Atomic Norm Minimisation for Superresolution

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Seminar Optimal Transport

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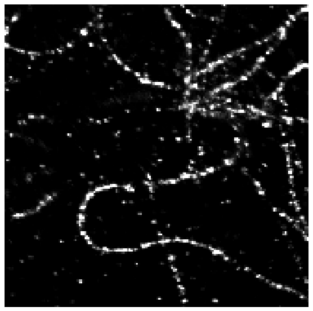
18.06.2020

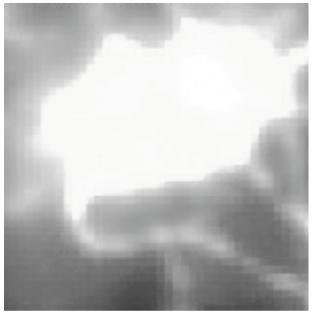
"Zoom and enhance" cliché in TV & movies

- In TV & movies: wrong representation of image enhancing
- **But what if we could actually manually increase the resolution?**
- This challenge is called Superresolution.

Agenda



is observed as



Ground truth

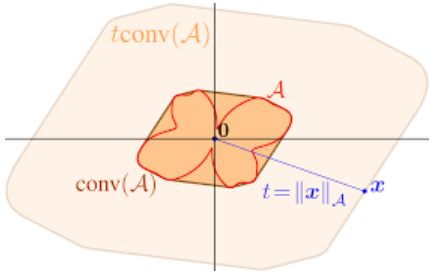
$$x = \sum_{k=1}^{r} c_k \delta_{\tau_k}$$

Observed data $\psi(z) = (1, z, z^2, \ldots)$

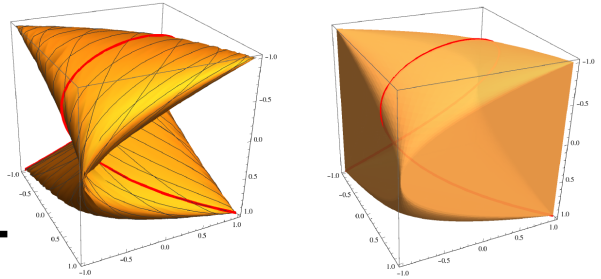
$$\tilde{x} = \sum_{k=1}^{r} |c_k| e^{-2\pi i \varphi_k} \psi(e^{-2\pi i \tau_k})$$

recover?

Atomic Norm $\|\tilde{x}\|_{\mathcal{A}}$



Atomic Set $\mathcal{A}$



induces

$$\inf\{r > 0 : x \in r\,\mathrm{co}(\mathcal{A})\}$$

$$\left\{e^{-2\pi i \varphi} \psi(e^{-2\pi i w})\right\}_{w,\varphi \in [0,1)}$$

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The ground truth - a spike train

Consider the spike train

$$x := \sum_{k=1}^{r} c_k \delta_{\tau_k} \in \mathcal{M}(\mathbb{T}),$$

supported on $T := (\tau_k)_{k=1}^r \subseteq \mathbb{T} = [0,1)$, where $(c_k)_{k=1}^r \in \mathbb{C} \setminus \{0\}$ are the amplitudes of the spikes.

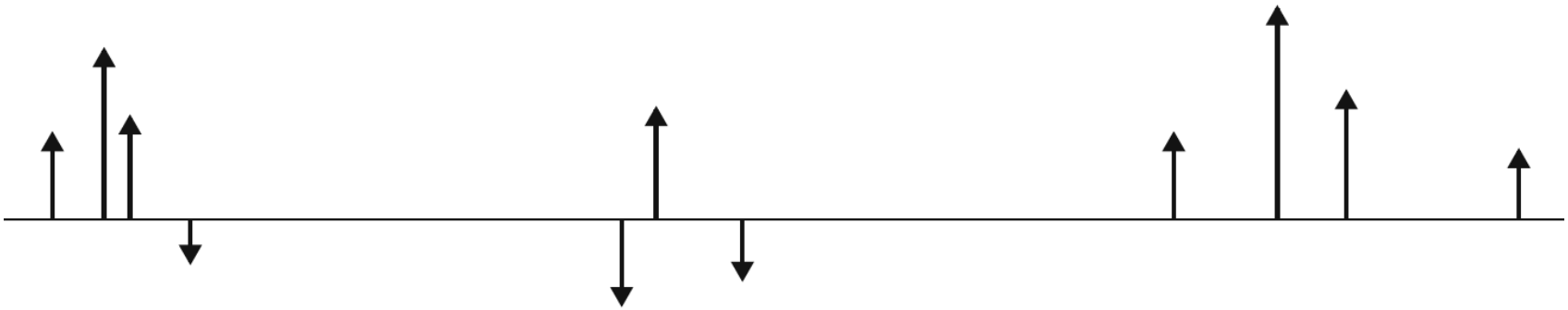

Fig. 1: A spike train with $r = 11$ spikes and real weights $(c_k)_{k=1}^r$.

The observed signal

Resolution limited $\implies$ observed signal is

$$x_{\text{obs}}\colon \mathbb{T} \to \mathbb{C}, \qquad t \mapsto (x * g)(t) = \sum_{k=1}^{r} c_k g(t - \tau_k),$$

where $g \in \mathcal{C}(\mathbb{T})$ is such that $\hat{g}(j) = 0$ if $|j| > f_c \in \mathbb{N}$, where $\hat{g}\colon \mathbb{Z} \to \mathbb{C}$ is the Fourier transform of g.

Assume $\hat{g}(j) \equiv 1$ for $|j| \leq f_c$.

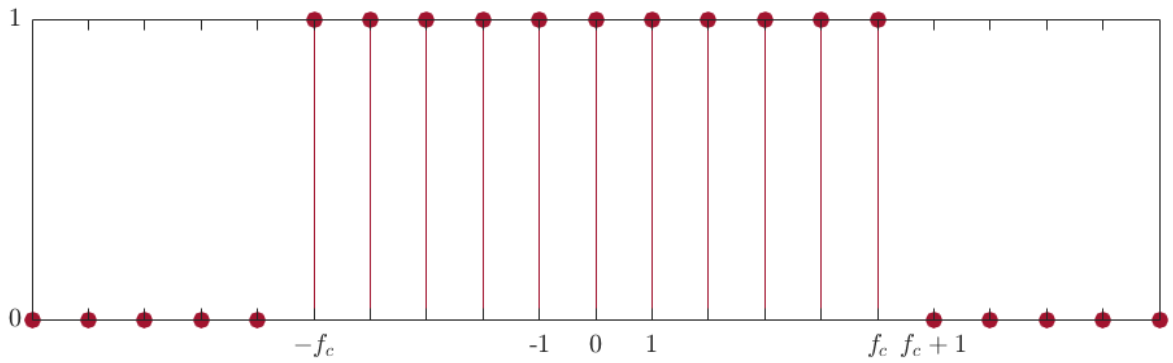

Fig. 2: The Fourier transform of g.

The Fourier transform of the observed signal

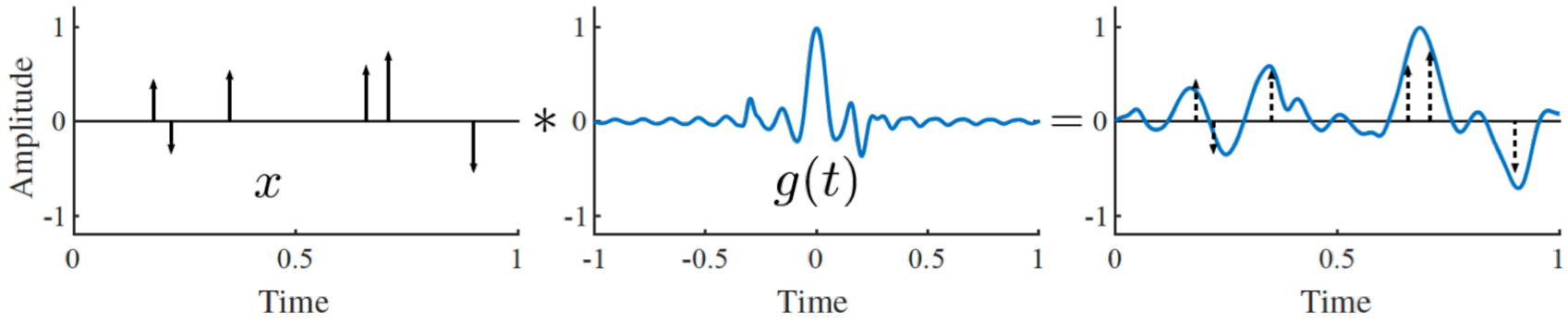

Fig. 3: The observed signal is the convolution of the spike train with a bandlimited function.

The Fourier transform of x_{obs} is

$$\widehat{x_{\text{obs}}}\colon \mathbb{Z} \to \mathbb{C}, \qquad j \mapsto \hat{x}(j)\hat{g}(j) = \left(\sum_{k=1}^{r} c_k e^{-2\pi i j \tau_k}\right) \hat{g}(j).$$

Interim conclusion

As $\hat{g}(j) = 0$ if $|j| > f_c$, we have $d := 2f_c + 1$ equidistant low frequency measurements $\widehat{x_{\text{obs}}}(j)$ for $|j| \leq f_c$.

$\rightsquigarrow$ Convolution with g erases high frequencies of x.

Let $\tilde{x} := (\widehat{x_{\text{obs}}}(j))_{j=-f_c}^{f_c}$ and $\psi(z) := \left(1, z, z^2, \ldots, z^{d-1}\right)^{\mathsf{T}}$.

Interim conclusion:

$$x = \sum_{k=1}^{r} c_k \delta_{\tau_k} \in \mathcal{M}(\mathbb{T}) \xrightarrow[\text{measurement}]{\text{yields the}} \tilde{x} = \sum_{k=1}^{r} c_k \psi(e^{-2\pi i \tau_k}) \in \mathbb{C}^d$$

$$= \sum_{k=1}^{r} |c_k| e^{-2\pi i \varphi_k} \psi(e^{-2\pi i \tau_k})$$

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Signal decomposition

- Goal: decompose signal $\tilde{x} \in \mathbb{K}^d$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) into finite nonnegative linear combination with respect to $\mathcal{A} \subseteq \mathbb{K}^d$:

$$\tilde{x} = \sum_{a \in \mathcal{A}} c_a a, \qquad c_a \geq 0.$$

$\rightsquigarrow$ $\exists$ infinitely many expansions of $\tilde{x}$. How to choose?

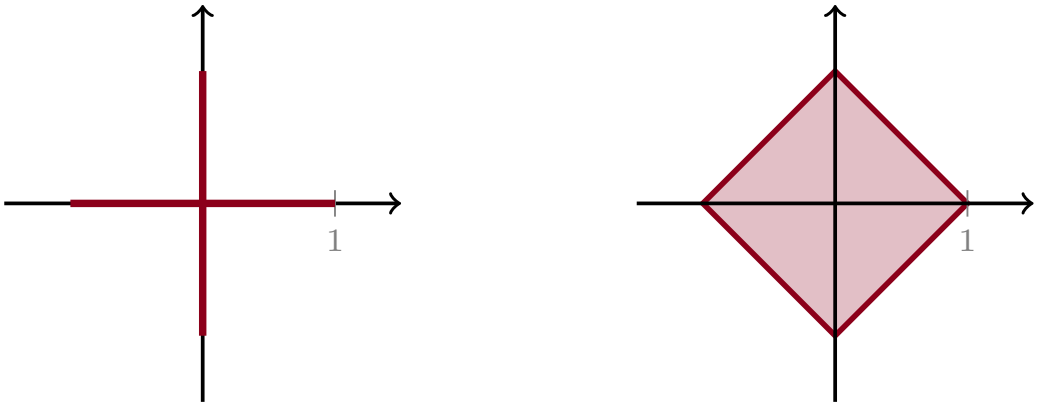
- Solve

$$\min_u \|c\|_0 \quad \text{such that} \quad \tilde{x} = \sum_{a \in \mathcal{A}} c_a a, \quad c_a \geq 0,$$

where $\|c\|_0 := \#\{a \in \mathcal{A} : c_a \neq 0\}$.

- But $\|\cdot\|_0$ is not convex and "not robust".

From cardinality minimisation to ℓ_1 minimisation



(a) The unit ball of the $\|\cdot\|_0$ function. (b) The unit ball of the ℓ_1 norm.

Fig. 4: The convex hull of (a) is (b).

$\rightsquigarrow$ instead solve

$$\min_c \|c\|_1 \quad \text{such that} \quad \tilde{x} = \sum_{a \in \mathcal{A}} c_a a, \quad c_a \geq 0.$$

Goal of next section: show that $\|\tilde{x}\|_{\mathcal{A}}$ is that minimum value

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When is the gauge a norm?

Let X be a *normed space*.

Definition (Gauge)

The **gauge** of a subset $A \subseteq X$ is

$$p_A \colon X \to [0, \infty], \qquad x \mapsto \inf\{r > 0 : x \in rA\},$$

where $\inf(\emptyset) := \infty$.

Theorem (Norm Properties)

If $A \subseteq X$ is a nonempty, convex, bounded, rotation invariant, fulldimensional set, then p_A is a norm on X.

rotation invariant: $rA = A \ \forall |r| = 1$.

fulldimensional: A contains open neighbourhood of 0.

Proof of the Theorem about Norm Properties

(i) $p_A(x) < \infty \; \forall x \in X$.

A fulldimensional $\implies \exists \rho > 0$ such that $B_\rho(0) \subseteq A$. Let $x \in X$ and $r := \frac{2}{\rho} \cdot \|x\|$, then $x \in rB_\rho(0) \subseteq rA$, as $\left\|\frac{x}{r}\right\| = \frac{\rho}{2} < \rho$.

(ii) $p_A(x) = 0 \implies x = 0$.

Take $x \in X$ with $p_A(x) = 0$. $\exists (r_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$, $r_n \to 0$ such that $x \in r_n A \; \forall n \in \mathbb{N}$. Assume $\exists \varepsilon > 0$ with $\|x\| > \varepsilon$.

$$\lim_{n\to\infty} \frac{1}{r_n}\|x\| \geq \lim_{n\to\infty} \frac{\varepsilon}{r_n} = \infty,$$

$\lightning$ to the boundedness of A.

Proof of the Theorem about Norm Properties

(iii) $p_A(x+y) \leq p_A(x) + p_A(y)\ \forall x, y \in X$.

Note: $x \in \lambda A \implies p_A(x) \leq \lambda$.

Let $x, y \in X$ and $\varepsilon > 0$. $\exists \lambda, \mu > 0$ such that

$$\lambda \leq p_A(x) + \frac{\varepsilon}{2}, \quad \mu \leq p_A(y) + \frac{\varepsilon}{2} \quad \text{and} \quad \frac{x}{\lambda}, \frac{y}{\mu} \in A.$$

A convex, so

$$\frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} = \frac{x+y}{\lambda+\mu} \in A.$$

Thus

$$p_A(x+y) \leq \lambda + \mu \leq p_A(x) + p_A(y) + \varepsilon \xrightarrow{\varepsilon \searrow 0} p_A(x) + p_A(y).$$

Proof of the Theorem about Norm Properties

(iv) $p_A(\lambda x) = |\lambda| p_A(x)\ \forall \lambda \in \mathbb{K},\ x \in X.$

Let $x \in X$. A fulldim. $\implies 0 \in A \implies 0 \in rA\ \forall r > 0$, so

$$p_A(0 \cdot x) = p_A(0) = \inf\{r > 0\} = 0 = |0| p_A(x).$$

For $\lambda > 0$

$$\begin{aligned} p_A(\lambda x) &= \inf\{r > 0 : \lambda x \in rA\} = \inf\left\{r > 0 : x \in \frac{r}{\lambda} A\right\} \\ &= \inf\{\lambda r > 0 : x \in rA\} = \lambda \inf\{r > 0 : x \in rA\} \\ &= \lambda p_A(x). \end{aligned}$$

A rotation invariant $\implies \lambda A = |\lambda| A$, so

$$\lambda x \in rA \iff |\lambda| x \in rA.$$

Extreme points of convex sets

Definition (Extreme point of a convex set)

A point $x \in C$ in a *convex* subset $C \subseteq X$ is an extreme point of C and we write $x \in \mathrm{ex}(C)$ if there does not exist an open line segment contained in C that contains x, that is, the relations $x = \lambda y + (1-\lambda)z$ for $y, z \in C$, $y \neq z$ and $\lambda \in [0,1]$ imply that $\lambda = 0$ or $\lambda = 1$ and thus $x = y$ or $x = z$.

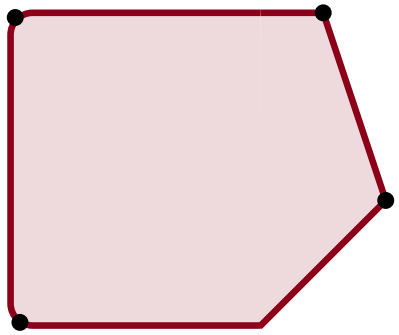

Fig. 5: The black dotes are some extreme points of the set.

Atomic Sets and the Atomic Norm

Definition (Atomic Set)

A set $\mathcal{A} \subseteq \mathbb{K}^d$ is an Atomic Set if $\mathcal{A}$ is compact, rotation invariant and a subset of $\mathrm{ex}(\mathrm{co}(\mathcal{A}))$ and $\mathrm{co}(\mathcal{A})$ is fulldimensional.

$\{a(w,\varphi) := e^{-2\pi i\varphi}\psi(e^{-2\pi i w}) : w,\varphi \in [0,1)\}$ is an Atomic Set. $\square$

Definition (Atomic Norm)

The Atomic Norm induced by an Atomic Set $\mathcal{A} \subseteq \mathbb{K}^d$ is the gauge on $\mathrm{co}(\mathcal{A})$:

$$\|\cdot\|_{\mathcal{A}} \colon \mathbb{K}^d \to [0,\infty), \qquad x \mapsto \inf\{r > 0 : x \in r\,\mathrm{co}(\mathcal{A})\}.$$

Representation of the Atomic Norm

Atomic norm solves the decomposition problem:

Theorem (Representation of the Atomic Norm)

For an Atomic Set $\mathcal{A} \subseteq \mathbb{K}^d$ and $\tilde{x} \in \mathbb{K}^d$ we have

$$\|\tilde{x}\|_{\mathcal{A}} = \inf \left\{ \|c\|_1 : \tilde{x} = \sum_{a \in \mathcal{A}} c_a a,\ c_a \geq 0 \right\}.$$

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Duality

- Goal: find $\|\tilde{x}\|_{\mathcal{A}}$
- Dual problem (we have strong duality):

$$\max_{p\in\mathbb{C}^d} \langle \tilde{x}, p\rangle_{\Re} \qquad \text{subject to} \qquad \|p\|_{\mathcal{A}}^* \leq 1,$$

where $\langle x, y\rangle_{\Re} := \Re(\langle x, y\rangle)$ and

$$\|p\|_{\mathcal{A}}^* := \sup_{\substack{a\in\mathbb{K}^d:\\ \|a\|_{\mathcal{A}}\leq 1}} \langle p, a\rangle_{\Re} = \sup_{a\in\mathcal{A}} \langle p, a\rangle_{\Re}.$$

- Plugging in the form of the atoms $a \in \mathcal{A}$ we obtain

$$\|p\|_{\mathcal{A}}^* = \max_{w\in[0,1)} \left|\langle \psi(e^{2\pi i w}), p\rangle\right|$$

Semidefinite formulation for $\|p\|_{\mathcal{A}}^* \leq 1$

Theorem (Nonnegative trigonometric polynomials and Hermitian Gram matrices)

For $p \in \mathbb{C}^d$, the following are equivalent.

1. *We have $\left|\langle \psi(e^{2\pi i w}), p\rangle_{\Re}\right| \leq 1$ for all $w \in [0,1)$.*
2. *There exists a Hermitian matrix $Q \in \mathbb{C}^{d\times d}$ such that*

$$\begin{pmatrix} Q & p \\ p^{\mathsf{H}} & 1 \end{pmatrix} \succeq 0 \qquad \text{and} \qquad T^*(Q) = e_0,$$

where $T^*(Q)_k = \mathrm{Tr}[\Theta_k Q]$ and Θ_k is the Toeplitz matrix whose first row is the k-th unit vector e_k, where $k \in \{0, \ldots, d-1\}$.

$\rightsquigarrow$ Dual problem can easily be solved by convex solvers

Localising the frequencies

– Let $\tilde{p}$ be the solution of the dual problem. Then

$$\{\tau_k\}_k = \{w \in [0,1) : |\langle a(w,0), \tilde{p}\rangle| = 1\}.$$

$\rightsquigarrow$ the spike locations are the extrema of $|\langle a(\cdot,0), \tilde{p}\rangle|$

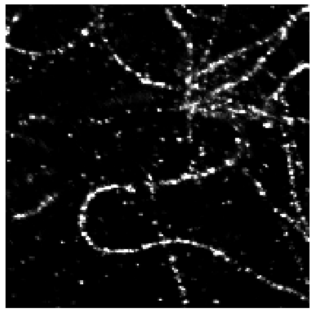
$\rightsquigarrow$ find roots of a polynomial on the unit circle.

– Using support estimate T_{est}, the c_j can be reconstructed by solving the system

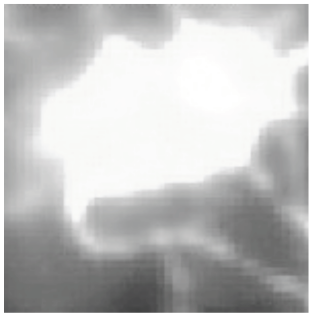
$$\sum_{\tau_j \in T_{\text{est}}} c_j e^{-2\pi i k \tau_j} = \tilde{x}_k, \qquad |k| \leq f_c$$

using *least squares*.

Conclusion



is observed as



Ground truth

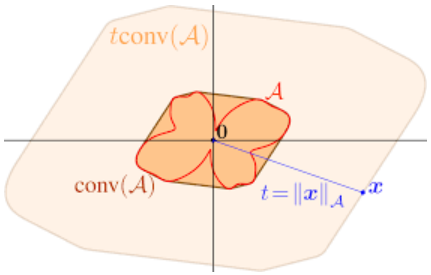
$$x = \sum_{k=1}^{r} c_k \delta_{\tau_k}$$

Observed data $\psi(z) = (1, z, z^2, \ldots)$

$$\tilde{x} = \sum_{k=1}^{r} |c_k| e^{-2\pi i \varphi_k} \psi(e^{-2\pi i \tau_k})$$

recover?

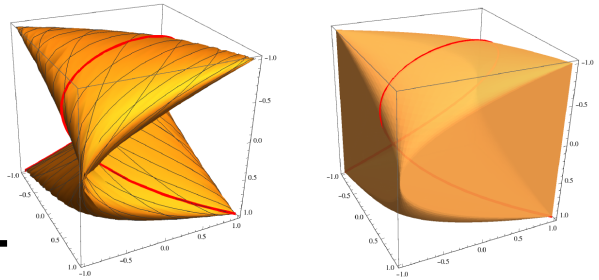
Atomic Norm $\|\tilde{x}\|_{\mathcal{A}}$



$$\inf\{r > 0 : x \in r\,\mathrm{co}(\mathcal{A})\}$$

induces

Atomic Set $\mathcal{A}$



$$\{e^{-2\pi i \varphi} \psi(e^{-2\pi i w})\}_{w,\varphi \in [0,1)}$$

Thank you for your attention!

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