any distortion of angles, since holomorphic maps are angle-preserving by Theorem 1.7.1, but there will be a huge distortion of area by this mapping. ○

Remark 9.0.7 (Uniqueness of Riemann maps) The Riemann mapping theorem asserts the existence of a Riemann map $f\colon U \to D$. How unique is it? If f and $\tilde{f}$ are biholomorphic maps from U onto D, then $\tilde{f} \circ f^{-1}$ is a bijective holomorphic map from D onto D. By Theorem 1.8.7, $\tilde{f} \circ f^{-1}$ is the restriction of a Möbius transformation m. But $\tilde{f} \circ f^{-1} = m$ implies $\tilde{f} = m \circ f$.

Hence two Riemann maps $U \to D$ differ by post-composition with a Möbius transformation mapping D onto D.

How can one make the Riemann mapping unique? One can prescribe a point $z_0 \in U$, that is should be mapped to 0 and we can also describe an angle α, with which the horizontal direction is mapped at z_0 and this is the argument of the derivative of f at 0: for any $z_0 \in U$ and any $\alpha \in [0, 2\pi)$ there is a unique Riemann map $f\colon U \to D$ satisfying

$$f(z_0) = 0 \qquad \text{and} \qquad f'(z_0) = e^{i\alpha}|f'(z_0)|. \qquad \circ$$

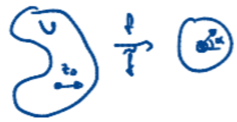

Fig. 90: The specifications needed for the uniqueness of a Riemann map.

Remark 9.0.8 (Proof of Theorem 9.0.1) There are many ways to prove the Riemann mapping theorem. The proof we will show here uses only complex analytic methods and is due to Carathéodory.

He also proved: a Riemann map $U \to D$ extends continuously to a map $\overline{U} \to \overline{D}$ (continuous on $\overline{U}$ and holomorphic on U) if and only if U is a Jordan domain, that is, the boundary of U is a Jordan curve - a simple closed curve.○

In order to prove Theorem 9.0.1, we will first prove the following Lemma.

Lemma 9.0.9 (Global root function on simply connected domains)
If $U \subset \mathbb{C}$ is a simply connected domain and $0 \notin U$, then there exists an injective holomorphic function ϱ on U such that $(\varrho(z))^2 = z$ for all $z \in U$.

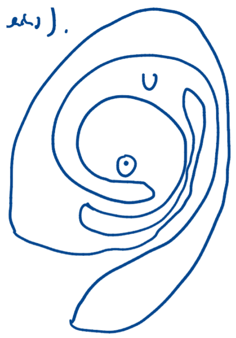

Fig. 91: A simply connected domain not containing the origin.

Remark 9.0.10 This holds for all $n > 0$, not just $n = 2$. In fact, using analytic continuation, one can show a global version of the inverse function theorem: If f is holomorphic on a domain U and f' has no zeros in U and $f(U)$ is simply connected, then there is an inverse function $f(U) \to U$ of f. ○

Proof. We will first construct the logarithm function λ and then construct the square root by considering $e^{\frac{1}{2}\lambda}$. Choose $z_0 \in U$ and $w_0 \in \mathbb{C}$ such that $e^{w_0} = z_0$. In a neighbourhood U_0 of z_0, let λ_0 be an antiderivative (the logarithm!) of the function $z \mapsto \frac{1}{z}$ with $\lambda_0(z_0) = w_0$. Since $z \mapsto \frac{1}{z}$ is holomorphic on U (because $0 \notin U$) and can therefore be trivially extended along any path in U starting at z_0, the same is true for the local antiderivative (λ_0, U_0). Since U is simply connected, the analytic continuation does not depend on the path but only on the endpoint. Hence this defines a holomorphic function λ on U with $\lambda'(z) = \frac{1}{z}$. Now,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z}e^{\lambda(z)}\right) = -\frac{1}{z^2}e^{\lambda(z)} + \frac{1}{z}\lambda'(z)e^{\lambda(z)} = 0$$

implies by Theorem 1.4.2 that there exists a constant $c \in \mathbb{C}$ such that $e^{\lambda(z)} = cz$ and $e^{\lambda(z_0)} = e^{w_0} = z_0$ implies that $c = 1$. So λ is a holomorphic function on U satisfying $e^{\lambda(z)} = z$ for all $z \in U$. Now let $\varrho := e^{\frac{1}{2}\lambda}$, which is holomorphic with $(\varrho(z))^2 = z$. Furthermore, ϱ is injective:$\varrho(z_1) = \varrho(z_2)$ implies $z_1 = (\varrho(z_1))^2 = (\varrho(z_2))^2 = z_2$. ☐

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Proof. (of Theorem 9.0.1) ① **The main argument.** Consider the case that U is bounded. We'll deal with the other case later. Then we may also assume that $U \subset D$ and $0 \in U$ (otherwise translate and scale U appropriately, which are biholomorphic operations). Then main idea of Carathéodory was considering the set of functions

$$\mathcal{F} := \{f\colon U \to \mathbb{C} : f \text{ is holomorphic, injective, } f(U) \subset D, f(0) = 0\}.$$

Claim. There exists a function $f \in \mathcal{F}$ for which $|f'(0)|$ is maximal among functions in $\mathcal{F}$. This is a biholomorphic map onto D.

Proof. 1.1 **A criterion for surjectivity.** Proposition. If for $f \in \mathcal{F}$ the value $|f'(0)|$ is maximal among all functions in $\mathcal{F}$, then $f(U) = D$.

Proof. We will show: if $f(U) \neq D$, then $|f'(0)|$ is not maximal. So assume $z_0 \in D \backslash f(U)$

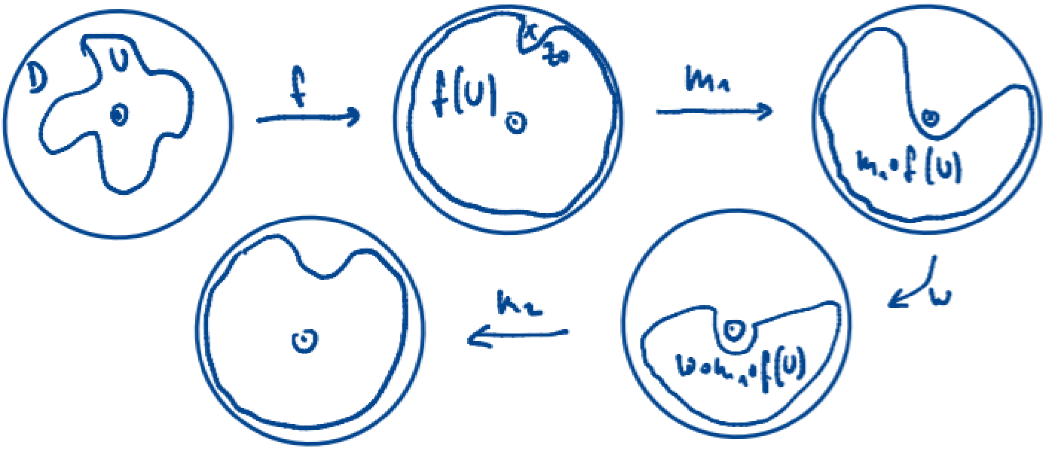

Fig. 92: The different maps in the order they appear in the proof of the proposition.

Let m_1 be a MÖBIUS transformation with $m_1(D) = D$ and $m(z_0) = 0$ (which exists by Theorem 1.8.7), so $0 \notin (m_1 \circ f)(U)$. Now let w be a square root function on $(m_1 \circ f)(U)$ (this set is simply connected because f and m_1 are injective and continuous; this is theorem from topology), i.e. a holomorphic injective function with $w(z)^2 = z$, which exists by lemma 9.0.9. Finally, let m_2 be another MÖBIUS transformation with $m_2(D) = D$, mapping $(w \circ m_1 \circ f)(0)$ to 0. Then $\tilde{f} := \underbrace{m_2 \circ w \circ m_1}_{=:g} \circ f \in \mathcal{F}$ because $\tilde{f}(0) = 0$ by construction and holomorphic and injective as a composition of holomorphic and injective maps. It remains to show that $|\tilde{f}'(0)| > |f'(0)|$. Note that $\tilde{f}'(0) = g'(\underbrace{f(0)}_{=0}) \cdot f'(0)$ by the chain rule. We will show that $|g'(0)| > 1$ and then we are done.

Note that g is an injective holomorphic function on $f(U) \subset D$ and $g(0) = 0$. The inverse,

$$g^{-1}(z) = (m_1^{-1} \circ w^{-1} \circ m_2^{-1})(z) = m_1^{-1}((m_2^{-1}(z))^2)$$

is a restriction of the entire function

$$h\colon \mathbb{C} \to \mathbb{C}, \qquad z \mapsto m_1^{-1}((m_2^{-1}(z))^2).$$

Now $h(D) \subset D$ and $h(0) = 0$, but h is not a MÖBIUS transformation (otherwise $z \mapsto z^2$ would have to be MÖBIUS transformation), in particular, h is not a rotation $z \mapsto az$ with $|a| = 1$. By SCHWARZ's Lemma, $|h'(0)| < 1$, so $|g'(0)| = \frac{1}{|h'(0)|} > 1$. □

1.2 **The existence statement.** The set $\mathcal{F}$ of functions contains the identity $z \mapsto z$, so it is nonempty. Also, the set of values

$$\{|f'(0)| : f \in \mathcal{F}\} \subset \mathbb{R}$$

is bounded. To see this, use CAUCHY's integral formula for the derivative: if $r > 0$ is small enough so $\{z \in \mathbb{C} : |z| < r\} \subset U$, then

$$|f'(0)| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{(z-0)^2} \, \mathrm{d}z \right| \leqslant \frac{1}{2\pi} 2\pi r \frac{1}{r^2} = \frac{1}{r},$$

as $|f(z)| < 1$. Hence

$$s_0 := \sup\{|f'(0)| : f \in \mathcal{F}\} \leqslant \frac{1}{r} < \infty.$$

Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ be a sequence of functions such that $\lim_{n\to\infty} |f_n'(0)| = s_0$. All functions in the sequence are bounded by 1, because their image is a subset of D. By MONTEL's theorem there is a subsequence of $(f_n)_{n\in\mathbb{N}}$ that converges uniformly on compact sets. Its limit f is holomorphic and $|f'(0)| = s_0$ by the theorem about uniform convergence on compact sets. Also $f(0) = 0$, as $f_n(0) = 0$ for all $n \in \mathbb{N}$. Since $f'(0) \neq 0$, f is not constant, so by the Corollary to HURWITZ theorem, f is injective because all f_n are injective. Finally, $f(U) \subset \overline{D}$, since $f_n(U) \subset D$ for all $n \in \mathbb{N}$. By the theorem on preservation of domain $f(U) \subset D$. By the surjectivity criterion 1.1, $f(U) = D$. □

2 Suppose U is not bounded. By assumption $U \neq \mathbb{C}$, so there is a point $z_0 \in \mathbb{C}\setminus U$. We may assume $0 \notin U$ (otherwise apply the translation $z \mapsto z - z_0$). Let w be a square root function on U, i.e. an injective holomorphic function on U with $(w(z))^2 = z$ for all $z \in U$.

Proposition. The set $w(U)$ does not contain a pair of diametrically opposed points p and $-p$.

Proof. $w(z_1) = -w(z_2)$ implies

$$z_1 = (w(z_1))^2 = (-w(z_2))^2 = z_2,$$

therefore $w(z_1) = -w(z_1)$, so $w(z_1) = 0$, so $z_1 = 0 \in U$, which is a contradiction to $0 \notin U$. □

Pick a point $z_0 \in w(U)$ and let $r > 0$ be small enough that $U_0 := \{z \in \mathbb{C} : |z - z_0| < r\} \subset w(U)$. Let $-U_0 := \{-z : z \in U_0\}$. Then $(-U_0) \cap w(U) = \varnothing$ by the above proposition.

The inversion $\varrho(z) = \frac{1}{z+z_0}$ maps $\mathbb{C}\setminus(-U_0) \supset U$ into the bounded set $\{z \in \mathbb{C} : |z+z_0| \leqslant \frac{1}{r}\}$. Therefore $\varrho(w(U))$ is bounded. □

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References

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