Definition

Complex differentiability, Holomorphy, Entire

Complex Analysis I

Theorem w/o proof

Real and complex differentiability

Complex Analysis I

Definition, Proposition & Theorem w/o proofs

Harmonic Functions

Complex Analysis I

Definition & 3 Theorems w/o proofs

Conformal Maps

Complex Analysis I

Definition & Remark

The Riemann sphere

Complex Analysis I

Definition, Remark & Theorem w/o proof

Möbius transformation & group

Complex Analysis I

Theorem

3 points + their images determine Möb uniquely

Complex Analysis I

Definition & Theorem

Cross-ratio

Complex Analysis I

Theorems w/o proofs

Möbius transformations preserving the unit ball / upper half-plane

Complex Analysis I

Definition

Contour integral

Complex Analysis I

A function $f\colon U \to \mathbb{C}$ is complex differentiable in $z_0 \in \mathbb{C}$ if it is *differentiable in the real sense* and one (and hence both) of the following two conditions hold:

- The *derivative* $\mathrm{d}_{z_0} f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{C}$*-linear* as a map on $\mathbb{C}$.
- The CAUCHY-RIEMANN *differential equations* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ hold in $z_0$.

In this case we have $f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$.

Let $U \subset \mathbb{C}$ be an *open* subset and $z_0 \in U$. A function $f\colon U \to \mathbb{C}$ is **(complex) differentiable** on $U$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \in \mathbb{C}\,.$$

exists. In that case, $f'(z_0)$ is the *derivative* of $f$ at $z_0$. If $f$ is differentiable for all $z_0 \in U$, then it is **holomorphic** or (complex) analytic. A holomorphic function on $\mathbb{C}$ is an **entire** function.

Holomorphic functions with nonvanishing derivative are **conformal**, that is, *angle-preserving.*
For an *invertible* $\mathbb{R}$*-linear* map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ TFAE

1. *$F$ preserves angles.*
2. $F$ preserves orthogonal angles: if $z$ and $w$ are *orthogonal*, then $F(z)$ and $F(w)$ are also orthogonal.
3. $F$ is $\mathbb{C}$-linear (that is, $F(iz) = iF(z)$ for all $z \in \mathbb{C}$) or $F$ is $\mathbb{C}$*-antilinear* (that is, $F(iz) = -iF(z)$).

A *real differentiable* map on a domain is *holomorphic* if its derivative in the real sense is *everywhere angle and orientation preserving.*

A function $f$ defined on an *open* subset $U \subset \mathbb{C}$ that satisfies the LAPLACE *equation* $\Delta f = 0$ is a **harmonic** function.

On a *simply connected* domain $U \subset \mathbb{C}$, every *harmonic function is the real part of a holomorphic function.*

Let $f\colon U \to \mathbb{C}$ be *holomorphic* and $h\colon f(U) \to \mathbb{R}$ *harmonic.* Then $h \circ f$ is *harmonic.*

A **Möbius transformation** is a function $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$.
We can (but do not need to) require that $ad - bc = 1$. Then the MÖBIUS transformation determines the coefficients up to a global sign change, i.e. a factor of $\pm 1$.
Our way out of this is to consider the MÖBIUS transformations as functions from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ instead of from $\mathbb{C}$ to $\mathbb{C}$ by defining: (if $c \neq 0$) $f\left(-\frac{d}{c}\right) := \infty$ and $f(\infty) = \frac{a}{c}$ and $f(\infty) = \infty$ if $c = 0$.
The MÖBIUS transformations form a *group* of *bijective* functions from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ under *composition.*

The RIEMANN *sphere* (or: extended complex plane)

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

is the complex plane $\mathbb{C}$ with the extra point $\infty$ added.
The point $\infty$ corresponds to the north pole of $\mathbb{S}^2$ under stereographic projection. The stereographic projection is a bijective map from $\mathbb{S}^2$ to $\hat{\mathbb{C}}$. Since $\mathbb{S}^2$ has a topology induced by the ambient $\mathbb{R}^3$, the *stereographic projection induces a topology on* $\hat{\mathbb{C}}$.

The **cross-ratio** of four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is $\operatorname{cr}(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_2 - z_3} \frac{z_3 - z_4}{z_4 - z_1}$. If one of the points is $\infty$, this is supposed to be evaluated by cancelling infinities.
The cross-ratio of four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is real if and only if the four points lie on a MÖBIUS circle.
For $f \in$ Möb and $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ we have $\operatorname{cr}(z_1, z_2, z_3, z_4) = \operatorname{cr}(f(z_1), f(z_2), f(z_3), f(z_4))$. Conversely, MÖBIUS are the *only* transformation that preserve the cross ratio: if $\operatorname{cr}(z_1, z_2, z_3, z_4) = \operatorname{cr}(w_1, w_2, w_3, w_4)$, there there exists a $f \in$ Möb with $f(z_j) = w_j$ for $j \in \{1, \ldots, 4\}$.

If $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ are three points and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ are three points, then there is a *unique* MÖBIUS transformation $f$ satisfying $f(z_i) = w_i$ for $i \in \{1, 2, 3\}$.
**Existence.** Let $g$ and $h$ be the MÖBIUS transformations sending $z_1, z_2, z_3$ and $w_1, w_2, w_3$ to $0, 1$ and $\infty$ respectively. Then $f := h^{-1} \circ g$ satisfies $f(z_i) = w_i$ for $i \in \{1, 2, 3\}$.
**Uniqueness.** 1. Suppose $f \in$ Möb and $f(z_i) = z_i$ for $i \in \{1, 2, 3\}$. Then $f = \mathrm{id}$. Indeed let $g \in$ Möb be the map with $g(z_1) = 0$, $g(z_2) = 1$ and $g(z_3) = \infty$. Then $h := g \circ f \circ g^{-1} \in$ Möb satisfies $h(0) = 0$, $h(1) = 1$, $h(\infty) = \infty$. By previous Lemma, $h = \mathrm{id}$ and thus $f = g^{-1} \circ h \circ g = \mathrm{id}$.
2. Suppose $f_1$ and $f_2$ are MÖBIUS transformations with $f_j(z_i) = w_i$, $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$. Then $f_2^{-1} \circ f_1 \in$ Möb fixed $z_1, z_2, z_3$, so by the previous step, $f_2^{-1} \circ f_1$, hence $f_2 = f_1$.

Let $U \subset \mathbb{C}$ be *any* subset, $f\colon U \to \mathbb{C}$ be continuous. If $\gamma\colon [t_0, t_1] \to U$ is only *piecewise continuously differentiable*, i.e. if there is a *subdivision* $t_0 = \tau_0 < \tau_1 < \ldots < \tau_n = t_1$ such that $\gamma \in \mathcal{C}([t_0, t_1])$ is *continuously differentiable on* $[\tau_j, \tau_{j+1}]$ for $j \in \{0, \ldots, n-1\}$, then $\int_\gamma f(z)\,\mathrm{d}z := \sum_{j=0}^{n-1} \int_{\gamma|_{[\tau_j, \tau_{j+1}]}} f(z)\,\mathrm{d}z$. If $\gamma\colon [t_0, t_1] \to U$ be a *continuously differentiable* curve, then the *(contour) integral of $f$ along $\gamma$* is $\int_\gamma f(z)\,\mathrm{d}z := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,\mathrm{d}t$.

The MÖBIUS transformations that map the unit disk

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

onto itself are precisely the MÖBIUS transformations of the form

$$f(z) = e^{i\varphi} \frac{z - z_0}{1 - \overline{z_0} z},$$

where $z_0 \in D$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.
The MÖBIUS transformations $f(z) = \frac{az+b}{cz+d}$ with $f(H) = H$ are characterised by $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

Theorem w/ proof

Cauchy's Integral Theorem of a Rectangle

Complex Analysis I

Theorem w/ proof

Cauchy's integral theorem for $\mathcal{C}^1$ images of rectangles

Complex Analysis I

Corollaries w/ proofs

Cauchy's theorem for triangles and disks

Complex Analysis I

Proof

Cauchy's Integral Theorem of a Rectangle

Complex Analysis I

Definition, Theorem w/ proof

Cauchy's integral theorem for $\mathcal{C}^1$-homotopic curves

Complex Analysis I

Definition, Theorem w/ proof

Cauchy's Theorem for freely $\mathcal{C}^1$-homotopic curves

Complex Analysis I

Corollary w/ proof

Cauchy's integral theorem for annuli

Complex Analysis I

Theorem w/o proof

Cauchy's integral formula for disks

Complex Analysis I

Proof

Cauchy's integral formula for disks

Complex Analysis I

Corollary w/ proof

Mean value property of holomorphic functions

Complex Analysis I

Let $f$ be a *holomorphic* function on $U \subset \mathbb{C}$, let $Q \subset \mathbb{C}$ be a *closed rectangular region*, let $\gamma$ be a $\mathcal{C}^1$ *parametrisation* of its boundary and let $\Phi\colon W \to \mathbb{C}$ be a *continuously differentiable* map on some domain $W$ containing $Q$ with $\Phi(Q) \subset U$. Then $\int_{\Phi\circ\gamma} f(z)\,\mathrm{d}z = 0$.

We construct a sequence of rectangles $Q \supset Q_1 \supset Q_2 \supset \ldots$ as before with $\left|\int_{\Phi\circ\gamma} f(z)\,\mathrm{d}z\right| \leqslant 4^k \left|\int_{\Phi\circ\gamma_k} f(z)\,\mathrm{d}z\right|$ with $\gamma_k := \partial Q_k$ and $\gamma := \partial Q$. But now we need to estimate $\operatorname{diam}(\Phi(Q_k))$ and $\operatorname{len}(\Phi\circ\gamma_k)$. To this end, we observe that since $\Phi$ *is a* $\mathcal{C}^1$ *function*, $\mathrm{d}\Phi$ is *continuous on the compact set* $Q$, so there exists a $C > 0$ s.t. $\|\mathrm{d}\Phi_z\| \leqslant C$ for all $z \in Q$. Hence $\operatorname{diam}(\Phi(Q_k)) \leqslant C\operatorname{diam}(Q_k) = C2^{-k}\operatorname{diam}(Q)$ and $\operatorname{length}(\Phi\circ\gamma_k) \leqslant C\operatorname{len}(\gamma_k) = C2^{-k}$.
Let $\varepsilon > 0$ and let $z_0$ be the image of the point contained in $\bigcap_{k\in\mathbb{N}} Q_k \in U$ under $\Phi$. Choose $\delta > 0$ so small that $|R_{z_0}(z)| < \varepsilon|z - z_0|$ holds for all $z$ with $|z - z_0| < \delta$ but now choose $k \in \mathbb{N}$ large enough that $C2^{-k}\operatorname{diam}(Q) \leqslant \delta$ holds, we have $\left|\int_{\Phi\circ\gamma} f(z)\,\mathrm{d}z\right| \leqslant 4^k \left|\int_{\Phi\circ\gamma_k} f(z)\,\mathrm{d}z\right| \leqslant \cancel{4^k 2^{-k} \cdot 2^{-k}} C^2 \operatorname{len}(\gamma)\operatorname{diam}(Q)\cdot\varepsilon$.

① We show: for $\varepsilon > 0$, $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant \varepsilon$. Since $f$ is *holomorphic* on $U$, for any $z \in U$ we have $f(z) = f(z_0) + f'(z_0)\cdot(z - z_0) + R_{z_0}(z)$, w/ $\lim_{z\to z_0} \frac{R_{z_0}(z)}{|z-z_0|} = 0$ $(\star)$. Since $z \mapsto f(z_0) + f'(z_0)\cdot(z - z_0)$ is entire and thus has a global *antiderivative*, its integral along the closed curve $\gamma$ is zero by the FTOC. Therefore $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma R_{z_0}(z)\,\mathrm{d}z$.
② Let $\varepsilon > 0$. Divide $Q$ into four equal subrectangles $Q_1, \ldots, Q_4$ and let $Q_1$ be that subrectangle for which the integral along the boundary, $\gamma_1$, is largest in absolute value. $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant 4\left|\int_{\gamma(1)} f(z)\,\mathrm{d}z\right|$ Now subdivide the rectangle $Q_1$ into four equal subrectangles and let $Q_2$ be the rectangle for which the integral along the boundary $\gamma_2$ is the largest. Continuing this process we obtain a infinite sequence of rectangles $Q_k$ and boundary curves $\gamma_k$ s.t. $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant 4^k\left|\int_{\gamma_k} f(z)\,\mathrm{d}z\right| = 4^k\left|\int_{\gamma_k} R_{z_0}(z)\,\mathrm{d}z\right|$. $\bigcap_{k=1}^\infty Q_k = \{z_0\}$.
③ $\left|\int_{\gamma_k} R_{z_0}(z)\,\mathrm{d}z\right| \leqslant \operatorname{len}(\gamma_k)\cdot\sup_{z\in Q_k}|R_{z_0}(z)|$. $\operatorname{len}(\gamma_k) = 2^{-k}\operatorname{len}(\gamma)$. By $(\star)$, $\exists\delta > 0$ s.t. $|R_{z_0}(z)| < \tilde\varepsilon|z - z_0|$ for all $z$ with $|z - z_0| < \delta$, where $\tilde\varepsilon := \frac{\varepsilon}{\operatorname{len}(\gamma)\operatorname{diam}(Q)}$.
Choose $k \in \mathbb{N}$ so large that $\operatorname{diam}(Q_k) = 2^{-k}\operatorname{diam}(Q) < \delta$, then $\sup_{z\in Q_k}|R_{z_0}(z)| \leqslant \varepsilon\sup_{z\in Q_k}|z - z_0| \leqslant \varepsilon\operatorname{diam}(Q_k) = \varepsilon\cdot 2^{-k}\operatorname{diam}(Q)$.
$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant \cancel{4^k 2^{-k}}\cdot\operatorname{len}(\gamma)\cdot\tilde\varepsilon\cdot\cancel{2^{-k}}\cdot\operatorname{diam}(Q) = \operatorname{len}(\gamma)\cdot\tilde\varepsilon\cdot\operatorname{diam}(Q) = \varepsilon$.

Two *closed* curves $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are *freely* $\mathcal{C}^1$*-homotopic in* $U \subset \mathbb{C}$ ($U$ only needs to be a subset) if there is a $\mathcal{C}^1$-function $H\colon [0,1]^2 \to U$ such that $H(0,\cdot) = \alpha$, $H(1,\cdot) = \beta$ and $H(\cdot,0) = H(\cdot,1)$.
If $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are *freely* $\mathcal{C}^1$*-homotopic* curves in $U$ and $f$ is *holomorphic* on $U$, then $\int_\alpha f(z)\,\mathrm{d}z = \int_\beta f(z)\,\mathrm{d}z$.

We apply Cauchy's Theorem for $\mathcal{C}^1$-images of rectangles. The image of the boundary of $[0,1]^2$ under $H$ is the curve $\alpha$ traced in the opposite direction, a segment connecting it to $\beta$, the curve $\beta$ and the segment traced in the other direction.

Let $f$ be holomorphic in the domain $U \subset \mathbb{C}$, which contains the closed disk

$$\{z \in \mathbb{C} : |z - z_0| \leqslant r\}$$

for $z_0 \in \mathbb{C}$. Then for every point in the interior of this disk, i.e. every $a \in \mathbb{C}$ with $|a - z_0| < r$,

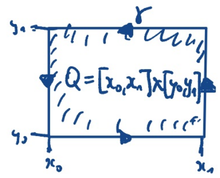
$$f(a) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - a}\,\mathrm{d}z.$$

If $f$ is holomorphic on a domain containing the closed disk with centre $z_0$ and radius $r$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

With the parametrisation $z = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and using Cauchy's Formula for $a = z_0$ we obtain $f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \frac{1}{2\pi\cancel{i}}\int_0^{2\pi} \frac{f(z_0+re^{it})}{\cancel{z_0}+\cancel{re^{it}}-\cancel{z_0}}\cdot\cancel{i}\cdot\cancel{re^{it}}\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})$.

Let $Q \subset \mathbb{C}$ be a closed *rectangular region* with *sides parallel* to the real and imaginary axes and let $\gamma$ be a *piecewise* $\mathcal{C}^1$ parametrisation of the boundary of $Q$ with orientation like here:




If $f$ is holomorphic on $U \supset Q$, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

If $f$ is *holomorphic* on $U$ and $\gamma$ is the boundary curve of a *triangular region* that is contained in $U$, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.
Apply Cauchy's theorem for $\mathcal{C}^1$ images of rectangles to

$$\Phi\colon [0,1]^2 \to U, \quad (s,t) \mapsto (1-t)\left((1-s)A + sB\right) + t\left((1-s)A + sC\right).$$

If $f$ is *holomorphic* on $U$ and $\gamma$ is the boundary circle of a closed disk that is contained in $U$, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.
Let $z_0 \in U$ be the centre and $r > 0$ the radius of the closed disk. Apply Cauchy's theorem for $\mathcal{C}^1$ images of rectangles to

$$\Phi\colon [0,2\pi]\times[0,r] \to U, \quad (s,t) \mapsto z_0 + te^{is}$$

A single point does not contribute to the integral and the two paths cancel each other out.

Two curves $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are $\mathcal{C}^1$*-homotopic in* $U \subset \mathbb{C}$ if $\exists$ $\mathcal{C}^1$-function $H\colon [0,1] \to U$, called *homotopy*, such that

- $H(0,t) = \alpha(t)$, $H(1,t) = \beta(t)$ for all $t \in [0,1]$,
- $H(s,0) = \alpha(0) = \beta(0)$, $H(s,1) = \alpha(1) = \beta(1)$ $\forall s \in [0,1]$,

If $\alpha, \beta\colon [0,1] \to \mathbb{C}$ are $\mathcal{C}^1$*-homotopic* curves in $U$ and $f$ is *holomorphic* on $U$, then $\int_\alpha f(z)\,\mathrm{d}z = \int_\beta f(z)\,\mathrm{d}z$.

Choosing $\Phi = H$, Cauchy's theorem for $\mathcal{C}^1$ images of rectangles implies $\int_\alpha f(z)\,\mathrm{d}z - \int_\beta f(z)\,\mathrm{d}z = 0$.

If two nested (that is, one is contained in the other and they don't intersect) circles with centres $z_0$ and $z_1$ and radii $r_0$ and $r_1$ are contained in $U$ together with the region between them, then for all holomorphic functions $f$ on $U$ we have

$$\int_{|z-z_0|=r_0} f(z)\,\mathrm{d}z = \int_{|z-z_1|=r_1} f(z)\,\mathrm{d}z.$$

A special case occurs if $z_0 = z_1$, and then the concentric circles in $U$ bound an annulus in $U$.
This is a special case of Cauchy's Theorem for freely $\mathcal{C}^1$-homotopic curves.

Choose $\varepsilon > 0$ so small that $B_\varepsilon(a) \subset B_r(z_0)$. By Cauchy's Theorem for Annuli, $\int_{|z-z_0|=r} \frac{f(z)}{z-a}\,\mathrm{d}z = \int_{|z-a|=\varepsilon} \frac{f(z)}{z-a}\,\mathrm{d}z$, $(\star)$ because the integrand is nevertheless holomorphic on the annulus (not containing $a$) bounded by the circles $|z - z_0| = r$ and $|z - a| = \varepsilon$ as it is the quotient of two holomorphic functions. We have $\int_{|z-a|=\varepsilon} \frac{f(z)}{z-a}\,\mathrm{d}z = \int_{|z-a|=\varepsilon} \frac{f(a)+f(z)-f(a)}{z-a}\,\mathrm{d}z$
$= \underbrace{\int_{|z-a|=\varepsilon} \frac{f(a)}{z-a}\,\mathrm{d}z}_{=:A} + \underbrace{\int_{|z-a|=\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z}_{=:B}$. $A = f(a)\int_{|z-a|=\varepsilon} \frac{1}{z-a}\,\mathrm{d}z =$
$f(a)\int_0^{2\pi} \frac{1}{\cancel{a}+\cancel{\varepsilon e^{it}}-\cancel{a}} i\cancel{\varepsilon e^{it}}\,\mathrm{d}t = \int_0^{2\pi} i = 2\pi i$. using the parametrisation $\gamma(t) = a + \varepsilon e^{it}$.
It remains to show that $B = 0$. Note that $B$ does not depend on $\varepsilon$ as long as $\varepsilon > 0$ is small enough: one can immediately see this from Cauchy's theorem for annuli with concentric circles because if we change $\varepsilon$ then we get the same result. Hence it is enough to show that $\lim_{\varepsilon\searrow 0}\int_{|z-a|=\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = 0$. We have $\int_{|z-a|=\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z =$
$\int_0^{2\pi} \frac{f(a+\varepsilon e^{it})-f(a)}{\cancel{a}+\cancel{\varepsilon e^{it}}-\cancel{a}} i\cancel{\varepsilon e^{it}}\,\mathrm{d}t = i\int_0^{2\pi} \underbrace{f(a+\varepsilon e^{it}) - f(a)}_{=:h_\varepsilon(t)}\,\mathrm{d}t$. Since $f$ is continuous at $a$, $\lim_{\varepsilon\searrow 0} h_\varepsilon(t) = 0$ *uniformly* in $t \in [0, 2\pi]$, because continuous functions on compact sets are uniformly continuous. Hence $\lim_{\varepsilon\searrow 0}\int_0^{2\pi} h_\varepsilon(t)\,\mathrm{d}t = 0$.

Theorem w/o proof

Complex Version of the Fundamental Theorem of Calculus

Complex Analysis I

Theorem & Corollary

Holomorphic functions can be represented by power series

Complex Analysis I

Theorem w/ proof

Liouville

Complex Analysis I

Theorem w/ proof

Cauchy's Integral Formula for Derivatives

Complex Analysis I

Definition

Order of a zero of a holomorphic function

Complex Analysis I

Theorem w/o proof

Isolated singularities

Complex Analysis I

Theorem w/ proof

Identity Theorem for Holomorphic Functions

Complex Analysis I

Theorem w/o proof

Local behaviour of a holomorphic function near a zero

Complex Analysis I

Theorem w proof

Preservation of Domain

Complex Analysis I

Theorem w/ proof

Maximum Principle (Version I)

Complex Analysis I

Let $f$ be a *holomorphic* function on $U$. For $z_0 \in U$ there exists a *unique* power series $f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ with positive convergence radius representing $f$ in some neighbourhood of $z_0$. The coefficients $c_k$ are determined by CAUCHY*'s coefficient formula* $c_k = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}} \, dz$, where the only condition on $r$ is to be small enough such that $\{z : |z - z_0| \leqslant r\} \subset U$.
The radius of convergence is not smaller than the radius of the largest open disk around $z_0$ contained in $U$.
Since power series are differentiable and their derivatives are again power series, we get (GOURSAT): every holomorphic function is arbitrarily often complex differentiable, in particular it is $\mathcal{C}^\infty$ in the real sense.

Let $f$ be *holomorphic* on a *domain* $U$ which is *star-shaped* with respect to $z_0 \in U$. Define

$$F : U \to \mathbb{C}, \qquad z \mapsto \int_{z_0}^{z} f(u) \, du,$$

where we write $\int_a^b$ for the integral along the straight line segment from $a$ to $b$ parametrised by $\gamma(t) = a + t(b - a)$ for $t \in [0, 1]$. Then $F$ is an antiderivative of $f$, that is, $F$ is holomorphic and $F' = f$.

Under the same conditions as in CAUCHY's Integral Formula for $f(a)$, we have

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-a)^{k+1}} \, dz.$$

By the Power Series Expansion Theorem, $f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ in some open disk around $z_0$ and we have two equations for the coefficients:

$$c_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}} \, dz.$$

A *bounded entire* function (that is $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$) is *constant.*

The function $f$ is represented by a power series and we can choose 0 as its centre: for all $z \in \mathbb{C}$ we have

$$f(z) = \sum_{k=0}^{\infty} c_k z^k.$$

By CAUCHY's estimate for the coefficients we have

$$|c_k| \leqslant \frac{M}{r^k},$$

for all $r > 0$, so $c_k = 0$ unless $k = 0$.

Let $U$ be a domain and let $z_0 \in U$ be a zero of order $k \in \mathbb{N} \cup \{\infty\}$. Then either ($k = \infty$ and $f = 0$) or there is a *holomorphic* function $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^k g(z).$$

In particular, zeros of finite order are *isolated* ($x \in S$ is isolated in $S \subset \mathbb{C}$ if there exists a neighbourhood of $x$ in $\mathbb{C}$ that doesn't contain any other points of $S$.).

The **order** or *multiplicity* of a *zero* $z_0 \in U$ of $f$ is $\operatorname{ord}(f, z_0) := \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$ or $\operatorname{ord}(f, z_0) = \infty$ if $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$.

Let $f$ be a holomorphic function on $U$, let $f(z_0) = 0$ and $n := \operatorname{ord}(f, z_0) < \infty$. Then there is an open neighbourhood $U_0$ of $z_0$ and an *biholomorphic* function $h$ on $U_0$ such that $h(z_0) = 0$ and $f|_{U_0} = h^n$.
In particular, the function $f$ takes any non-zero value $w \in f(U_0)$ exactly $n$ times in $U_0$.

Let $U$ be a *domain* and $f_1$ and $f_2$ be *holomorphic* on $U$. If the set $M := \{z \in U : f_1(z) = f_2(z)\}$ has an *accumulation point* in $U$, then $f_1 = f_2$.

The set $M$ is the set of zeros of the holomorphic function $f_1 - f_2$. If it has an accumulation point in $U$, that is if there is a sequence $(z_j)_{j \in \mathbb{N}} \subset M$ with limit in $U$, then that is a zero of infinite order as the set of finite order zeros is isolated. Hence $f_1 - f_2 = 0$ by the Theorem of Isolated Singularitities.

If $f$ is *holomorphic* and *nonconstant* on a *domain* $U$, then $|f|$ *does not attain a supremum* on $U$.

Let $z_0 \in U$ and $w_0 := f(z_0)$. As $f(U)$ is open by the Open Mapping Theorem, it contains an open disk of radius $\varepsilon > 0$ around $w_0$ which is not contained in the closed disk $\{w \in \mathbb{C} : |w| \leqslant |w_0|\}$. Hence the $\varepsilon$-disk contains the point $w_1 = f(z_1)$ with $|f(z_1)| = |w_1| > |w_0|$.

If $f$ is *holomorphic* and *not constant* on a *domain* $U$, then $f(U)$ is also a *domain.*

The image $f(U)$ *is connected* because it is the image of the connected set $U$ under the *continuous* function $f$.
Suppose $w_0 = f(z_0) \in f(U)$. We have to show that $f(U)$ contains an open neighbourhood of $w_0$. Since $f$ is *not constant*, the function $g(z) := f(z) - f(z_0)$ has a zero of finite order at $z_0$. Hence there is an open neighbourhood $W$ of $z_0$ such that $g$ takes any nonzero value in $W$ at least once. So $f$ takes any value in the open neighbourhood $f(z_0) + W$ at least once.

Theorem w/o proof

Schwarz's Lemma



Proof

Schwarz's Lemma



Defintions

Isolated / removable singularity



Theorem w/ partial proof

Riemann'scher Hebbarkeitssatz



Theorem w/o proof

3 types of singularities



Proof

3 types of singularities



Theorem w/o & Remark

Casorati-Weierstrass



Proof

Casorati-Weierstrass



Definition & Remark

Order of any point



Definition

Meromorphic / holomorphic except for ...

Let $f\colon D \to D$ be *holomorphic* with $f(0) = 0$. Then

1. $|f'(0)| \leqslant 1$,
2. $|f(z)| \leqslant |z|$.

If we have $|f'(0)| = 1$ or there is a point $z_0 \in D$ where $|f(z_0)| = |z_0|$, then $f$ is a *rotation*, that is $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

If $z_0$ is an isolated singularity of a holomorphic function $f\colon U \to \mathbb{C}$, then the following statements are equivalent.

1. The singularity $z_0$ is removable.
2. $f$ is *bounded in a neighbourhood* of $z_0$: there is a $\varepsilon > 0$ and a $M \geqslant 0$ such that $|f(z)| \leqslant M$ for all $z \in U$ with $|z - z_0| < \varepsilon$.
3. We have $\lim\limits_{z\to z_0} (z - z_0)f(z) = 0$.

"1 $\implies$ 2": If $z_0$ is removable, then by Definition there exists a holomorphic continuation $\tilde{f}$, which is bounded in a neighbourhood of $z_0$ because it is continuous. As $f = \tilde{f}|_U$, the statement follows.
"2 $\implies$ 3": is clear by the normal rules of doing limits.
"3 $\implies$ 1": more tricky.

Let $f$ be holomorphic on $U$. A point $z_0 \in \mathbb{C}\backslash U$ is a **isolated singularity** *of* $f$ if there is an *open neighbourhood* $U_0$ of $z_0$ such that $U_0 \cap U = U_0\backslash\{z_0\}$, that is, there is an $\varepsilon > 0$ such that

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\} \subset U.$$

An isolated singularity is "point-shaped holeïn the domain of definition.
An *isolated singularity* $z_0$ of $f\colon U \to \mathbb{C}$ is **removable** if there is a *holomorphic* function $\tilde{f}$ on $U \cup \{z_0\}$ (still open!) such that $f = \tilde{f}|_U$.

We only have to prove that only at most one of the possibilities can hold, since by construction of 3, every isolated singularity must fall in one of the three categories.
The statement 1 holds by the RIEMANN'scher Hebbarkeitssatz.
2: Suppose $\lim_{z\to z_0} |f(z)| = \infty$. Then $\frac{1}{f}$ is bounded in a neighbourhood of $z_0$, as $\lim_{z\to z_0} \frac{1}{|f(z)|} = 0$. Hence $z_0$ is a removable singularity of $\frac{1}{f}$. After removing the singularity, one obtains a holomorphic function $g := \frac{1}{f}$ and $g(z_0) = 0$. If $m$ is the order of the zero, $g(z) = (z - z_0)^m h(z)$, where $h$ is a holomorphic function with $h(z_0) \neq 0$. Hence $(z - z_0)^m f(z) = (z - z_0)^m \frac{1}{(z-z_0)^m h(z)} = \frac{1}{h(z)}$ has a removable singularity at $z_0$. (We also see that the order of the pole is the order of the zero of $\frac{1}{f}$ after the singularity has been removed.)

Let $z_0$ be an isolated singularity of a holomorphic function $f$. There are three possibilities:

1. $f$ is bounded in a neighbourhood of $z_0$ and $z_0$ is a *removable* singularity.
2. $\lim_{z\to z_0} |f(z)| = \infty$. Then $z_0$ is a **pole** of $f$ and there exist a number $m \in \mathbb{N}$ such that $z \mapsto (z-z_0)^m f(z)$ has a removable singularity at $z_0$. The smallest such exponent $m$ is the *order of the pole.*
3. If none of the above holds, $z_0$ is an **essential singularity**.

We will show: if there is a neighbourhood $U_0$ of $z_0$ such that $f(U_0\backslash\{z_0\})$ is not dense in $\mathbb{C}$, then $z_0$ is a removable singularity or a pole of $f$. By assumption, there is a complex number $w_0 \in \mathbb{C}$ that is not a a limit point of $f(U_0\backslash\{z_0\})$. Hence there is a $\varepsilon > 0$ such that $|f(z) - w_0| > \varepsilon$ for all $z \in U_0\backslash\{z_0\}$. This implies that $g(z) := \frac{1}{f(z)-w_0}$ is holomorphic on $U_0\backslash\{z_0\}$ and bounded. Hence $g$ has a removable singularity at $z_0$ by the RIEMANN'scher Hebbarkeitssatz. Hence $f(z) = \frac{1}{g(z)} + w_0$ has a removable singularity at $z_0$ or a pole by the Theorem of the 3 types of isolated singularities (depending on whether $\lim_{z\to z_0} g(z) \neq 0$ (removable) or not (pole)).

If $z_0$ is an *essential singularity* of a holomorphic function $f$ on $U$, then the set of values that $f$ takes on any open neighbourhood of $z_0$ is dense in $\mathbb{C}$.
Great PICARD: In any neighbourhood of an essential singularity, a holomorphic function takes all values in $\mathbb{C}$ or all values in $\mathbb{C}$ except for one.
Whereas for poles, where the function values tend to infinity when approaching a singularity, near an essential singularity, the set of values of the function is dense, that is, no matter how small a neighbourhood of the singularity we choose, we can come arbitrarily close to any complex number. In a sense, any small neighbourhood of the essential singularity gets splatted over the whole complex plane.

Let $U \subset \mathbb{C}$ be an open subset. A function $f$ is *holomorphic on $U$ except for isolated singularities* if $f$ is holomorphic on $U\backslash S$ for some subset $S \subset U$ and all points in $S$ are isolated singularities of $f$. If all points in $S$ are removable singularities or poles, then $f$ is *holomorphic on $U$ except for poles* or **meromorphic**.

Let $f$ be holomorphic on $U$ and let $z_0$ be an isolated singularity of $f$ or a just $z_0 \in U$. The order of $f$ at $z_0$ is $\operatorname{ord}(f, z_0) := \sup\left\{m \in \mathbb{Z} : z \mapsto \frac{f(z)}{(z-z_0)^m} \text{ has rem. sing. at } z_0\right\} \in \mathbb{Z} \cup \{\pm\infty\}$ with the convention $\sup(\mathbb{Z}) = \infty$ and $\sup(\varnothing) = -\infty$.

The meromorphic functions on $\hat{\mathbb{C}}$ are precisely the rational functions.

Consistency of the Definition: if $\operatorname{ord}(f, z_0) = m \geqslant 0$, then $f$ has at most a removable singularity at $z_0$. After removing the singularity (if necessary), $f$ has a zero of order $m$ at $z_0$. If $\operatorname{ord}(f, z_0) = m < 0$ and $m \neq -\infty$, then $f$ has a pole of order $-m > 0$. If $\operatorname{ord}(f, z_0) = -\infty$, then $f$ has an essential singularity at $z_0$.

Definitions & Remark

Types of isolated singularities at $\infty$

Complex Analysis I

Definition

Laurent series

Complex Analysis I

Theorem w/ proof

Cauchy formula for Laurent coefficients

Complex Analysis I

Theorem

Cauchy's Integral Formula for Annuli

Complex Analysis I

Definitions & Remark

Function element and Direct analytic continuation

Complex Analysis I

Definitions & Remark

Analytic continuation along a sequence of domains, Global analytic function, branch

Complex Analysis I

Lemma w/ proof

Analytic continuation of local inverse of a holomorphic function

Complex Analysis I

Definition

Analytic continuation along curves

Complex Analysis I

Lemma w/o proof

From direct continuation to continuation along a curve

Complex Analysis I

Lemma w/o proof

Analytic continuation of the derivative

Complex Analysis I

A **Laurent series** with *centre* $z_0$ is a series of the form $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$. More precisely, a LAURENT series is composed of two ordinary series: the nonsingular part $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and the principal part $\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} = \sum_{k=-\infty}^{-1} a_k(z-z_0)^k$. If both series converge, then $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ also denotes the sum of the limits.

Let $f$ be holomorphic on some domain $U$. Then $\infty \in \hat{\mathbb{C}}$ is an isolated singularity of $f$ *if there is a number* $R \geqslant 0$ *such that* $\{z \in \mathbb{C} : |z| > R\} \subset U$ (equivalently: if $\mathbb{C}\backslash U$ is bounded and hence compact).

**Motivation.** To classify the isolated singularities at $\infty$, note the following. If $z_0 \in \mathbb{C}^*$ is a removable singularity, a pole of order $m$ or a essential singularity of $f$, then $\frac{1}{z_0}$ is a singularity of the same type of the function $g(z) := f\left(\frac{1}{z}\right)$.

If $\infty$ is an isolated singularity of a holomorphic $f$, then we say that $f$ has a removable singularity / pole of order $m$ / essential singularity at $\infty$ if $z \mapsto f\left(\frac{1}{z}\right)$ has a removable singularity / pole of order $m$ / essential singularity at 0.

One can differentiate and integrate LAURENT series term by term.

Let $z_0 \in \mathbb{C}$ and let $f$ be *holomorphic* on the *annulus* $A := \{z \in \mathbb{C} : r < |z-z_0| < R\}$ for $0 \leqslant r < R \leqslant \infty$. If $z \in \mathbb{C}$ is such that $r < \rho_1 < |z-z_0| < \rho_2 < R$, then

$$f(z) = \frac{1}{2\pi i}\left( \int\limits_{|z-z_0|=\rho_2} \frac{f(u)}{u-z}\,\mathrm{d}u - \int\limits_{|z-z_0|=\rho_1} \frac{f(u)}{u-z}\,\mathrm{d}u \right)$$

If the LAURENT series $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ converges on the domain $\{z \in \mathbb{C} : r < |z-z_0| < R\}$ and represents a holomorphic function $f$ there, then $a_n = \frac{1}{2\pi i} \int\limits_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$ for all $n \in \mathbb{N}$ and any $\rho \in (r, R)$.

Assume $z_0 = 0$. As we can integrate LAURENT series term-by-term, for $\xi \in (0, R)$ we get $\int\limits_{|z|=\xi} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z = \int\limits_{|z|=\xi} \sum_{k=-\infty}^{\infty} a_k \frac{z^k}{z^{n+1}}\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} a_k \int\limits_{|z|=\xi} z^{k-n-1}\,\mathrm{d}z$, so every summand except the $n$-th one vanishes and we get $\int\limits_{|z|=\xi} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z = 2\pi i a_n$.

Function elements $(f, U)$ and $(\tilde{f}, U)$ are **analytic continuation**s *of each other*, if there exists a *finite sequence* $(f, U) = (f_1, U_1), (f_2, U_2), \ldots, (f_n, U_n) = (\tilde{f}, \tilde{U}_n)$ of function elements such that $(f_j, U_j)$ and $(f_j, U_{j+1})$ are *direct analytic continuations* of each other for all $j \in \{1, \ldots, n-1\}$. In this case we say that $(\tilde{f}, \tilde{U})$ is an *analytic continuation* of $(f, U)$ *along the sequence of domains* $U_1, \ldots, U_n$.

This defines an *equivalence relation* on the set of function elements, where $(f, U) \sim (\tilde{f}, \tilde{U})$ if and only if $(f, U)$ and $(\tilde{f}, \tilde{U})$ are analytic continuations of each other.

An equivalence class of $\sim$ as described above is a **global analytic function**. A function element of an equivalence class is a **branch** of the global analytic function.

A **function element** is a pair $(f, U)$ consisting of a *domain* $U \subset \mathbb{C}$ and a *holomorphic* function $f$ on $U$.

Function elements $(f, U)$ and $(\tilde{f}, \tilde{U})$ are **direct analytic continuation**s *of each other* if $U \cap \tilde{U} \neq \varnothing$ and $f \equiv \tilde{f}$ *on* $U \cap \tilde{U}$.

This definition of direct analytic continuation is inherently symmetric.

Let $\gamma\colon [t_0, t_1] \to \mathbb{C}$ be a *continuous curve*. A function element $(\tilde{f}, \tilde{U})$ is an *analytic continuation* of a function element $(f, U)$ *along* $\gamma$ if there is a family of function elements $((f_t, U_t))_{t \in [t_0, t_1]}$ such that

1. $(f_{t_0}, U_{t_0}) = (f, U)$ and $(f_{t_1}, U_{t_1}) = (\tilde{f}, \tilde{U})$,
2. $\gamma(t) \in U_t$ for all $t \in [t_0, t_1]$ (In particular, $\gamma(t_0) \in U$ and $\gamma(t_1) \in \tilde{U}$.) and there exists a $\varepsilon > 0$ such that for each $t' \in [t_0, t_1]$ with $|t - t'| < \varepsilon$ we have $\gamma(t') \in U_t$ and $f_{t'}$ agrees with $f_t$ on $U_t \cap U_{t'}$.

Let $f$ be an *entire* function and $(g, U)$ be a *function element* such that $f(g(z)) = z$ for all $z \in U$. If $(\tilde{g}, \tilde{U})$ is a analytic continuation of $(g, U)$, then $f(\tilde{g}(z)) = z$ for all $z \in \tilde{U}$.

The general case follows directly from the special case that $(\tilde{g}, \tilde{U})$ is a *direct* analytic continuation of $(g, U)$, because any non-direct analytic continuation is a sequence of direct analytic continuations and if the property of being a local inverse of $f$ is preserved from one direct continuation to the other, then it is preserved for all steps. So assume $(\tilde{g}, \tilde{U})$ is a direct analytic continuation of $(g, U)$, that is $U \cap \tilde{U} \neq \varnothing$ and $g \equiv \tilde{g}$ on $U \cap \tilde{U}$. Hence for $z \in U \cap \tilde{U}$ we have $f(\tilde{g}(z)) = f(g(z)) = z$. So $f \circ g$ and the identity function $z \mapsto z$ agree on $U \cap \tilde{U} \subset \tilde{U}$. By Identity Theorem for Holomorphic Functions $f \circ \tilde{g}$ and $z \mapsto z$ agree of the domain $\tilde{U}$.

If the derivative $(f', U)$ of a function element $(f, U)$ can be analytically continued along a curve $\gamma\colon [t_0, t_1] \to \mathbb{C}$, then $(f, U)$ can be analytically continued along $\gamma$.

Suppose there is a finite family $(f, U) = (f^{(0)}, U^{(0)}), (f^{(1)}, U^{(1)}) \ldots (f^{(n)}, U^{(n)}) = (\tilde{f}, \tilde{U})$ such that

1. $(f^{(j)}, U^{(j)})$ and $(f^{(j+1)}, U^{(j+1)})$ are direct analytic continuations of each other for every $j \in \{0, \ldots, n-1\}$,
2. there is a *subdivision* $t_0 = \tau_0 < \tau_1 < \ldots < \tau_n = t_1$ such that $\gamma(\tau_j) \in U^{(j)}$ for all $j \in \{0, \ldots, n\}$ and $\gamma([\tau_j, \tau_{j+1}]) \subset U^{(j)} \cup U^{(j+1)}$ for all $j \in \{0, \ldots, n-1\}$.

Then $(\tilde{f}, \tilde{U})$ is an analytic continuation of $(f, U)$ along $\gamma$.

Definition & Theorem w/o proof

Integral along a continuous curve

Complex Analysis I

Definitions

Homotopy and null homotopic curve

Complex Analysis I

Definitions & Lemma w/o proof

Concatenation and Inverse of curves

Complex Analysis I

Definition & Theorem w/o proof

Loop, fundamental group

Complex Analysis I

Definition & Theorem w/o proof

Simply connected

Complex Analysis I

Theorem & Corollary w/o proofs

Monodromy Theorem and the Homotopy-Version of Cauchy's Integral Theorem

Complex Analysis I

Definition

1-chain, $C_1$, Integral of ??

Complex Analysis I

Definition

Free Abelian group

Complex Analysis I

Definitions

0-chain, $C_0$, boundary map, cycle, support of a 1-chain

Complex Analysis I

Definition & Theorem w/o proof & Lemma w/ proof

Winding number of a closed curve is constant on connected components

Complex Analysis I

Two curves $c_0, c_1\colon [0,1] \to X$ in a topological space $X$ are **homotopic (in $X$)** if there exists a **homotopy** between them, that is, a continuous map $H\colon [0,1] \times [0,1] \to X$ for which $H(\cdot, 0) = c_0$ and $H(\cdot, 1) = c_1$ as well as $H(0, \cdot) = c_0(0) = c_1(0)$ and $h(1, \cdot) = c_0(1) = c_1(1)$ (same starting- and endpoint).

Let $f$ be a holomorphic function on $U$, $\gamma\colon [t_0, t_1] \to U$ be a continuous curve in $U$, $D_0 \subset U$ be an open disk around $\gamma(t_0)$ and $F_0$ be an antiderivative of $f$ on $D_0$ (which exists because $f$ is represented by a power series on $D_0$). Let $(F_1, D_1)$ be an *analytic continuation* of $(F_0, D_0)$ along $\gamma$ (which exists by a Lemma because $(F_0', D_0) = (f|_{D_0}, D_0)$ can be trivially continued along $\gamma$). Define the integral of $f$ along $\gamma$ by $\int_\gamma f(z)\,\mathrm{d}z := F_1(\gamma(t_1)) - F_0(\gamma(t_0))$. The RHS does not depend on any choice involved in the construction.

A closed curve $c\colon [0,1] \to X$ is **null homotopic** if it is homotopic to the constant curve at $c_1(t) = c(0) = c(1)$.

If $\gamma$ is piecewise continuously differentiable, then the above integral agrees with our original Definition.

Let $X$ be a topological space and $x_0 \in X$ a (base)point. A curve $c\colon [0,1] \to X$ is a *loop at $x_0$* if $c(0) = x_0 = c(1)$. Then homotopy is an *equivalence relation* on the set of loops at $x_0$. The set of equivalence classes, $\pi_1(X, x_0)$, together with the well-defined operation $[c_1c_2] = [c_1][c_2]$, where $c_1$ and $c_2$ are loops at $x_0$, is the **fundamental group** *of $X$ with base point $x_0$*. The neutral element is the class of constant curves $[x_0]$, i.e. the set of null-homotopic loops at $x_0$. The inverse of $[c]$ is $[c^{\mathrm{inv}}]$.
$\pi_1(X, y)$ depends on $y$ if $X$ is not path-connected.

The *composition* of $c_1, c_2\colon [0,1] \to X$ with $c_1(1) = c_2(0)$ is

$$c_1c_2\colon [0,1] \to X, \qquad t \mapsto \begin{cases} c_1(2t), & \text{for } t \in [0, \frac{1}{2}], \\ c_2(2t-1), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

The *inverse of a curve* $c\colon [0,1] \to X$ is the curve $c^{\mathrm{inv}}\colon [0,1] \to X$, $t \mapsto c(1-t)$.

Let $c_1, c_2, c_3\colon [0,1] \to X$ be curves with $c_1(1) = c_2(0)$ and $c_2(1) = c_3(0)$. Then $(c_1c_2)c_3$ is homotopic to $c_1(c_2c_3)$.

Let $U \subset \mathbb{C}$ be a domain and let $(f_0, U_0)$ be a function element, $z_0 \in U \cap U_0$ and suppose $(f_0, U_0)$ can be continued analytically along every curve in $U$ starting at $z_0$. If $c$ and $\tilde{c}$ are homotopic curves starting at $z_0$ and $(f_1, U_1)$ and $(\tilde{f}_1, \tilde{U}_1)$ are analytic continuations of $(f_0, U_0)$ along $c$ and $\tilde{c}$ respectively, then $f_1$ *and $\tilde{f}_1$ agree in some open neighbourhood of $z_1 := c(1) = \tilde{c}(1)$.*

Corollary: If $f$ is holomorphic on $U \subset \mathbb{C}$ and $c_1$ and $c_2$ are *homotopic* curve in $U$, then $\int_{c_1} f(z)\,\mathrm{d}z = \int_{c_2} f(z)\,\mathrm{d}z$. In particular, $\int_c f(z)\,\mathrm{d}z = 0$ if $c$ is *null homotopic.*

Let $X$ be a nonempty *path-connected* topological space, e.g. a domain. Then the following are equivalent:

1. Every closed curve $c\colon [0,1] \to X$ is null homotopic in $X$.
2. For every $x_0 \in X$, $\pi_1(X, x_0) = \{1\}$.
3. There is a point $x_0 \in X$ such that $\pi_1(X, x_0) = \{1\}$.
4. Any curves $c_1, c_2\colon [0,1] \to X$ with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$ are homotopic.

If one of the above statements hold, $X$ is **simply connected**.

If $B$ is some set, then one can define the *free* ABELIAN *group generated by $B$* as the group $(\mathbb{Z}^{(B)}, +)$, where $\mathbb{Z}^{(B)}$ is the set of functions $B \to \mathbb{Z}$ (mapping a shopping item to its multiplicity), which are zero for all but finitely many elements and $+$ means pointwise addition.
The confusing part: interpret an element $b_0 \in B$ also as the characteristic function $\varphi_{b_0}\colon B \to \mathbb{Z}$, with $\varphi_{b_0}(b) = 1$ if $b = b_0$ and 0 else. Then we can write any element in the free ABELIAN group generated by $B$ as a finite "formal" linear combination $\sum_{j=1}^k n_j b_j$ for $(n_j)_{j=1}^k \subset \mathbb{Z}$.

A **1-chain** $c$ in an open set $U \subset \mathbb{C}$ is a *formal linear combination* $c = n_1 \odot c_1 \oplus \ldots \oplus n_k \odot c_k$ of curves $c_j\colon [0,1] \to U$, where $n_J \in \mathbb{Z}$ for $j \in \{1, \ldots, k\}$. The ABELIAN *group* of 1-chains in $U$ is $C_1(U)$.
For a holomorphic function $f$ on $U$, the *integral of $f$ along $c$* is $\int_c f(z)\,\mathrm{d}z := \sum_{j=1}^k n_j \int_{c_j} f(z)\,\mathrm{d}z$.

The **winding number** or *winding index* of a *closed* curve $\gamma\colon [0,1] \to \mathbb{C}$ *around a point* $z_0 \in \mathbb{C} \backslash \gamma([0,1])$ is $\nu_\gamma(z_0) := \mathrm{Ind}_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z \in \mathbb{Z}$.

The winding number $\mathrm{Ind}_\gamma$ is constant on connected components of $\mathbb{C} \backslash \gamma([0,1])$.

For $z_0 \in \mathbb{C} \backslash \gamma([0,1])$, the winding number depends continuously on $z_0$ and takes integer values. Hence it is constant on connected components of its image. To see continuity, note that $\left|\frac{1}{z - z_0} - \frac{1}{z - z_1}\right| = \frac{|z_0 - z_1|}{|z - z_0||z - z_1|}$ and that $\frac{1}{|\gamma - z_0||\gamma - z_1|}$ is bounded on $[0,1]$.

A **0-chain** in $U$ is a formal linear combination $\bigoplus_{j=1}^k n_j \odot z_j$ of points $(z_j)_{j=1}^k \subset U$ with integer coefficients $(n_j)_{j=1}^k \subset \mathbb{Z}$. The ABELIAN group of 0-chains in $U$ is $C_0(U)$.

The **boundary** map $\partial\colon C_1(U) \to C_0(U)$ is the group homomorphism, where the 1-chain $\bigoplus_{j=1}^k n_j \odot c_j$ is mapped to $\partial c := \bigoplus_{j=1}^k n_j \oplus \big(c_j(1) \ominus c_j(0)\big)$.

A 1-chain $c$ is **closed** if $\partial c = 0$. A **cycle** is a **closed 1-chain**. The **support** $|c|$ of a 1-chain in $U$ is $\bigcup_{\substack{j=1 \\ n_j \neq 0}}^k c_j([0,1]) \subset U$.

Definition

Winding number of a cycle

Complex Analysis I

Definition

Zero homogolous cycle

Complex Analysis I

Theorem w/o proof

Cauchy's Integral Theorem (Winding number / homology version)

Complex Analysis I

Theorem w/o proof

Cauchy's Integral Formula (Winding number version)

Complex Analysis I

Definition

Bounding cycle

Complex Analysis I

Definition & Theorem

Residue

Complex Analysis I

Theorem & Corollary w/o proofs

Residue Theorem

Complex Analysis I

Proof

Residue Theorem

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Examples

Residue calculus

Complex Analysis I

Theorem w/o proof

Zero and pole counting integral

Complex Analysis I

A cycle $c$ in an open set $U \subset \mathbb{C}$ is **zero homogolous** *in* $U$ if $\operatorname{Ind}_c(z) = 0$ for all $z \in \mathbb{C} \backslash U$.

Let $f$ be a holomorphic function on $U \subset \mathbb{C}$, let $a \in U$ and let $c$ be a *cycle in* $U \backslash \{a\}$ that is *zero-homologous in* $U$. Then

$$\frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} \,\mathrm{d}z = \operatorname{Ind}_c(a) \cdot f(a).$$

1. Suppose the holomorphic function $f$ has an isolated singularity at $z_0$ (or is holomorphic at $z_0$, too). The ***residue*** *of $f$ at $z_0$* is

$$\operatorname{Res}_{z_0}(f) := \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z) \,\mathrm{d}z,$$

where $\varepsilon > 0$ is so small that $\{z \in \mathbb{C} : 0 < |z - z_0| \leqslant \varepsilon\} \subset U$.

2. Equivalently, if the LAURENT series around $z_0$ representing $f$ is $\sum_{k \in \mathbb{Z}} a_k (z-z_0)^k$, then $\operatorname{Res}_{z_0}(f) = a_{-1}$.

Let $f$ be *meromorphic* on $U \subset \mathbb{C}$ and $c$ be a *cycle that bounds a compact set* $K \subset U$ such that $\partial K$ doesn't contain any zero or poles of $f$. Then

$$\frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} \,\mathrm{d}z = Z - P,$$

where $Z$ is the *number of zeros* of $f$ in $\mathring{K}$ and $P$ the *number of poles*, each *counted with multiplicity* according to their order.
Proof: Apply Residue Theorem to $\frac{f'}{f}$, as $\operatorname{Res}(\frac{f'}{f}, z) = \operatorname{ord}(f, z)$.

The **winding number** of a cycle $c$ in $\mathbb{C}$ around a point $z_0 \in \mathbb{C} \backslash |c|$ is $\operatorname{Ind}_c(z_0) = \frac{1}{2\pi i} \int_c \frac{1}{z-z_0} \,\mathrm{d}z \in \mathbb{Z}$.

Let $U$ be a domain in $\mathbb{C}$ and $c$ be a cycle in $U$. The following statements are equivalent.

1. $c$ is zero homologous in $U$
2. $\int_c f(z) \,\mathrm{d}z = 0$ for all holomorphic functions $f$ on $U$.

Let $K \subset \mathbb{C}$ be a *compact* set. A cycle $c$ *bounds* $K$ if $|c| \subset \partial K$ and if

$$\operatorname{Ind}_c(z) = \begin{cases} 1, & \text{if } z \in \mathring{K}, \\ 0, & \text{if } z \notin K. \end{cases}$$

Let $f$ *holo* on $U$ except for a set $S \subset U$ of *isolated singularities* and $c$ a *0-homologous cycle* in $U$ with $|c| \cap S \neq \varnothing$.

$$\frac{1}{2\pi i} \int_c f(z) \,\mathrm{d}z = \sum_{a \in S} \operatorname{Ind}_c(a) \operatorname{Res}_a(f),$$

where the *sum is finite* because $\operatorname{Ind}_c(a) \neq 0$ only for finitely many $a \in S$.
Corollary: If $c$ bounds a compact subset $K \subset U$, then

$$\frac{1}{2\pi i} \int_c f(z) \,\mathrm{d}z = \sum_{a \in S \cap \mathring{K}} \operatorname{Res}_a(f).$$

If $f$ has a poles of order 1 at $z_0$, then $\lim_{z \to z_0} (z-z_0) f(z) = \operatorname{Res}_f(z_0)$.

If $f = \frac{g}{h}$, where $h$ has a simple zero at $z_0$ and $g(z_0) \neq 0$, then $f$ has a first order pole at $z_0$ and $\operatorname{Res}_f(z_0) = \frac{g(z_0)}{h'(z_0)}$.

If $f$ has a pole of order $n$ at $z_0$, then

$$\operatorname{Res}_f(z_0) = a_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \left[(z-z_0)^n f(z)\right].$$

Lemma w/o proof

Dog on a leash

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Theorem w/ proof

Rouché's Theorem

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Definition & Lemma w/o proof

Uniform convergence on compact sets

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Theorem w/ proof

Uniform convergence on compact sets

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Theorem w/o proof idea & Corollary

Multiplicities of values in the limit (Hurwitz)

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Definition

Locally bounded function sequence

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Theorem & preparatory Lemmas w/o proofs

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Definition & Examples

Conformally equivalent domains

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Theorem w/o proof

Riemann mapping theorem

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Proof

Riemann mapping theorem

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Let $\gamma$ be a *closed* curve bounding a compact region $K \subset U$ and $f$ and $g$ be holomorphic functions on $U$ such that $|g(z)| < |f(z)|$ for all $z \in |\gamma|$. Then $f$ and $f+g$ have the same number of zeros (counted with multiplicities) in $\mathring{K}$.

Let $c_1, c_2 \colon [0,1] \to \mathbb{C}$ be two closed curves and $z_0 \in \mathbb{C} \setminus (|c_1| \cup |c_2|)$ Furthermore assume that for all $t \in [0,1]$:

$$|c_1(t) - c_2(t)| < |c_1(t) - z_0|. \tag{1}$$

Then $\operatorname{Ind}_{c_1}(z_0) = \operatorname{Ind}_{c_2}(z_0)$.

Since the functions have no poles, the numbers of zeros are winding numbers of $c_1 := f \circ \gamma$ and $c_2 := f \circ \gamma + g \circ \gamma$ around 0 (by zero and poles counting integral thm). But since $|c_1 - c_2| = |g \circ \gamma| < |f \circ \gamma| = |c_1|$, the winding numbers are equal by the Dog-on-a-leash Lemma.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of *holomorphic* functions on $U$ that *converges uniformly on compact sets* to the function $f$. Then $f$ is also *holomorphic* on $U$ and the sequence $(f'_n)_{n \in \mathbb{N}}$ *converges uniformly on compact sets to* $f'$.

We show that $\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$ for every closed triangular region $\Delta \subset U$. By Cauchy's Theorem and uniform convergence $\int_{\partial\Delta} f(z)\,\mathrm{d}z = \int_{\partial\Delta} \lim_{n\to\infty} f_n(z)\,\mathrm{d}z = \lim_{n\to\infty} \underbrace{\int_{\partial\Delta} f_n(z)\,\mathrm{d}z}_{=0} = 0$. Cauchy's integral formula for the derivative yields $|f'_n(z) - f(z)| = \left| \frac{1}{2\pi i} \int\limits_{|u-z_0|=r} \frac{f_n(u)-f(u)}{(u-z)^2}\,\mathrm{d}u \right| \leqslant \frac{\cancel{2\pi} r}{\cancel{2\pi}} \frac{\max\{|f_n(u)-f(u)| : |u-z_0|=r\}}{\min\{|u-z|^2 : |u-z_0|=r\}} = \frac{r}{|r-|z-z_0||^2} \max\limits_{\substack{u\in\mathbb{C}:\\ |u-z_0|=r}} |f_n(u) - f(u)|$, where $z_0 \in U$ and $r > 0$ are chosen such that $|z - z_0| < r$ and $\{u \in \mathbb{C} : |u - z_0| = r\} \subset U$,

A *sequence* $(f_n \colon U \to \mathbb{C})_{n \in \mathbb{N}}$ of functions **converges uniformly on compact sets** to a function $f \colon U \to \mathbb{C}$ if one of the following conditions is satisfied.

- For any compact subset $K \subset U$, we have $f_n \to f$ uniformly on $K$.
- $(f_n)_{n \in \mathbb{N}}$ **converges locally uniformly** to $f$, that is, for any $z_0 \in U$, there exists an open neighbourhood on which $f_n \to f$ converges uniformly.

A sequence $(f_n \colon U \to \mathbb{C})_{n \in \mathbb{N}}$ is **locally bounded** if every $z_0 \in U$ has an open neighbourhood $U_0$ so that there is a number $m \in \mathbb{R}$ for which $|f_n(z)| \leqslant M$ for all $z \in U_0$ and $n \in \mathbb{N}$.

Suppose $a \in \mathbb{C}$ and $(f_n)_{n \in \mathbb{N}}$ is a sequence of *holomorphic* functions on $U$ that *converges uniformly on compact sets* to the function $f$. Suppose further that each function $f_n$ takes the value $a$ at most $m$ times (counting multiplicities). Then $f$ takes the value $a$ at most $m$ times (counting multiplicities) or $f$ is constant.
Corollary: The limit function of a sequence of injective holomorphic functions than converges uniformly on compact sets is also injective or constant.
Use Rouche's Theorem.

Two domains $U$ and $\tilde{U}$ in $\mathbb{C}$ are *biholomorphically* or **conformally equivalent** if there is a *bijective holomorphic* function $f \colon U \to \tilde{U}$.
(In this case $f^{-1}$ is also holomorphic.)

Every *locally bounded* sequence of *holomorphic* functions has a *subsequence that converges uniformly on compact sets.*

By Liouville, $\mathbb{C}$ and $D$ aren't conformally equivalent, while $D$ and $H$ are (as there is a Möbius transformation between them).

First we prove: A locally bounded sequence of *holomorphic* functions $(f_n \colon U \to \mathbb{C})_{n \in \mathbb{N}}$ is locally equi-Lipschitz-continuous.
If $(f_n \colon U \to \mathbb{C})_{n \in \mathbb{N}}$ is a locally bounded sequence of holomorphic functions, which *converges pointwise on a dense subset* $A \subset U$, then $(f_n)_{n \in \mathbb{N}}$ converges *uniformly on compact sets.*

Every nonempty *simply connected* domain $U \subsetneq \mathbb{C}$ is conformally equivalent to the open unit disk $D$.

Two Riemann maps $U \to D$ differ by post-composition with a Möbius transformation mapping $D$ onto $D$.

Preparatory Lemma: if $U \subset \mathbb{C}$ is a *simply connected* domain and $0 \notin U$, then there exists an *injective holomorphic* function $\rho$ on $U$ such that $(\rho(z))^2 = z$ for all $z \in U$.

Proof idea: $\mathcal{F} := \{f \colon U \to \mathbb{C} : f \text{ is holomorphic, injective, } f(U) \subset D, f(0) = 0\}$. Claim. There exists a function $f \in \mathcal{F}$ for which $|f'(0)|$ is maximal among functions in $\mathcal{F}$. This is a biholomorphic map onto $D$.